Three-loop gap equation for a spontaneously broken 3d-Yang-Mills theory

Master's thesis

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1. Introduction

Quantum mechanics and special relativity belong to the most important cornerstones for the understanding of nature. The description of elementary particles and their interactions has to be done on a quantum level as well as in a special relativistic framework. Quantum field theory turned out to be a very successful way to combine these two concepts. But due to it's complexity all realistic calculations are constrained to approximation methods, like perturbation theory or lattice simulations. Perturbation theory has indeed proved it's worth in the small coupling regime in many applications from the high-precision predictions for cross sections and decay rates to the calculation of the anomalous magnetic moment.

Many properties of matter shortly after the Big Bang, in heavy ion collisions and in astroparticle physics can be understood by predictions of finite temperature field theories. But perturbation theory at finite temperature T in non-Abelian gauge theories breaks down due to the appearance of prohibitive infrared (IR) divergences, when massless fields are used [19]. The IR divergences are of so called *electric* and *magnetic* type [13]. The first one can be cured by introducing an effective infrared cut-off $m_{el} \propto gT$, the so called *electric mass*. The second type of infrared divergences is believed to be curable by an effective *magnetic* infrared cut-off $m_{mag} \propto g^2T$, the so called *magnetic mass*. The magnetic mass cannot be calculated at a finite order of perturbation theory, due to the so called *Linde problem* [14] and is therefore non-perturbative.

But estimates for the magnetic mass have been made perturbatively up to two-loop with resummed Yang-Mills theories, where a mass term is added to the massless Yang-Mills Lagrangian and subtracted again one order higher in perturbation theory [19, 22, 23, 25, 26, 27, 28], leading to so called gap equations, which determine the mass self-consistently.

At very high temperatures T one can work in a *dimensional reduced* setting, where the equilibrium properties of a four dimensional finite temperature field theory are approximated by a three dimensional effective zero-temperature field theory [15, 17].

Here we work with a recently proposed three dimensional Yang-Mills theory [22], where gauge boson masses are generated by a gauge invariant interaction with an auxiliary $SU(N)$ field. Together with a resummation scheme a gap equation is derived, whose solution should give a gauge invariant estimate for the non-perturbative magnetic mass. Indeed the gap equation leads to a gauge invariant solution in one- and two-loop, but we show that the approach used in this work leads to a gauge parameter dependent solution at higher order of perturbation theory.

This work is structured as follows. At first we will give a short introduction to the methods of perturbative quantum field theory and non-Abelian gauge theories. The diagrammatic approach in perturbation theory leads to large mathematical expressions in higher orders which can only be managed by automatized computer algebraic techniques. We will present here the integration-by-parts technique [43] and the so called Laporta algorithm [36] for the reduction of Feynman-integrals to so called *master-integrals*. These techniques will be applied to the gauge boson

self-energy up to three-loop in the above-named three dimensional spontaneously broken Yang-Mills theory.

Furthermore we will present a method to solve master-integrals by means of difference equations and apply it to a sunset class of vacuum bubble integrals.
Finally we will give a short review of the one- and two-loop results of the magnetic mass gap and will give some arguments why the gap equation cannot yield a gauge invariant solution at three-loop and higher orders.

The following notation is used in this work. For Lorentz indices Greek letters are used. The metric convention is $(\eta_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1, ...)$ in Minkowski space. For Minkowskian four-vectors x and y the summation convention $x_\mu y^\mu = \sum_{\mu,\nu} \eta_{\mu\nu} x^\mu x^\nu$ is used. In the Euclidean case we write $x_\mu y_\mu = \sum_\mu x_\mu y_\mu$. Color indices are written as letters from the beginning of the Latin alphabet like $a, b, c...$. It will not be distinguished between raised and lowered color indices. This means $T^a = T_a$.

2. Basics of perturbative Quantum Field Theory

In this chapter the basic concepts of perturbation theory we will summarized, followed by a brief discussion of regularization schemes and renormalization. We follow here basically the approach of [1, 2, 3].

