Evading the IR Problem of Thermal QCD

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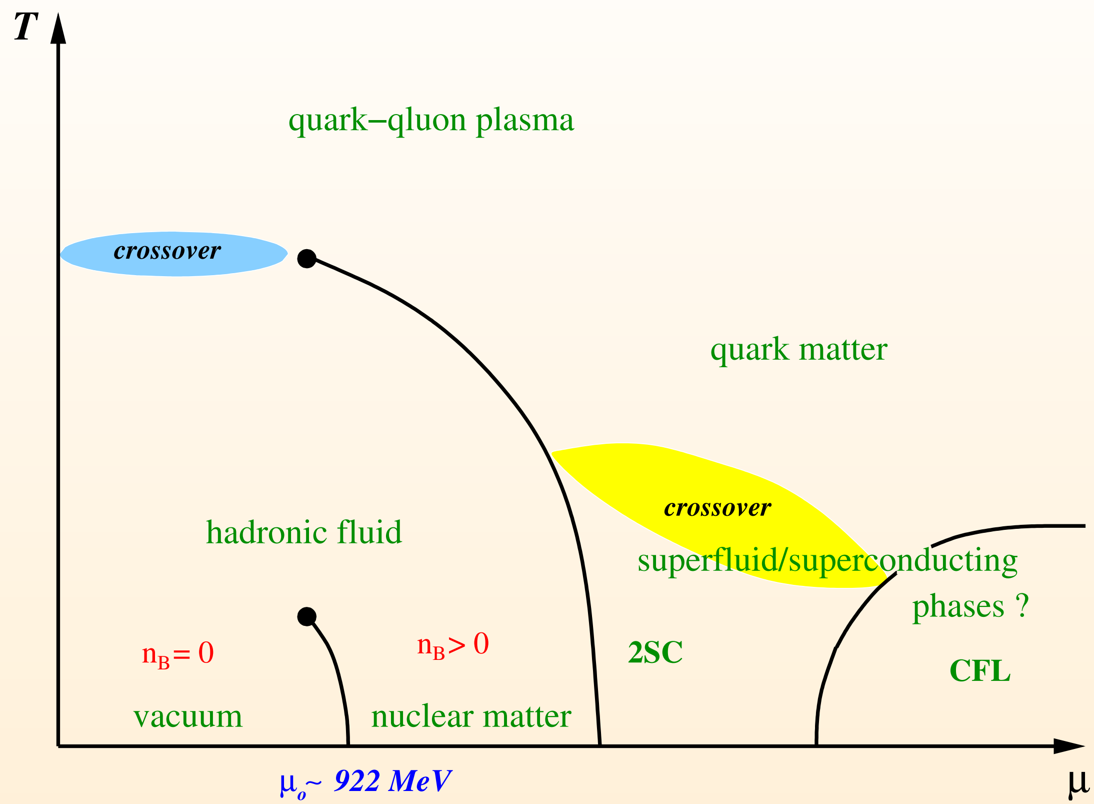
The QCD (equilibrium) phase diagram

$\mathcal{L}(\psi, A_\mu) \to p, n$

squeeze (μ_B) / heat (T) $\to$ simpler?

The QCD (equilibrium) phase diagram



nature: early univ, μ tiny ($\sim \frac{\#baryons}{entropy}$), $T_c \sim 170 MeV \sim 10\mu s$
neutron/quark stars

lab expt.: SPS / RHIC $\mu_B \sim \frac{\#baryons}{pions} \sim 45 MeV$ / LHC / GSI

Outline

The IR problem

QCD pressure: lattice

QCD pressure: perturbative

Methods

- Diagram generation
- reduction
- integration
- lattice MC
- lattice: perturbative

Conlusions

The IR problem

from now on: high T, $\mu_B \equiv 0$

$\Rightarrow$ asymptotic freedom!

$$g(Q \sim T) \overset{ren}{\propto} \frac{1}{\ln \frac{T}{\Lambda_{\overline{\text{MS}}}}} \overset{T\to\infty}{\longrightarrow} 0$$

$\Rightarrow$ QGP = ideal gas of quarks/gluons?
+ perturbative corrections?

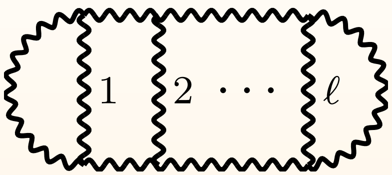
$T > 0$: $\int d^4p \to T \sum_{p_0} \int d^3p$

Matsubara frequencies: $p_0 = 2\pi n T$ (gluons, ghosts)
$p_0 = 2\pi(n + 1/2)T$ (fermions)

$\Rightarrow$ IR divergences from bosons, $n = 0$, *'zero mode'*

The IR problem

[Linde 1979; Gross/Pisarski/Yaffe 1981]



(ℓ+1) loops, 2ℓ vert, 3ℓ propags

$$\sim \left(T \sum_n \int d^3p \right)^{\ell+1} \frac{(gp)^{2\ell}}{[(2\pi n T)^2 + p^2 + \Pi(2\pi n T, p)]^{3\ell}}$$

IR power counting: n=0, define $\Pi(0, p \to 0) \equiv m^2$

$$\sim \; T^{\ell+1} g^{2\ell} m^{3(\ell+1)+2\ell-6\ell} \;=\; g^6 T^4 \left(\frac{g^2 T}{m} \right)^{\ell-3}$$

- $\Pi_L(0,p) \sim (gT)^2 \Leftarrow$ OK: get series in g
- $\Pi_T(0,p) \sim (g^2T)^2 \Leftarrow$ all orders important!

Evading the IR problem

what now?

- resummations? $\longrightarrow$ not known
- change language!
  3 physical scales: $2\pi T$, gT, g^2T [think pure gauge for now]
  $\Rightarrow$ effective theories
  3 contributions [Braaten/Nieto 1995]
  3rd is long distance $\longrightarrow$ non-pert $\longrightarrow$ LAT

why not LAT from the outset?

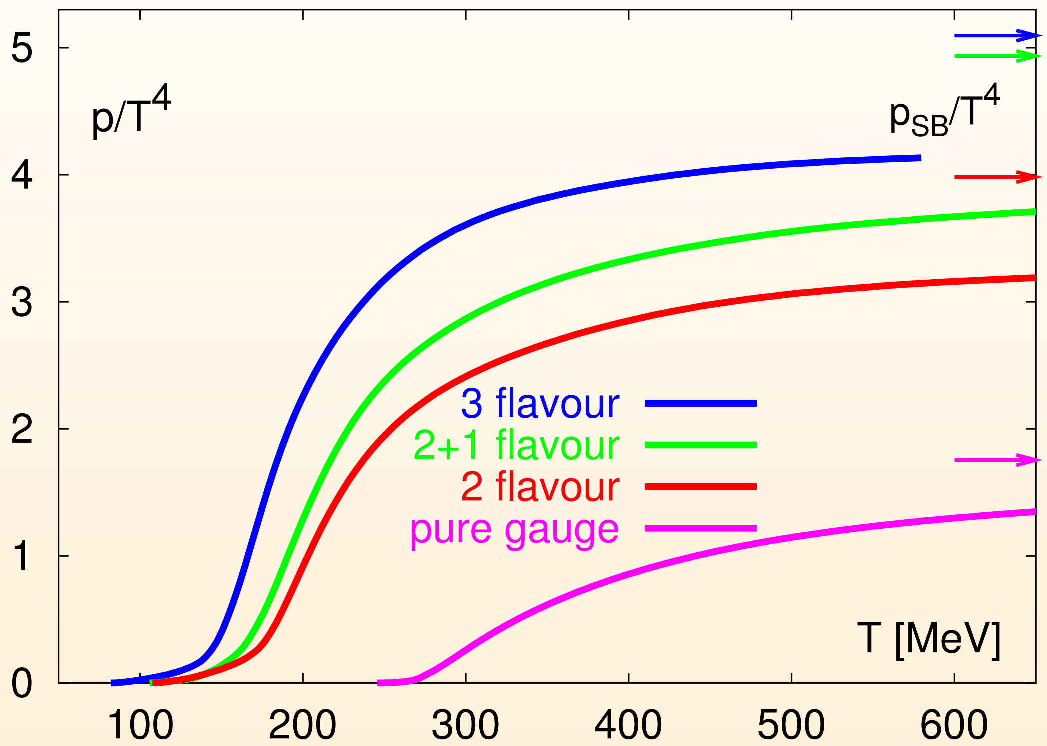
- original problem: 4d QCD, LAT
  - ▷ N_f *''hard''*
  - ▷ μ_B *''exp. hard''*
- effective theories: 4d QCD, pert
  +3d SU(3) + adj. Higgs, pert
  +3d SU(3), LAT
  - ▷ *3d bosonic theory!* N_f *OK*, μ_B *OK.*
  - ▷ *superrenormalizable!*

can I see an example?