2.1. Functional integrals and propagators

A natural way to arrange a relativistic theory is to start from a Lorentz invariant action

$$S[\phi] \equiv \int d^4x \mathcal{L}(\{\phi_l(x)\}, \{\partial_\mu \phi_l(x)\}), \tag{2.1}$$

with a Lagrangian density $\mathcal{L}(\{\phi_l(x)\}, \{\partial_\mu \phi_l(x)\})$ depending on a set $\{\phi_l(x)\}$ of quantum fields[1] as dynamical variables. At a fixed time t these fields provide a set of complete and orthonormal[2] eigenstates $\mid \{\phi_l\}\rangle$ with [4]

$$\hat{\phi}_l(\vec{x}) \mid \{\phi_l\}\rangle = \phi_l(\vec{x}) \mid \{\phi_l\}\rangle \tag{2.2}$$

as well as for the canonical momenta

$$\pi_l(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_l(x)} \tag{2.3}$$

with

$$\hat{\pi}_l(\vec{x}) \mid \{\pi_l\}\rangle = \pi_l(\vec{x}) \mid \{\pi_l\}\rangle. \tag{2.4}$$

The time evolution of a quantum mechanical system is generated by the Hamiltonian H defined via a Legendre transformation

$$H(\{\phi_l\}, \{\pi_l\}) \equiv \int d^3x \mathcal{H} = \int d^3x \left(\pi_l \frac{\partial \phi_l}{\partial t} - \mathcal{L} \right), \tag{2.5}$$

where $\frac{\partial \phi_l}{\partial t}$ has to be expressed in terms of π_l and ϕ_l by inverting eq.(2.3). Working in the Heisenberg picture and shortly omitting the label l for simplicity, the transition amplitude of an

[1] The index l distinguishes for instance the field type or different components of a field
[2] Orthonormal in the sense that $\langle \{\phi_l\} \mid \{\phi'_l\}\rangle = \prod_l \delta(\phi_l - \phi'_l)$

(i)nitial field configuration $\mid \phi_i, t_i\rangle$ at time t_i to a (f)inal configuration $\mid \phi_f, t_f\rangle$ at later time t_f is given by

$$\langle \phi_f, t_f \mid \phi_i, t_i\rangle = \langle \phi_f \mid e^{-iH(\phi,\pi)(t_f - t_i)} \mid \phi_i\rangle. \tag{2.6}$$

Dividing the time in $N+1$ small slices $t_0 = t_i$, $t_1 = t_i + \tau$,...,$t_N = t_f$ with $\tau = \frac{t_f - t_i}{N}$, and inserting unit operators, expressed in terms of intermediate field configurations $\mid \phi_n\rangle \equiv\mid \phi_n, t_n\rangle$ and $\mid \pi_n\rangle \equiv\mid \pi_n, t_n\rangle$ respectively, between the small slices one ends up with Feynman's famous path integral formula [2]

$$\begin{aligned}\langle \phi_f, t_f \mid \phi_i, t_i\rangle &= \int \mathcal{D}\pi \int_{\phi(\vec{x},t_i)=\phi_i(\vec{x})}^{\phi(\vec{x},t_f)=\phi_f(\vec{x})} \mathcal{D}\phi \\ &\quad \times \exp\left[i \int_{t_i}^{t_f} dt \int d^3x \left(\pi(\vec{x},t)\right) \frac{\partial \phi(\vec{x},t)}{\partial t} - \mathcal{H}(\pi(\vec{x},t), \phi(\vec{x},t))\right], \end{aligned} \tag{2.7}$$

where $\int \mathcal{D}\pi \int \mathcal{D}\phi \equiv \lim_{N\to\infty} \prod_{n=1}^{N} \int \frac{d\pi_n}{2\pi} \int d\phi_n$.

Many bosonic field theories have the exponent in eq.(2.7) quadratic in π without mixing terms of π and ϕ, so that one can integrate out the canonical momentum to obtain a functional integral over the fields ϕ only. Furthermore working in imaginary time $t \to \tau = -it$ and replacing the Lagrangian by an Euclidean Lagrangian $\mathcal{L} \to -\mathcal{L}_E$, one obtains Euclidean path integral [1]

$$\langle \phi_f, t_f \mid \phi_i, t_i\rangle = N \int \mathcal{D}\phi \exp\left(- \int_{\tau_i}^{\tau_f} d\tau \int d^3x \left[\mathcal{L}_E(\phi(x,\tau), \partial_\mu \phi(x,\tau))\right]\right), \tag{2.8}$$

where N is a field independent constant that will drop out in physical calculations. For large times $\tau_f \to \infty$ and $\tau_i \to -\infty$, due to exponential suppression of higher energy states, the amplitude $\langle \phi_f, \infty \mid \phi_i, -\infty\rangle$ is asymptotically proportional to the ground state transition amplitude $\langle 0, \infty \mid 0, -\infty\rangle$.