- yes - QCD pressure, $p \overset{V\to\infty}{=} -f$

$$\exp\left[-f(T,g)\frac{V}{T}\right] = \int \mathcal{D}[A\bar{\psi}\psi]\exp\left[-\int_0^{1/T} d\tau \int d^3x \mathcal{L}_E[A\bar{\psi}\psi]\right]$$

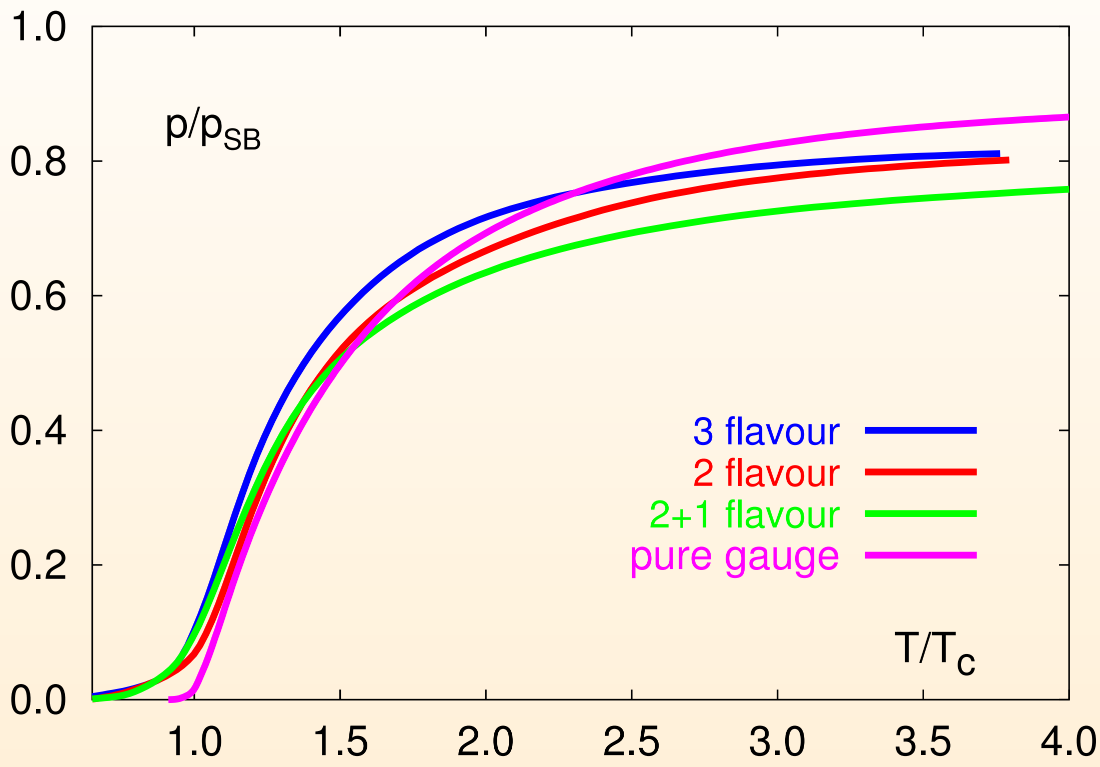
$p(T)$ on the lattice ($\mu_B = 0$)



[from Karsch et.al.]

at $T \to \infty$, expect ideal gas: $p_{\text{SB}} = \left(16 + \frac{21}{2} N_f\right) \frac{\pi^2 T^4}{90}$

$p(T)$ on the lattice ($\mu_B = 0$)



[from Karsch et.al.]

at $T \to \infty$, expect ideal gas: $p_{\text{SB}} = \left(16 + \frac{21}{2} N_f\right) \frac{\pi^2 T^4}{90}$

$p(T)$ in (thermal) perturbation theory

want to compute the QCD pressure

$$\begin{aligned}
p_{\mathsf{QCD}}(T) &\equiv \lim_{V\to\infty}\frac{T}{V}\ln\int\mathcal{D}A^a_\mu\,\mathcal{D}\psi\,\mathcal{D}\bar\psi\,\exp\Big(-\frac{1}{\hbar}S_{\mathsf{QCD}}\Big)\\
S_{\mathsf{QCD}} &= \int_0^{\beta\hbar}d\tau\int d^dx\,\mathcal{L}_{\mathsf{QCD}}\\
\mathcal{L}_{\mathsf{QCD}} &= \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}+\bar\psi\gamma_\mu D_\mu\psi+\mathcal{L}_{\mathsf{GF}}+\mathcal{L}_{\mathsf{FP}}
\end{aligned}$$

$p_{\mathsf{QCD}}(T)$ renormalised such that it vanishes at $T=0$.

$p(T)$ in (thermal) perturbation theory

integrate out $|p| \gtrsim 2\pi T$: ψ, $A_\mu(n \neq 0)$

$$
\begin{aligned}
p_{\mathsf{QCD}}(T) &\equiv p_{\mathsf{E}}(T) + \frac{T}{V} \ln \int \mathcal{D}A_k^a\, \mathcal{D}A_0^a \exp\Big(- \int d^d x\, \mathcal{L}_{\mathsf{E}} \Big) \\
\mathcal{L}_{\mathsf{E}} &= \frac{1}{2} Tr\, F_{kl}^2 + Tr\, [D_k, A_0]^2 + m_{\mathsf{E}}^2 Tr\, A_0^2 + \lambda_{\mathsf{E}}^{(1)} (Tr\, A_0^2)^2 + \lambda_{\mathsf{E}}^{(2)} Tr\, A_0^4 + \ldots
\end{aligned}
$$

five matching coefficients $p_{\mathsf{E}} \sim T^4$, $m_{\mathsf{E}}^2 \sim g^2 T^2$, $g_{\mathsf{E}}^2 \sim g^2 T$, $\lambda_{\mathsf{E}}^{(1/2)} \sim g^4 T$.

higher order operators

$$\delta \mathcal{L}_{\mathsf{E}} \sim g^2 \frac{D_k D_l}{(2\pi T)^2} \mathcal{L}_{\mathsf{E}}$$

could contribute (most conservative estimate)

$$\frac{\delta p_{\mathsf{QCD}}(T)}{T} \sim \delta \mathcal{L}_{\mathsf{E}} \sim g^2 \frac{(gT)^2}{(2\pi T)^2} (gT)^3 \sim g^7 T^3$$

$p(T)$ in (thermal) perturbation theory

integrate out $|p| \gtrsim gT$: A_0

$$\frac{T}{V}\ln\int \mathcal{D}A_k^a\,\mathcal{D}A_0^a\exp\Big(-S_\mathsf{E}\Big) \equiv p_\mathsf{M}(T)+\frac{T}{V}\ln\int \mathcal{D}A_k^a\exp\Big(-\int d^dx\,\mathcal{L}_\mathsf{M}\Big)$$

$$\mathcal{L}_\mathsf{M} = \frac{1}{2}Tr\,F_{kl}^2+\ldots$$

two matching coefficients $p_\mathsf{M}\sim m_\mathsf{E}^3 T$, $g_\mathsf{M}^2\sim g_\mathsf{E}^2$.