One of the most important quantities in this work will be the gauge boson two-point correlation function. A convenient way to define correlation functions is to introduce a generating functional by adding a source term $J\phi$ into the action in eq.(2.8) yielding[3] [3]

$$Z[J] \equiv \prod_l \int \mathcal{D}\phi_l \exp\left(- \int d^4x \left[\mathcal{L}_E(\{\phi_l(x)\}, \partial_\mu\{\phi_l(x)\}) + J_l(x)\phi_l(x)\right]\right). \tag{2.9}$$

The normalization N in eq.(2.8) can be chosen so that $Z[J=0] = \langle 0, \infty \mid 0, -\infty\rangle$.

The two-point correlation function is then simply defined by functional derivatives with respect to J [3]

$$\tau_{lm}(x,y) \equiv \frac{1}{Z[0]} \left(-\frac{\delta}{\delta J_l(x)}\right) \left(-\frac{\delta}{\delta J_m(y)}\right) Z[J]|_{J=0}. \tag{2.10}$$

In a non-interacting theory the action is quadratic in the fields and can be written in the form $S_0[\phi] = \int d^4x \phi_l(x) D_{lm,x} \phi_m(x)$, with some differential operator $D_{lm,x}$. Solving the Gaussian-like integral after completing the square in ϕ leads to the free generating functional [1]

[3] Now again with ϕ_l to be general

$$Z_0[J] = Z_0[0] \exp\left(-\frac{1}{2}\int d^4x \int d^4y J_l(x) G_{lm}(x-y) J_m(y)\right), \tag{2.11}$$

where $G_{lm}(x-y)$ is the inverse or *Green's function* of the differential operator $D_{lm,x}$ defined by $D_{lm,x}G_{mk}(x,y) = \delta_{lk}\delta^{(4)}(x-y)$. Applying the derivatives in eq.(2.10) for a free functional Z_0 leads to [1]

$$\tau_{lm,0}(x-y) = G_{lm}(x-y). \tag{2.12}$$

Thus the Green's function G_{lm} is the two-point correlation function in the free theory. $G_{lm}(x-y)$ is also called position space propagator.

Example: Scalar field theory A simple example is given by the free scalar field theory defined by the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2. \tag{2.13}$$

Integration-by-parts in the action S_0 and inverting the occurring differential operator[4] $D_x = \left(-\Box_E + m^2\right)$ leads to the propagator

$$G(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2+m^2} \tag{2.14}$$

2.2. Interactions and perturbation theory

For an interacting field theory it can be very difficult or even impossible to solve the functional integral $Z[J]$. One solution to this problem is to consider interactions as perturbations around the free theory and to expand $Z[J]$ in small coupling parameters. The action $S[\phi]$ can be written as a sum $S = S_0 + S_I$, where S_0 is the free part containing only quadratic terms in the fields and $S_I = \int d^4x \mathcal{L}_I(\{\phi_l\}, \{\partial_\mu\phi_l\})$ is the interaction part. A rather general expression for such interaction Lagrangian[5] is

$$\mathcal{L}_I(\{\phi_l\}, \{\partial_\mu\phi_l\}) = \sum_{k\geq 2} g_{k,i_1,,,i_k}\Phi_{i_1}...\Phi_{i_k}, \tag{2.15}$$

with coupling parameters $g_{k,i_1,...,i_k}$ and fields $\Phi \in \{\phi_l, \partial_\mu\phi_m\}$. The exponential in the full generating functional eq.(2.9) can then be expanded in the parameters $g_{k,i_1,...,i_k}$. For perturbation theory a very useful and easy-to-handle expression can be obtained by substituting $S_I[\phi] \to S_I[-\frac{\delta}{\delta J}]$ so that the generating functional $Z[J]$ can be written as [3]

$$Z[J] \quad = \quad \exp\left(-S_I\left[-\frac{\delta}{\delta J}\right]\right) Z_0[J]. \tag{2.16}$$

[4] $\Box_E = \sum_{\mu=0}^{3}\partial_\mu^2$

[5] Still working in imaginary time, but omitting the label E.