higher order operators

$$\delta\mathcal{L}_\mathsf{M}\sim g_\mathsf{E}^2\frac{D_kD_l}{m_\mathsf{E}^3}\mathcal{L}_\mathsf{M}\overset{e.g.}{\sim}\frac{g_\mathsf{E}^2}{T^3}Tr[D_i,F_{i,j}]^2$$

could contribute

$$\frac{\delta p_\mathsf{QCD}(T)}{T}\sim\delta\mathcal{L}_\mathsf{M}\sim g_\mathsf{E}^2\frac{(g^2T)^2}{m_\mathsf{E}^3}(g^2T)^3\sim g^9T^3$$

$p(T)$ in (thermal) perturbation theory

$\mathcal{L}_{\mathsf{M}}$ only has one (dimensionful) parameter

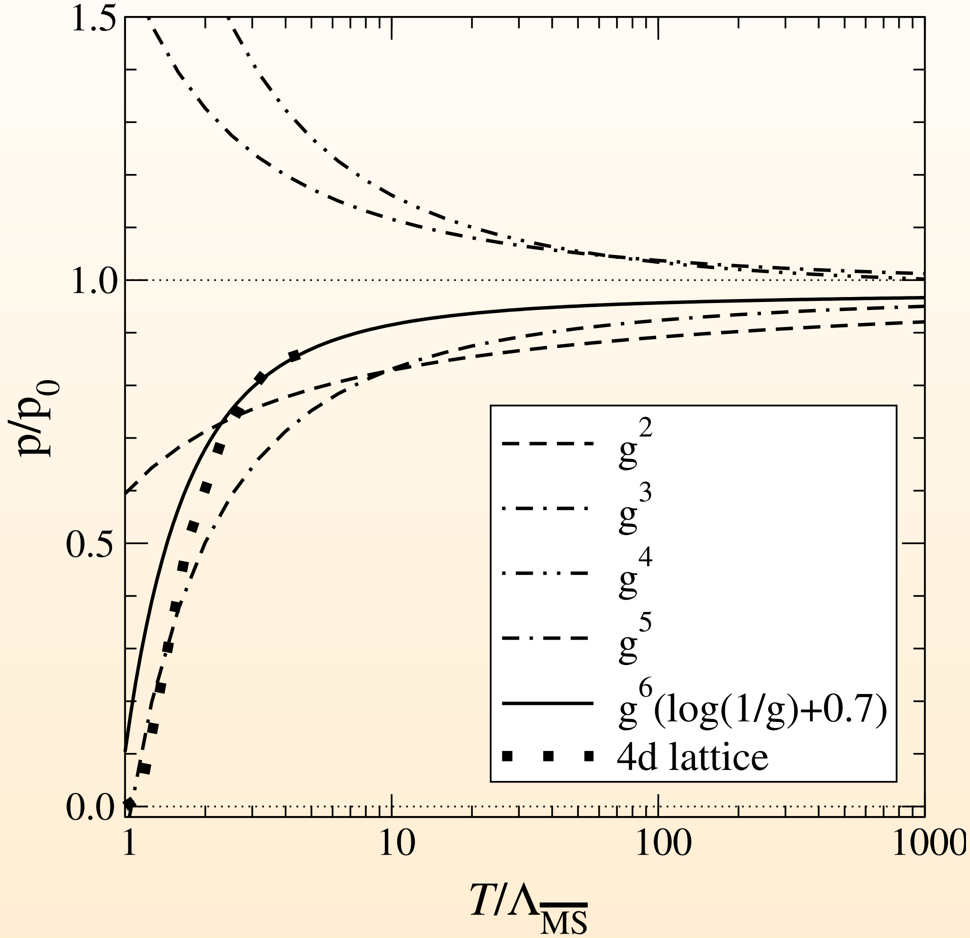
$$p_{\mathsf{G}}(T) \quad\equiv\quad \frac{T}{V}\ln\int\mathcal{D}A_k^a\exp\Big(-S_{\mathsf{M}}\Big)\sim Tg_{\mathsf{M}}^6$$

coefficient is non-perturbative!

$$\begin{aligned}
\frac{p_{\mathsf{QCD}}(T)}{p_{\mathsf{SB}}} &= \frac{p_{\mathsf{E}}(T)}{p_{\mathsf{SB}}}+\frac{p_{\mathsf{M}}(T)}{p_{\mathsf{SB}}}+\frac{p_{\mathsf{G}}(T)}{p_{\mathsf{SB}}}\\
&= 1+g^2 \quad +g^4 \quad +g^6 \quad +\dots && \Leftarrow \text{4d QCD}\\
&\quad +g^3+g^4+g^5+g^6+\dots && \Leftarrow \text{3d adj H}\\
&\quad +\frac{1}{p_{\mathsf{SB}}}\frac{T}{V}\int\mathcal{D}[A_i]\exp[-\int d^3x\mathcal{L}_{\mathsf{M}}] && \Leftarrow \text{3d YM}\\
&= c_0+c_2g^2+c_3g^3+(c_4'\ln g+c_4)g^4+c_5g^5+(c_6'\ln g+c_6)g^6+\mathcal{O}(g^7)
\end{aligned}$$

[c_2 Shuryak 78, c_3 Kapusta 79, c_4' Toimela 83, c_4 Arnold/Zhai 94, c_5 Zhai/Kastening 95, Braaten/Nieto 96, c_6' KLRS 03], c_6 ??

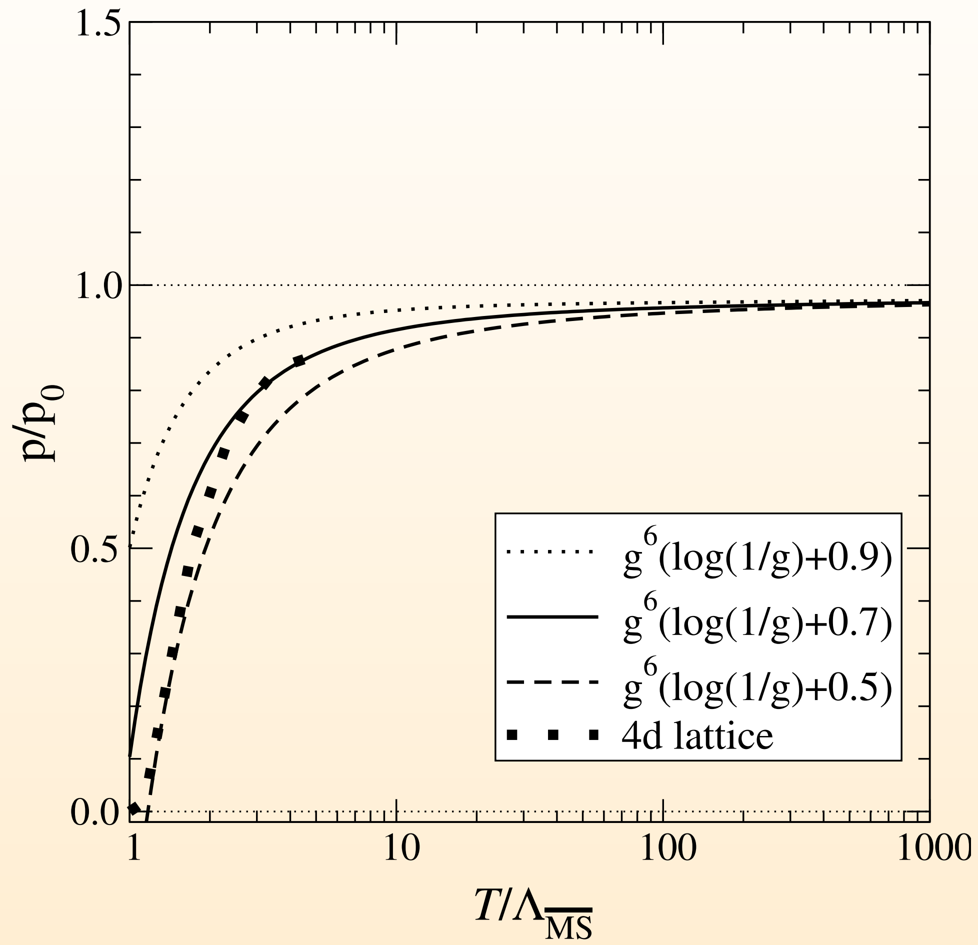
$p(T)$ in (thermal) perturbation theory



g^6 constant is a guess.

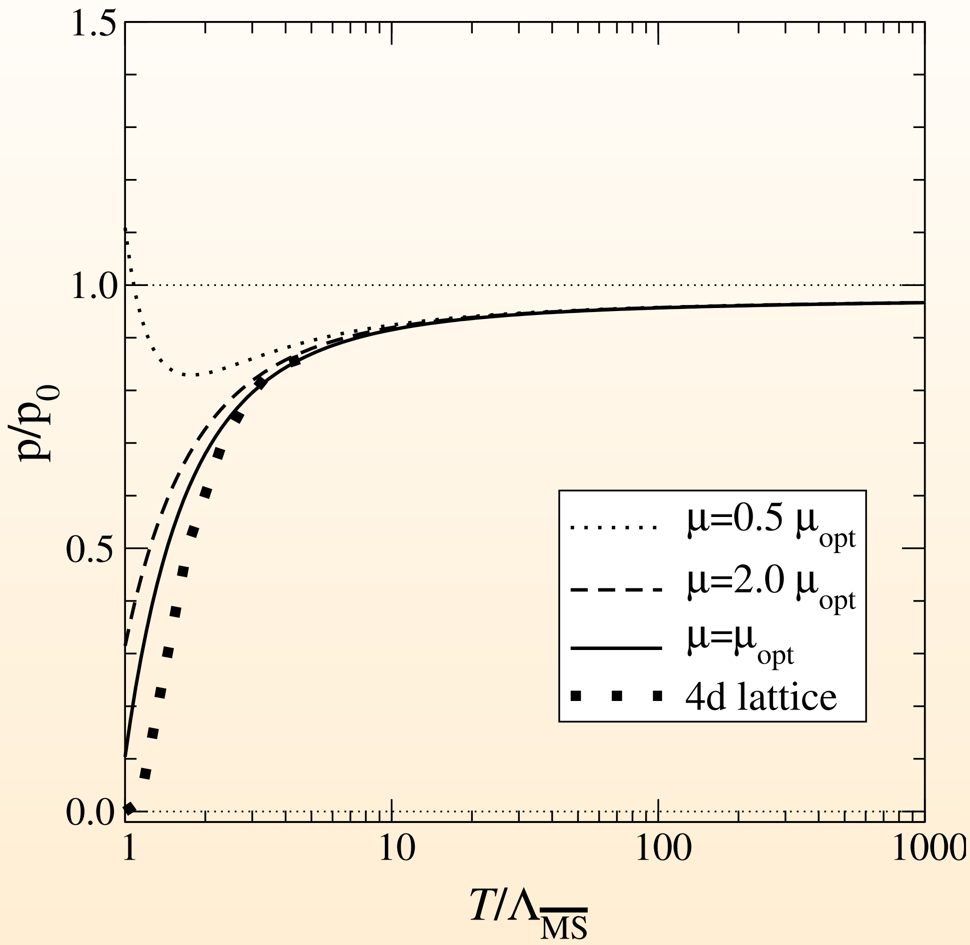
non-perturbative contrib not known, but computable!

$p(T)$ in (thermal) perturbation theory



depencence on g^6 constant

$p(T)$ in (thermal) perturbation theory



scale dependence

shopping list for c_6

$\ldots + g^6$

- 4-loop sum-integrals needed, const term
- DOABLE?! manpower OR brainpower?

matching coeffs

- 2-loop ϵ-terms for m_E^2, g_E^2 DONE. ML/YS, in prep.

$\ldots + g^6$

- 4-loop integrals needed DONE. KLRS: reduction, master ints

match $\overline{\mathsf{MS}}$/LAT

- 4-loop const in LAT reg
- DOABLE?! YS+Parma: NSPT

$\ldots + g^6$

- measure $<$ Plaquette $>$ in 3d SU(N) DONE. HKLRS, in prep.

Methods I: Diagram generation

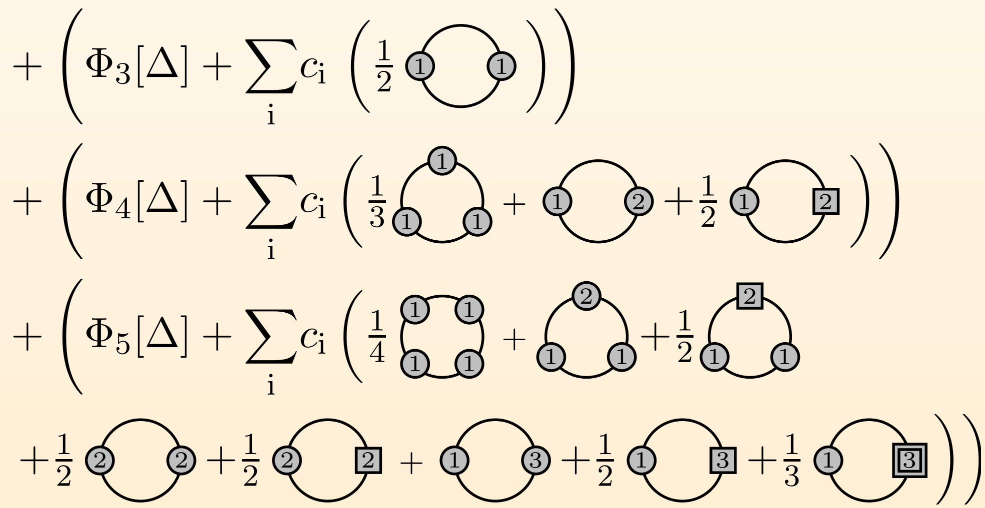
yet another generator? QGRAF [Nogueira], FeynArts [Denner/Hahn] n/a for 0-pt fcts.

skeleton (2PI) expansion [Luttinger/Ward, Baym, ...]

$$F[D] = \sum_i c_\mathrm{i} \left(Tr \ln D_\mathrm{i}^{-1} + Tr\, \Pi_\mathrm{i}[D] D_\mathrm{i} \right) - \Phi[D]$$

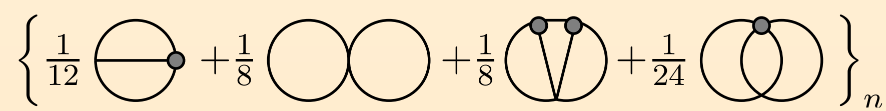
extremal property of partition function $\Rightarrow \delta_{D_\mathrm{i}} \Phi[D] = c_\mathrm{i} \Pi[D]$