The great advantage is that one only has to take derivatives of the simple functional Z_0 in order to do perturbation theory. Furthermore eq.(2.16) can be used to express the full two-point function in terms of derivatives of Z_0 [3]

$$\tau_{lm}(x,y) = \left.\frac{\exp\left(-S_I[-\frac{\delta}{\delta J}]\right)\left(-\frac{\delta}{\delta J_l(x)}\right)\left(-\frac{\delta}{\delta J_m(y)}\right)Z_0[J]}{\exp\left(-S_I[-\frac{\delta}{\delta J}]\right)Z_0[J]}\right|_{J=0}. \tag{2.17}$$

The two-point correlation function can more conveniently be calculated in momentum space with

$$\tilde{\tau}_{kl}(p) = \int d^4x e^{ipx}\tau_{kl}(x,0). \tag{2.18}$$

$$\tilde{G}_{kl}(p) = \int d^4x e^{ipx}G_{kl}(x) \tag{2.19}$$

After a successive application of eq.(2.17) it turns out that $\tilde{\tau}_{kl}(p)$ can be written as a geometric series

$$\tilde{\tau}_{lm}(p) = \tilde{G}_{lk}(p)\sum_{n=0}^{\infty}\left(\left(\tilde{G}_F(p)\Pi(p)\right)^n\right)_{km} = \left(\frac{1}{\tilde{G}^{-1}(p)-\Pi(p)}\right)_{lm}, \tag{2.20}$$

with a not yet specified function $\Pi_{lm}(p)$, called self-energy. The self-energy can be computed by an application of a set of graphical rules, the so called Feynman rules. Every propagator in Π_{lm} is represented by a line and every interaction by a vertex.

Example: Scalar field theory In scalar field theory with an interaction Lagrangian $\mathcal{L}_I(\phi) = \sum_{k\geq 3}\frac{g_k}{k!}\phi^k$ these rules are simply [1]

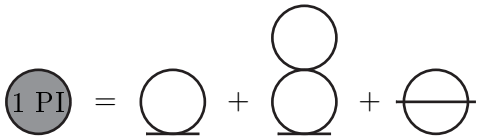
1. For every line carrying momentum p write a propagator:  $= \frac{1}{p^2+m^2}$
2. For every vertex write a coupling factor:  $= -g_3$  $= -g_4$, and so on for higher vertices.
3. All propagators have to carry momenta so that momentum is conserved at every vertex.
4. Integrate over all momenta in a closed loop: $\int \frac{d^4k}{(2\pi)^4}$
5. Divide by a symmetry factor according to the number of possibilities to combine vertices and propagators to a specific diagram

(2.21)

In order to compute the self-energy from the Feynman rules to order g_k^n, draw and sum up all possible *one-particle irreducible* (1PI) diagrams with two external lines and with n vertices

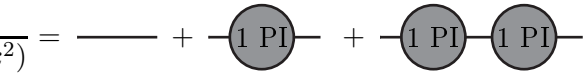
connected by propagators and truncate the external lines. A one-particle irreducible diagram is a diagram that cannot be split into two separated diagrams by cutting a single line.

Applied to the scalar field theory with $\mathcal{L}_I(\phi) = \frac{g}{4!}\phi^4$, this would yield diagrammatically the self-energy [1]

$\Pi(k^2) =$ 

$+ \ldots,$ (2.22)

where the short external lines are truncated and thus do not count propagators. The geometric series eq.(2.20) can be graphically represented by [1]

$\tilde{\tau}(k^2) = \dfrac{1}{k^2 + m^2 - \Pi(k^2)} =$ 

$+ \ldots,$ (2.23)

where the external lines now have to be interpreted as propagators. A one-loop example of this calculation will be given in section 2.3.

2.3. Divergences and regularization

2.3.1. Ultraviolet divergences

The first order correction to the self-energy in the $\frac{g}{4!}\phi^4$ theory at zero temperature according to the Feynman rules in eq.(2.21) leads to the one-loop *tadpole integral*

$$\Pi^{(1)} = -\frac{g}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}. \tag{2.24}$$

Using spherical coordinates in 4 dimensions $\left(\int \frac{d^4k}{(2\pi)^4} \to \frac{1}{(2\pi)^4}\int d\Omega_4 \int d|k||k|^3\right)$ yields an integral

$$\Pi^{(1)} = -\frac{g}{2}\frac{\Omega_4}{(2\pi)^4}\int_0^\infty d|k|\frac{|k|^3}{|k|^2+m^2}, \tag{2.25}$$

which is divergent for large values k. These kind of divergences are called *ultraviolet* (UV) divergences and appear in almost all relativistic Quantum field theories. This seems to be disastrous, but there is a solution to this problem called renormalization. Before renormalization is discussed, it will be useful to have an idea about regularization.