$$-F = -F_0 + \Phi_2[\Delta]$$



get skeletons from

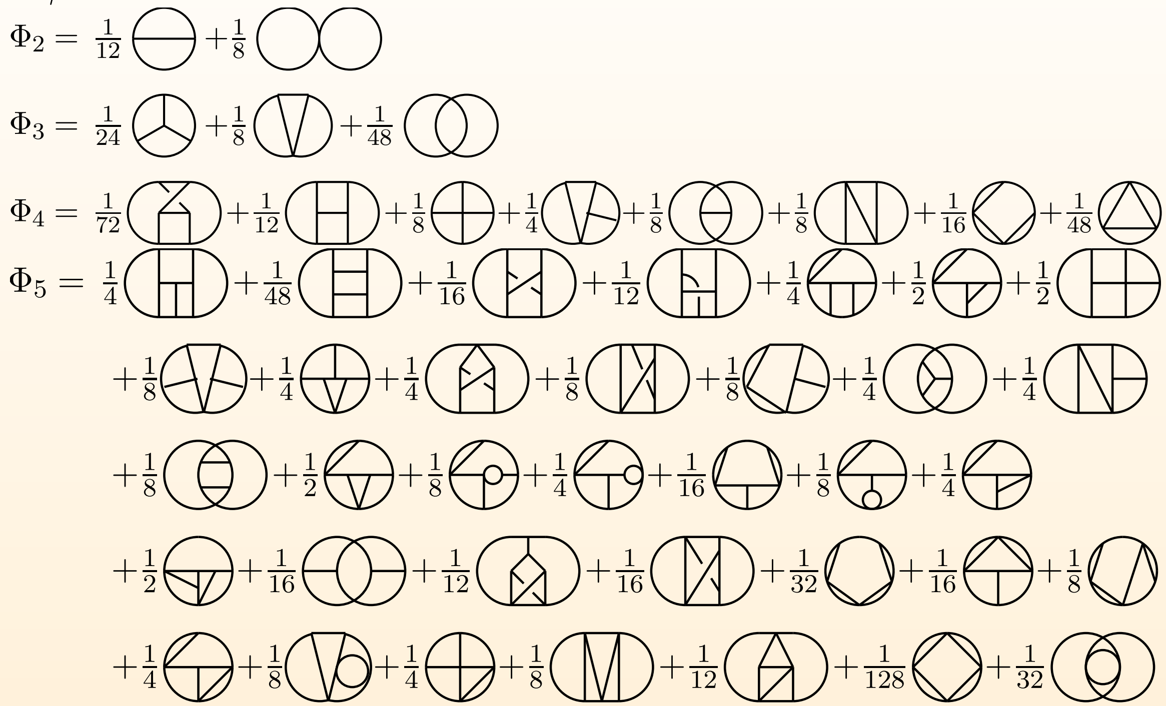
$$\Phi_n[\Delta] = \frac{1}{n-1}$$



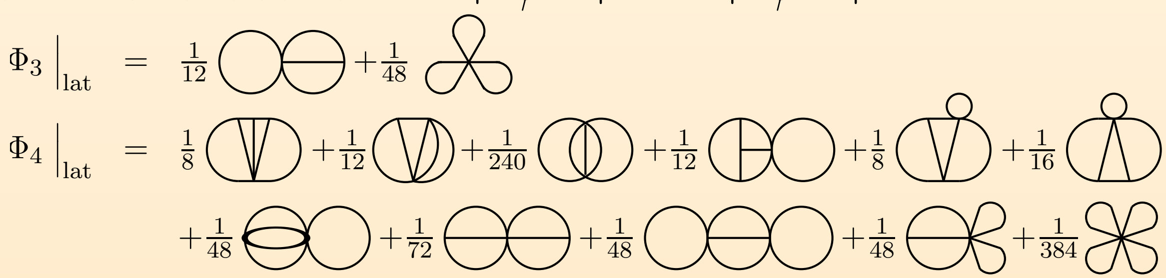
and SD eqs $\Gamma_n^{1PI} = \delta_\phi^{n-1} S'[\phi + D[\phi]\delta_\phi]\Big|_{\phi=0}$

Methods I: Diagram generation

generic $\phi^3 + \phi^4$ skeletons

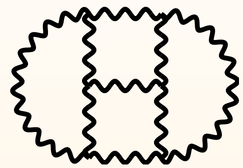


LAT: additional skeletons $\ldots + \phi^5 + \ldots + \phi^8 + \ldots$



Methods II: reduction, IBP

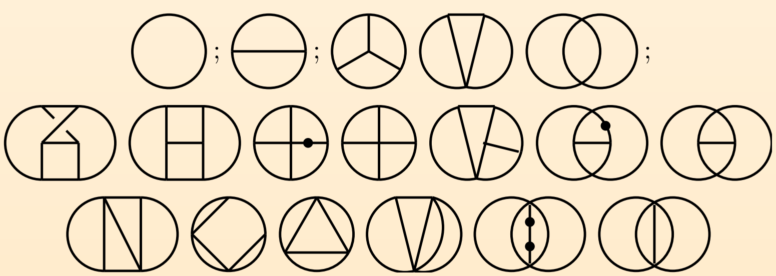
Laporta.. no need to detail it here in KA..

in a nutshell, reduces e.g.  25M integrals $(2^9 6^6)$

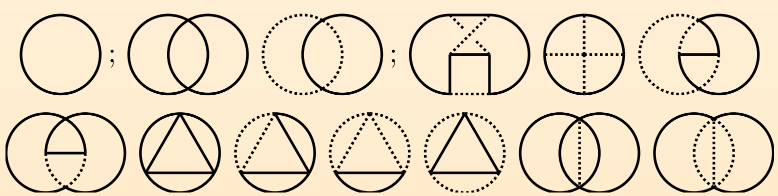
to

$$d_A C_A^3 \frac{g^6}{(4\pi)^4} \sum_i \frac{poly_i(d,\xi)}{poly_i(d)} Master_i(d)$$

18 fully massive master ints



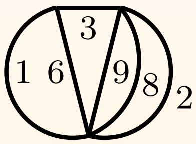
13 "QED" type master ints



Methods III a: analytic Integration

3d, Euclidean, massive, dim. reg., $\overline{\text{MS}}$, x-space, ...

one 3d example[YS,AV]:



$$= \left(\frac{\bar{\mu}}{m_{316}}\right)^{8\epsilon} \frac{1}{32} \left[\frac{1}{\epsilon^2} + \frac{8}{\epsilon} + 4\,S\left(\frac{m_{316}}{m_{16289}}, \frac{2m_1}{m_{316}}, \frac{2m_3}{m_{316}} - 1\right) + \mathcal{O}(\epsilon)\right]$$

$$\text{where} \quad S(x,y,z) = 13 + \frac{7}{12}\pi^2 + 2\mathrm{Li}_2(1-y) + 2\mathrm{Li}_2(y+z) + 2\mathrm{Li}_2(-z) - 4\,(\ln x)^2 + 8\frac{1-x}{x(1+z)}\,\mathrm{Li}_2(1-x) + 8\left(1 + \frac{1-x}{x(1+z)}\right)\left(\mathrm{Li}_2(-xz) + \ln(x)\ln(1+xz) - \frac{\pi^2}{6}\right)$$

a semi-analytic approach (need a difference equation, see next slides):

Harmonic Sums $\quad S_{\bar{m}}(x) \quad$ [Vermaseren]

Methods III b: numeric Integration, Deqs

very general setup [Laporta]

derive difference equation for generalized master $U(x) \equiv \int \frac{1}{D_1^x D_2 \ldots D_N}$

$$\sum_{j=0}^{R} p_j(x) U(x+j) \quad = \quad F(x)$$

solve via factorial series $U(x) = U_0(x) + \sum_{j=1}^{R} U_j(x)$, where

$$U_j(x) = \mu_j^x \sum_{s=0}^{\infty} a_j(s) \frac{\Gamma(x+1)}{\Gamma(x+1+s-K_j)}$$

plug in, get μ, $K_j(d)$, and recursion rels for $a_j(s)$.

need bc for fixing, say, $a_j(0)$

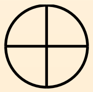
Methods III b: numeric Integration, Deqs

particularly simple bc at large x:

$$
\begin{aligned}
U(x) &= \int \frac{1}{(p_1^2+1)^x}\, g(p_1) \\
\lim_{x\to\infty} U(x) &= \left[\int \frac{1}{(p_1^2+1)^x}\right] \times \Big[g(0)\Big] \sim (1)^x x^{-d/2} g(0)
\end{aligned}
$$

while factorial series behaves as $\sum_j \mu_j^x x^{K_j} a_j(0)$

numerics: truncate sum. example:



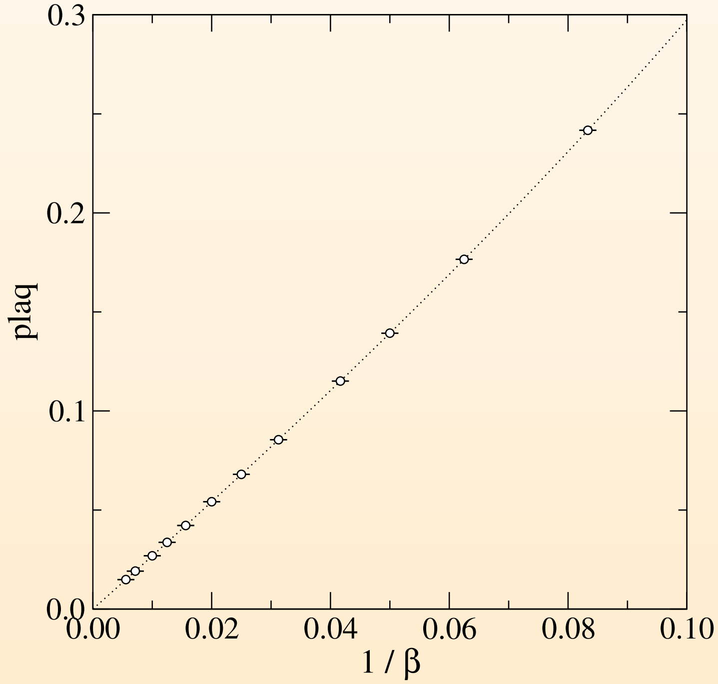
$$
\begin{aligned}
= \; & + 1.27227054184989419939788 - 5.67991293994853579036683\epsilon \\
& + 17.67972389481737323437 88\epsilon^2 - 46.57218466495432618640 19\epsilon^3 \\
& + 111.65852217621438536356 8\epsilon^4 - 252.46396390100217743236\epsilon^5 \\
& + 549.30166596161426941705\epsilon^6 - 1164.51205889715216235 46\epsilon^7 + \mathcal{O}(\epsilon^8)
\end{aligned}
$$

Methods IV: Lattice simulation

3d, finite box $(aL)^3$. infinite-volume ($L \to \infty$) and continuum ($\frac{1}{\beta} \equiv \frac{g_M^2 a}{2N_c} \to 0$) limits

$$\frac{1}{2g_M^2}\left\langle Tr\,[F_{kl}^2]\right\rangle_{\overline{MS}} \equiv g_M^2 \frac{\partial}{\partial g_M^2} p_{G,\overline{MS}} = 3g_M^6 \frac{d_A C_A^3}{(4\pi)^4}\left[\alpha_G\left(\ln\frac{\bar{\mu}}{2C_A g_M^2} - \frac{1}{3}\right) + B_G + \mathcal{O}(\epsilon)\right]$$

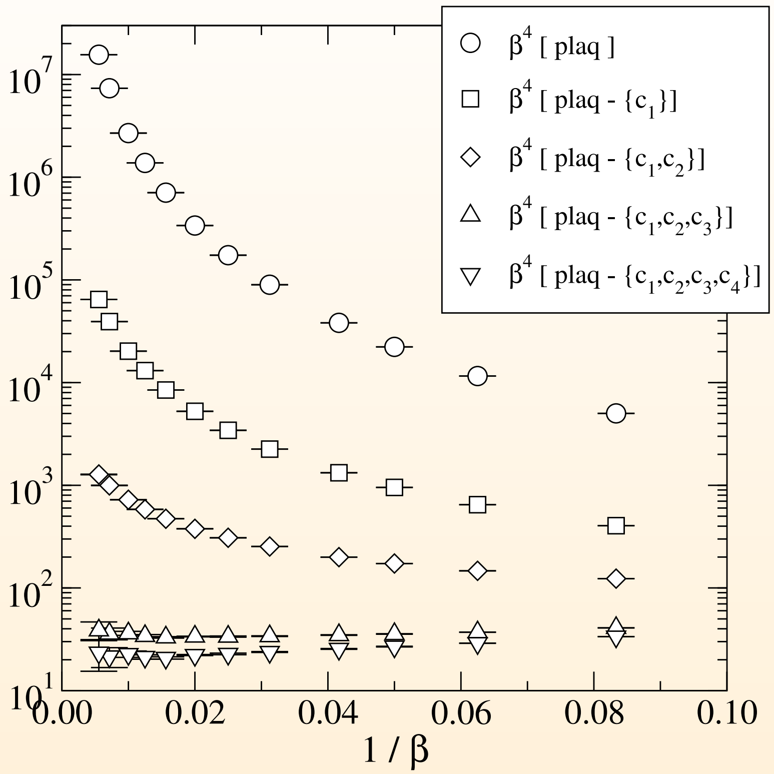
$$8\frac{d_A C_A^6}{(4\pi)^4} B_G = \lim_{\beta\to\infty} \beta^4 \left\{\left\langle 1 - \frac{1}{C_A} Tr\,[P_{12}]\right\rangle_a - \left[\frac{c_1}{\beta} + \frac{c_2}{\beta^2} + \frac{c_3}{\beta^3} + \frac{c_4}{\beta^4}\left(\ln\beta + c_4'\right)\right]\right\}$$

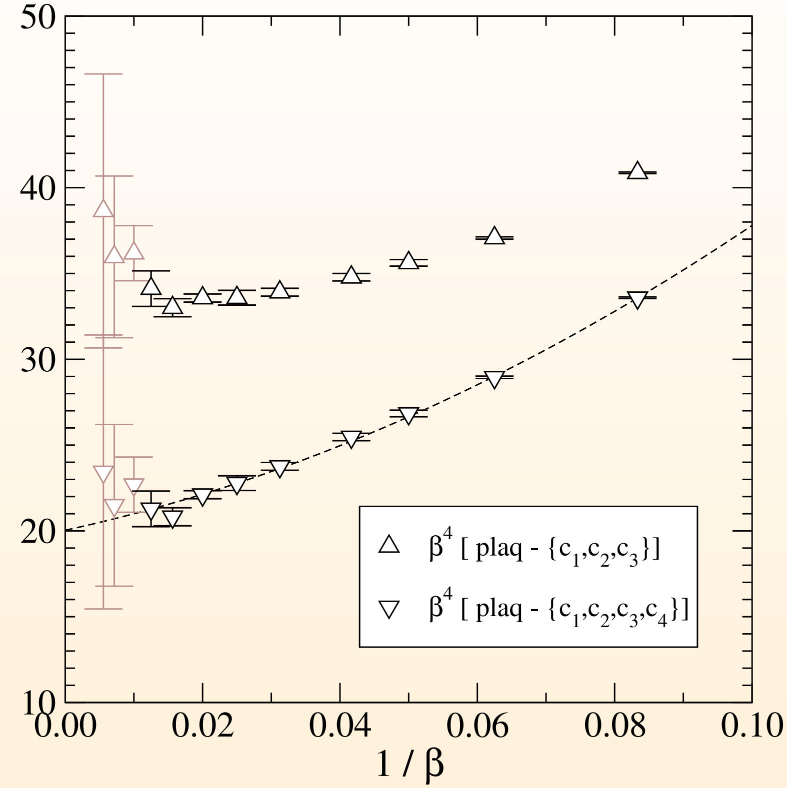


statistical errors are (much) smaller than the symbol sizes

Fit: $c_1/\beta + c_2/\beta^2 + c_3/\beta^3 + c_4 \ln\beta/\beta^4 \; + c_4'/\beta^4 \; + c_5/\beta^5 + c_6/\beta^6$

Methods IV: Lattice simulation





significance loss due to the UV subtractions

continum limit of infinite-volume extrapolated data

$$B_{\mathsf{G}} + \left(\frac{43}{12} - \frac{157}{768}\pi^2\right) c_4' \quad = \quad 10.7 \pm 0.4 \qquad (N_c = 3)$$

Methods V: Lattice perturbation theory

amusing: 1loop tadpole has elliptic integral in 3d [M.Shaposhnikov]

$$a^{2-d}\int_{-\pi}^{\pi}\frac{d^d\hat{k}}{(2\pi)^d}\frac{1}{\sum_{\mu=0}^{d-1}4\sin^2(\hat{k}_\mu/2)+\hat{m}^2}=\frac{1}{a}\sum_{n\geq 0}\hat{m}^{2n}\left(\{1,\Sigma,\xi\}+\{1\}\hat{m}\right)$$

where $\Sigma=4\pi G(0)=\frac{8}{\pi}(18+2\sqrt{2}-10\sqrt{3}-7\sqrt{6})K^2((2-\sqrt{3})^2(\sqrt{3}-\sqrt{2})^2)$

2loop example:

$$\kappa_5=\frac{1}{\pi^4}\int_{-\pi/2}^{\pi/2}d^3x\,d^3y\frac{\sum_i\sin^2 x_i\sin^2(x_i+y_i)\sin^2 y_i}{\sum_i\sin^2 x_i\sum_i\sin^2(x_i+y_i)\sum_i\sin^2 y_i}=1.013041(1)$$

$\longrightarrow$ classification? *very* little is known systematically.