Cut-off regularization The idea of the cut-off regularization is to integrate only up to a finite value Λ, leading to the finite result

$$\Pi^{(1)}_\Lambda = -\frac{g}{2}\frac{\Omega_4}{(2\pi)^4}\int_0^\Lambda dk_E \frac{k_E^3}{k_E^2+m^2} = \frac{g}{4}\frac{\Omega_4}{(2\pi)^4}\left(\left(\frac{\Lambda}{m}\right)^2 - \ln(1+\frac{\Lambda}{m^2}^2)\right). \tag{2.26}$$

The divergence is now parametrized by Λ, so that for $\Lambda \to \infty$ the old divergent integral is restored. But for finite Λ it is possible to write down a well defined result. This method has the disadvantage that it breaks translational invariance, which makes computations more difficult.

Lattice regularization When space-time is a discrete lattice with lattice spacing a, then the momentum is limited by $p \leq \frac{2\pi}{a}$. Furthermore when the volume V of the system is limited, then the momenta are discretized as well. Then one has to deal with finite dimensional matrices instead of differential operators appearing in the action. Instead of using perturbation theory, one can compute the correlation functions numerically for fixed values a and V. The advantage of this method is that non-perturbative effects like bound states are can be better analyzed, but an enormous computational power is needed.

Dimensional regularization The regularization scheme for (UV) divergences used in this work will be the dimensional regularization [5]. The idea of dimensional regularization is to integrate in a different dimension d, so that the results of the divergent loop integrals become analytic functions in $d \in \mathbb{C}$ with poles in specific integer values for d. The integration measure gets simply substituted by

$$\int \frac{d^4k}{(2\pi)^4} \to \int \frac{d^dk}{(2\pi)^d}. \tag{2.27}$$

Let us apply this method to the more general integral

$$J(x,d,m) \equiv \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+m^2)^x}. \tag{2.28}$$

Working in d-dimensional polar coordinates with $\int d\Omega_d = \Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ and integrating over $z = k^2/m^2$ yields

$$J(x,d,m) = \frac{1}{2}\frac{\Omega_d}{(2\pi)^d} \left(m^2\right)^{\left(\frac{d}{2}-x\right)} \int_0^\infty dz \frac{z^{\left(\frac{d}{2}-1\right)}}{(z+1)^x}. \tag{2.29}$$

Making the substitution $z = \frac{u}{1-u}$ leads then to the integral

$$J(x,d.m) = \frac{1}{2}\frac{\Omega_d}{(2\pi)^d} \left(m^2\right)^{\left(\frac{d}{2}-x\right)} \int_0^1 du u^{\left(\frac{d}{2}-1\right)} (1-u)^{\left(x-\frac{d}{2}-1\right)}. \tag{2.30}$$

This can be written in terms of the Euler Beta function

$$B(x,y) \equiv \int_0^1 du u^{x-1}(1-u)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{2.31}$$

so that [1]

$$J(x,d,m) = \frac{\left(m^2\right)^{\left(\frac{d}{2}-x\right)}}{2^d\pi^{d/2}} \frac{\Gamma(x-\frac{d}{2})}{\Gamma(x)}. \tag{2.32}$$

The first order ϕ^4 self-energy correction in dimensional regularization is then simply

$$\Pi^{(1)} = -\frac{g}{2} J(1, 4-2\varepsilon, m) = -\frac{g}{2} \frac{\left(m^2\right)^{\left(\frac{d}{2}-1\right)}}{(4\pi)^{d/2}} \Gamma(1-\frac{d}{2}). \tag{2.33}$$

The divergence appears now as the pole of the Gamma function in $d = 4$. Writing $d = 4 - 2\varepsilon$ and using the expansion[6] $\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon)$ one can write the result as a Laurent series in ε

$$\Pi^{(1)}(d = 4 - 2\varepsilon) = -\frac{g}{2} J(1, 4 - 2\varepsilon, m) = -\frac{gm^2}{8\pi^2} \left(\frac{1}{\varepsilon} + \ln(\frac{4\pi e^{-\gamma} \mu^2}{m^2}) + \mathcal{O}(\varepsilon) \right), \tag{2.34}$$

where the number $\gamma \approx 0.5772$ is the Euler–Mascheroni constant. The mass scale μ has been introduced to get a dimensionless argument in the logarithm. Since we will later work in a three dimensional theory it is interesting to have a look what happens for $d = 3 - 2\varepsilon$. Although the integral in eq.(2.25) would be infinite in three dimensions, it is finite in $d = 3 - 2\varepsilon$ with $\varepsilon \to 0$ where

$$J(1, 3 - 2\varepsilon, m) = \frac{m}{(4\pi)^{3/2}} \Gamma(-\frac{1}{2}) = -\frac{m}{4\pi} + \mathcal{O}(\varepsilon). \tag{2.35}$$

This is due to the fact, that limits have been interchanged. But in the sense of analytic continuation this gives a meaningful result. For an expansion up to the first order in ε see eq.(B.30).