1loop IBP + coordinate-space method [Lüscher/Weisz] [Becher/Melnikov]

or

Numerical Stochastic Perturbation Theory [with F. Di Renzo, V. Miccio]

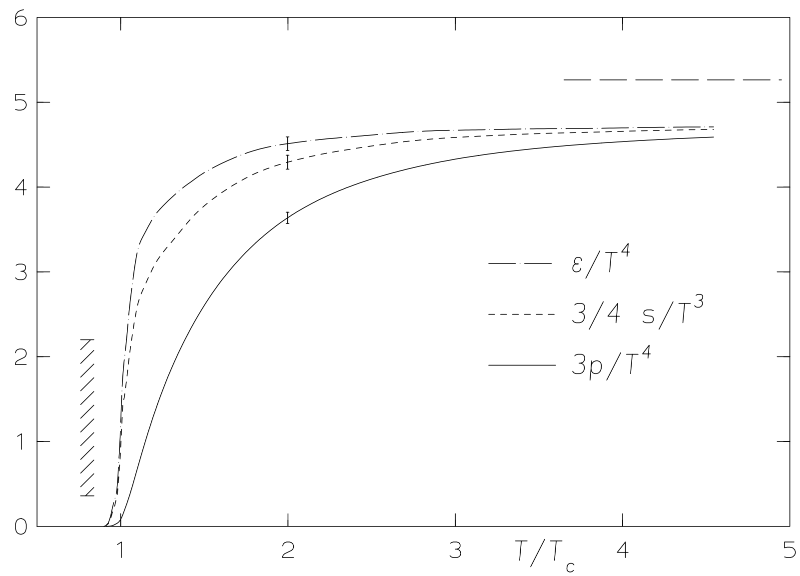
no diagrams!

Conclusions

- $p(T, \mu_B)$ is plagued by serious IR sensitivity, even at (asymptotically) high T.
- effective theory methods reduce the problem to a perturbative + 3d lattice MC setup for $T \gtrsim 2T_c$, $\mu_B \lesssim T$.
- 3d eff. th. brings major benefits: superren., cost, universality (Nf... μ... $i\mu N_f A_0^3$)
- a few well-defined (but hard) calculations left. Manpower and/or Brainpower (new methods)?
- on the way, tested a few nice techniques..

Backup slides

energy density in pure YM

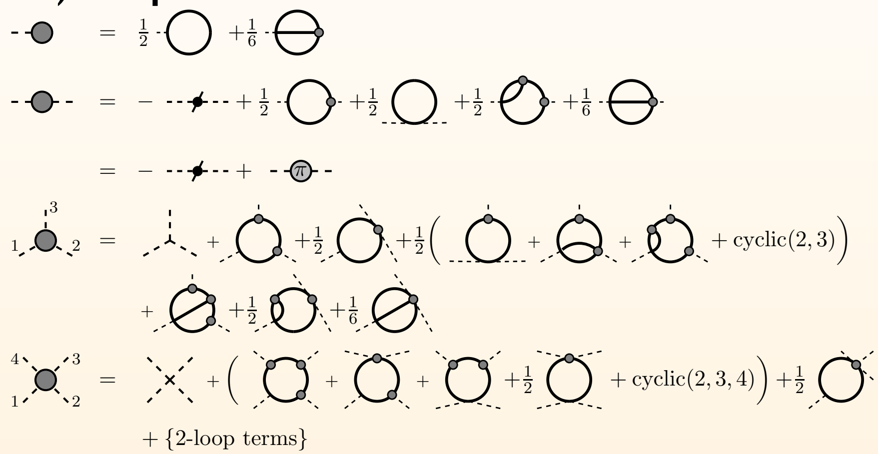


[pure YM, Boyd/Engels/Karsch/Laermann/Legeland/Lütgemeier/Petersson 1995]

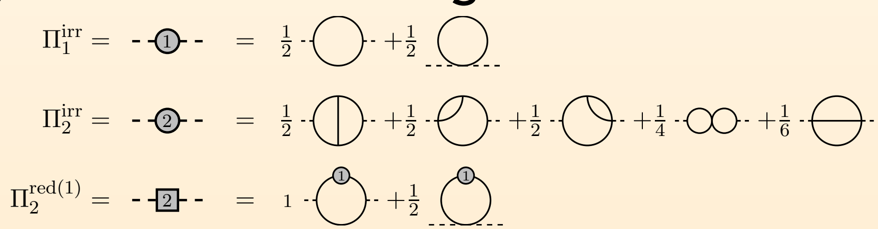
continuum extrapolated, $T_0 \approx 270MeV$

more on diagram generation

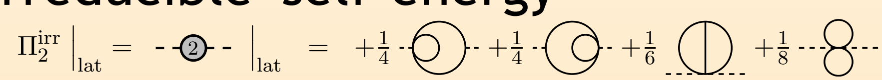
Schwiger-Dyson (SD) eqs



(2PI) skeletons generate self-energies. for bosonic particles ($c_i = \frac{1}{2}$):