Some important facts about dimensional regularization

- One can shift the integration variable: $\int d^d k f(k + p) = \int d^d k f(k)$
- The scale substitution $\int d^d k f(\lambda k) = |\lambda|^{-d} \int d^d k f(k)$, leads to the fact that mass scale independent integrals vanish: $\int d^d k \frac{1}{(k^2)^{\alpha \neq d/2}} . = 0$
- The trace of the metric tensor is $\eta_{\mu\nu} \eta^{\mu\nu} = d$
- Dimensional regularization preserves important symmetries like gauge invariance and unitarity [5]

2.3.2. Infrared divergences

In addition to the UV divergences there can be infrared (IR) divergences due to small momenta. These divergences appear when propagators are massless, so that the integrand has a pole at $k = 0$. Consider the integral $J(x, 3, 0)$. For $x \geq 2$ this integral is clearly infrared divergent in three dimensions. Unfortunately in many cases dimensional regularization does not work for IR divergences, because the integral $J(x, d \neq x/2, 0)$ vanishes. On the other hand there are many physical calculations where IR divergent integrals contribute. These integrals have to be regularized in a different way. The obvious solution to the problem is to introduce a infrared cut-off parameter m by the replacement $\frac{1}{p^2} \to \frac{1}{p^2 + m^2}$. We will later realize this regularization in a somewhat fancier way under the term *resummation,* which will be discussed in sections 4.3 and 4.4.

[6] This easy to obtain from the Weierstrass representation $\frac{1}{\Gamma(\varepsilon)} = \varepsilon e^{\gamma \varepsilon} \prod_{k=1}^{\infty} \left(1 + \frac{\varepsilon}{k}\right) e^{-\varepsilon/k}$ or from the functional equation$\Gamma(\varepsilon) = \frac{\Gamma(1+\varepsilon)}{\varepsilon}$ and $\gamma \equiv -\Gamma'(0)$ [8].

In contrast to the UV divergences, IR divergences cannot removed by renormalization. But for the computation of observables one usually has to add up diagrams so that the IR divergences cancel out in the end of the computation. This cancellation cannot always be realized in practice. This might for example be the case when an infinite number of diagrams has to be summed up in order to cancel the divergences.

2.4. Renormalization

A reasonable physical observable should be finite when the regularization parameter ε is set to zero. It would therefore be nice to somehow cancel the divergent terms in the ε-expanded result of $\Pi^{(1)}$ or other calculated quantities. This can be done by renormalization. The general idea behind this is to rescale the fields and all parameters by some constants Z_i which can absorb the divergence. For this purpose the fields ϕ_l and parameters λ in the Lagrangian are renamed to bare fields $\phi_{l,B}$ and bare parameters λ_B, where the parameter λ_B can be the mass-parameter m_B or the coupling g_B (or a gauge parameter ξ_B introduced later in this work). Then the renormalized fields and parameters are defined by [9]

$$\phi_{l,B} = \sqrt{Z_{\phi_l}}\phi_{l,R} \tag{2.36}$$

$$\lambda_B = Z_\lambda \lambda_R \tag{2.37}$$

and the bare Lagrangian $\mathcal{L}_B(\{\phi_{l,B}\}, \{\partial_\mu \phi_{l,B}\}, \{\lambda_B\})$ is expressed in terms of the renormalized quantities leading to a new Lagrangian $\mathcal{L}'(\{\phi_{l,R}\}, \{\partial_\mu \phi_{l,R}\}, \{\lambda_R\})$. Writing $Z_i = 1 + \delta Z_i$, one can split the new Lagrangian into two parts

$$\begin{aligned}\mathcal{L}'(\{\phi_{l,R}\}, \{\partial_\mu \phi_{l,R}\}, \{\lambda_R\}) = \mathcal{L}_{ren}(\{\phi_{l,R}\}, \{\partial_\mu \phi_{l,R}\}, \{\lambda_R\}) \\ + \mathcal{L}_{ct}(\{\phi_{l,R}\}, \{\partial_\mu \phi_{l,R}\}, \{\lambda_R\}, \{\delta Z_{\phi_i}\}, \{\delta Z_\lambda\}),\end{aligned} \tag{2.38}$$

where $\mathcal{L}_R$ is the old Lagrangian in terms of renormalized fields and parameters. $\mathcal{L}_{ct}$ is an additional *counter term* Lagrangian, which is dependent on renormalized fields and parameters and also on the new parameters δZ_i. The $\delta Z_i's$ are themselves dependent on the coupling g_R and have to be expanded in a series in g_R during the perturbative expansion.

Once a perturbative expansion has been done, one can fix the parameters δZ_i, so that the result for $\Pi_{lm}(p)$ or other quantities are UV-finite. This can be done by several possible renormalization schemes. One of these schemes is called *minimal subtraction* (MS) [6], where the parameters δZ_i are chosen such that they simply cancel the $\frac{1}{\varepsilon}$ term in divergent expressions like eq.(2.34). Another scheme is called *modified minimal subtraction* ($\overline{MS}$), where one also removes the terms γ and $\log(4\pi)$ from the final result [1]. For this purpose it is common to define $\bar{\mu}^2 = 4\pi e^{-\gamma}\mu^2$, so that these terms get absorbed into the renormalization scale. After the theory has been renormalized, i.e the δZ_is are fixed, all other observables computed in perturbation theory will be finite.

A theory has to be *renormalizable* in order to ensure that the above techniques work in all orders of perturbation theory with a finite set of parameters. In the framework of renormalization one can classify theories in three categories according to the mass dimension of the coupling [1]:

$$\begin{aligned} \text{Super-Renormalizable: } & [g] > 0 \\ \text{Renormalizable: } & [g] = 0 \\ \text{Non-renormalizable: } & [g] < 0 \end{aligned}$$

The above methods work only without limitations in (super-)renormalizable theories for all order of perturbation theories. Examples for renormalizable theories in four dimensions are the scalar ϕ^4 theory, QED and non-Abelian gauge theories. In this work we will discuss the latter one in detail in chapter 3.

The renormalized result can depend on the arbitrary mass scale μ introduced in eq.(2.34) in order to have a dimensionless argument in the logarithm. In fact, using dimensional regularization, one has to introduce this energy scale μ in the interaction Lagrangian, so that the coupling keeps its mass dimension for different space-time dimensions. Therefore one obtains a scale dependent renormalized coupling [3]

$$\hat{g}_R(\mu) = \mu^{(d-4)/2} Z_g^{-1} g_B. \tag{2.39}$$

In Yang-Mills theories with[7] $N_f < \frac{11}{2} N_c$ one can show that this *running coupling* becomes small for higher energy scales μ. For example the one-loop result calculated by Wilczek, Gross and Politzer reads [9, 10]

$$\hat{g}_R^2(\mu) = \frac{16\pi^2}{\beta_0 \ln(\frac{\mu}{\mu_0})}, \tag{2.40}$$

with some number $\beta_0 = \frac{11}{3} N_c - \frac{2}{3} N_f$, which is positive for $N_f < \frac{11}{2} N_c$, and some reference mass scale μ_0. The phenomenon is called asymptotic freedom and is an argument why perturbation theory works well for high energies in such theories.

The μ-dependence of the truncated perturbation series can be taken as an estimate of the size of higher order corrections [22].

[7] Number of colors N_c and number of flavors N_f

3. Non-Abelian gauge theories and spontaneous symmetry breaking

The last chapter was about a rather general treatment of functional methods in quantum field theory, leading to the definition of the two-point function and to the idea to compute this function with diagrammatic rules. Now these methods shall be more explicitly applied to a special class of theories, namely the non-Abelian gauge theories. In this chapter we follow basically the notation and the approach of [1] and [3].

3.1. Gauge invariance and the Yang-Mills Lagrangian

Additionally to the invariance of coordinate transformations, i.e invariance under Lorentz transformations and translations in space-time, there can be some internal symmetries relying on simple compact Lie groups G. Examples for such groups are $U(1)$, $SU(N)$ and $SO(N)$.

Suppose there is a Lagrangian that is locally invariant under the action of a unitary representation $U(\alpha) = e^{i\alpha(x)} \in G$ on complex n-component spin 0 fields ϕ

$$\phi(x) \to \phi'(x) = U(\alpha)\phi(x) = e^{i\alpha(x)}\phi(x), \tag{3.1}$$

where $\alpha(x)$ is a linear combination $\alpha(x) = \alpha_k(x)T_k$ of $n \times n$ matrix generators T_k determined by

$$T_k \equiv -i\frac{\partial U(\alpha)}{\partial \alpha_k}|_{\alpha=0}, \tag{3.2}$$

with real parameters $\alpha^k(x)$. The generators have some nice properties. Firstly they obey a Lie algebra [7]