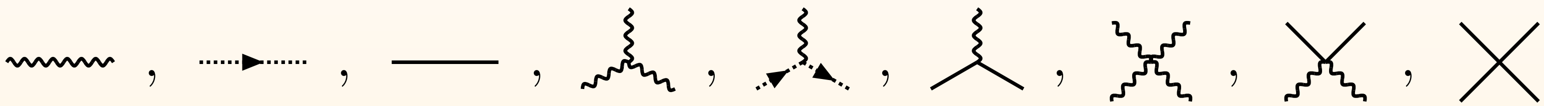


LAT: additional irreducible self-energy

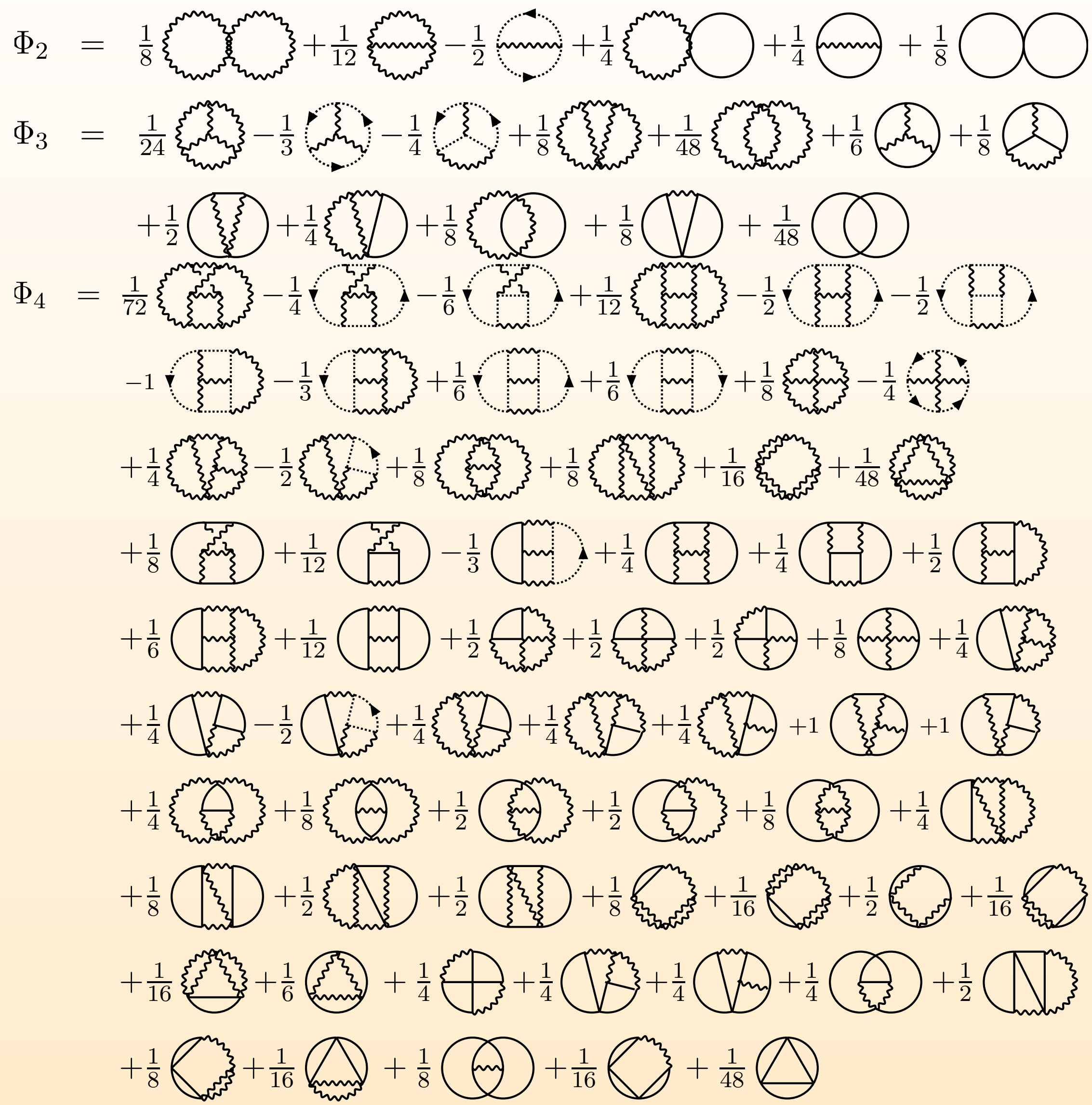


more on diagram generation

Now, consider SU(N) gauge theory with fermions and a scalar field. This class includes QCD, QED, EW, SQED

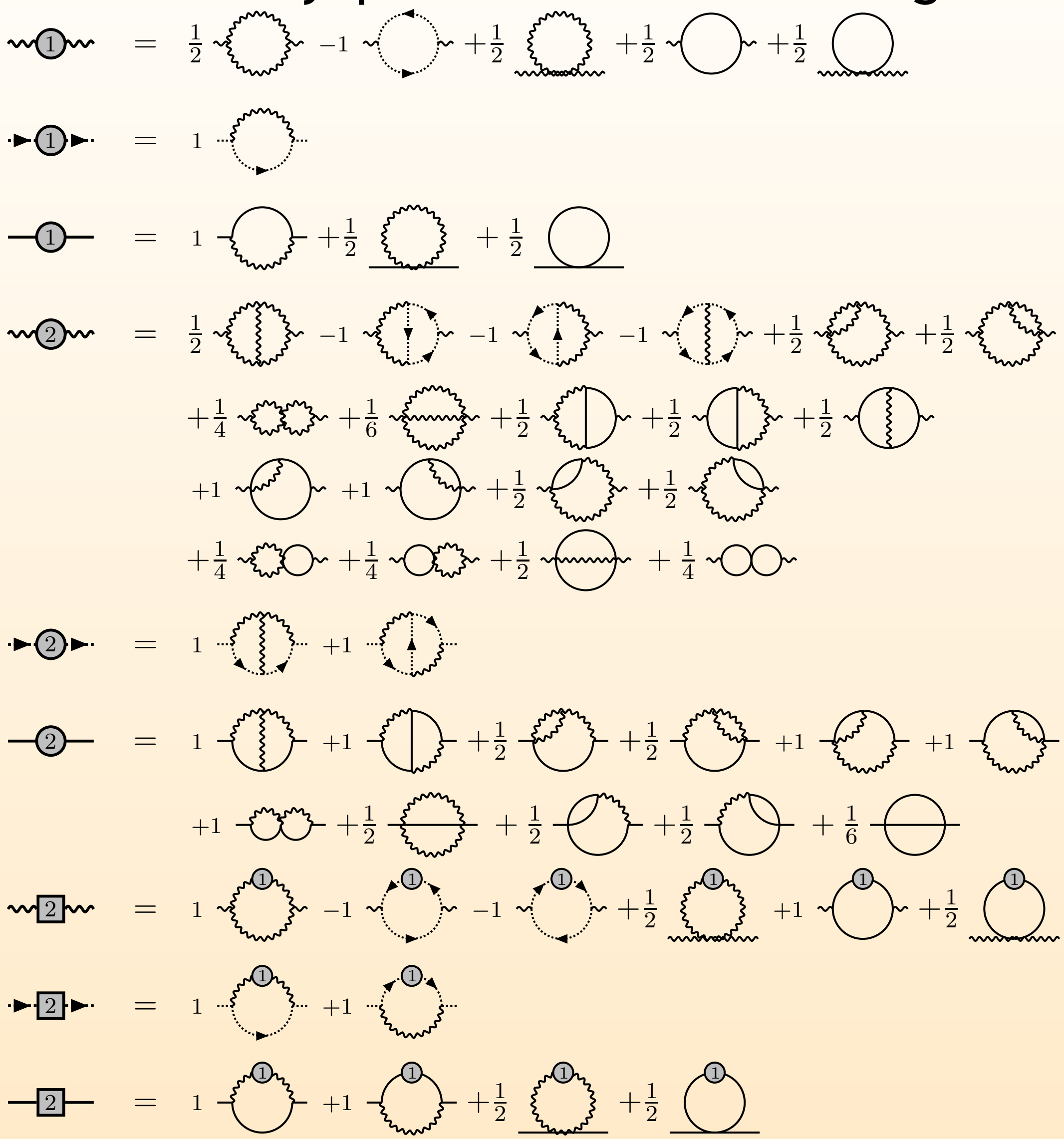


more on diagram generation



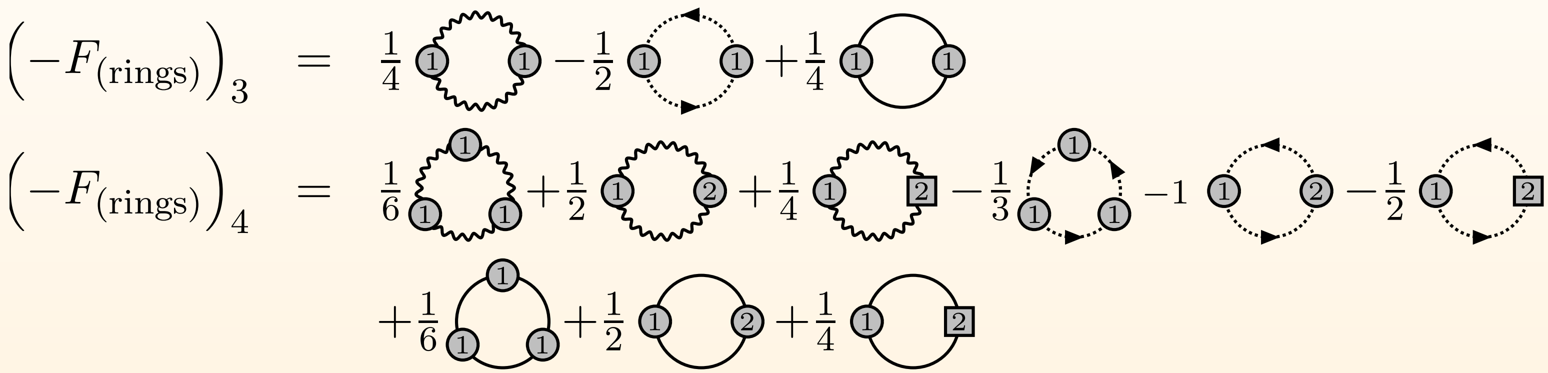
more on diagram generation

the skeletons immediately produce the self-energies of the model



more on diagram generation

ring diagrams for the model



extremely economic structure of the skeleton expansion:

the few ring diagrams above summarize

22 (276) 3-loop (4-loop) diagrams