$$[T_a, T_b] = if_{abc}T_c, \tag{3.3}$$

with real numbers f_{abc} called the structure constants. Once one has specified the structure constants the whole algebra is specified. Secondly they form a vector space with a scalar product given by the trace. Thus one can find a orthogonal basis conventionally normalized such that

$$(T_a, T_b) \equiv \mathrm{Tr}\,(T_aT_b) = \frac{1}{2}\delta_{ab}. \tag{3.4}$$

In general the generators are not Abelian, i.e the structure constants do not vanish. These theories are called non-Abelian gauge theories or Yang-Mills theories and are the fundamental background of the electroweak interactions ($U(\alpha) \in U(1) \times SU(2)$) and Quantum Chromodynamics ($U(\alpha) \in SU(3)$). In contrast electrodynamics is described by an Abelian $U(1)$ gauge symmetry containing only one single generator.

In order to describe dynamics, every theory must contain derivatives in the fields. The derivatives transform under local transformations as

$$\partial_\mu \phi \to \partial_\mu \left(U(\alpha)\phi(x) \right) = \left(\partial_\mu U(\alpha) \right) \phi(x) + U(\alpha)\partial_\mu \phi(x), \tag{3.5}$$

so that an extra term $\left(\partial_\mu U(\alpha) \right) \phi(x)$ arises due to the product rule. This term can be absorbed by introducing a so called covariant derivative [1]

$$D_\mu \equiv 1_{n\times n}\partial_\mu - A_\mu, \tag{3.6}$$

where A_μ is also some linear combination of the generators $A_\mu = igA^a_\mu T^a$. These new vector fields A^a_μ are called gauge fields. The covariant derivative shall transform in such away, that the term $D_\mu\phi$ transforms covariantly under unitary representation of the symmetry group. This is equivalent to demanding the transformation rule as

$$D_\mu \to D'_\mu = U(\alpha) D_\mu U^\dagger(\alpha). \tag{3.7}$$

Comparing eq.(3.5) and eq.(3.6) with eq.(3.7) fixes the transformation behavior of the gauge fields A_μ to be

$$A'_\mu = U(\alpha) A_\mu U^\dagger(\alpha) + (\partial_\mu U) U^\dagger. \tag{3.8}$$

The transformations in eq.(3.1) together with eq.(3.8) are called gauge transformations. Again the most straightforward way to get a gauge invariant theory is to construct a gauge invariant Lagrangian out of D_μ, A_μ and ϕ. Clearly the combinations $\phi^\dagger\phi$ and $(D_\mu\phi)^\dagger(D^\mu\phi)$ are gauge invariant because $U(\alpha)$ is unitary. In order to get dynamic gauge fields one has to introduce the field strength tensor $F_{\mu\nu}$ defined by the commutator [1]

$$F_{\mu\nu} \equiv -igF^a_{\mu\nu}T^a \equiv [D_\mu, D_\nu], \tag{3.9}$$

whose transformation behavior is

$$F'_{\mu\nu} = U(\alpha) F_{\mu\nu} U^\dagger(\alpha). \tag{3.10}$$

Using the cyclicity of the trace it is easy to see that $\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right)$ is also gauge invariant. A rather general gauge invariant Lagrangian containing the fields ϕ and A_μ is therefore[1]

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + (D_\mu\phi)^\dagger(D^\mu\phi) - \frac{1}{2}m^2\phi^\dagger\phi - V(\phi^\dagger\phi). \tag{3.11}$$

Some comments:

- An infinitesimal transformation is the first order expansion $U(\alpha) = 1 + i\alpha + \mathcal{O}(\alpha^2)$. Using the commutator of the generators yields the infinitesimal transformations of the fields [1]

$$A'^a_\mu = A^a_\mu + \frac{1}{g}\partial_\mu\alpha^a + f^{abc}A^b_\mu\alpha^c + \mathcal{O}(\alpha^2) \tag{3.12}$$

$$\phi' = (1 + i\alpha + \mathcal{O}(\alpha^2))\phi \tag{3.13}$$

[1] The Lagrangian is given here in real and not in imaginary time in order to keep the notation from the literature.

- The components of the field strength tensors read [1]

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{3.14}$$

- From now on we will restrict to the gauge group $SU(N)$ where $\det(U) = 1$ and equivalently $\mathrm{Tr}(T^a) = 0$ for all generators. The number of free parameters α^a and generators T^a is N^2-1 respectively. According to quantum chromodynamics N will be replaced by the number of colors N_c in equations. But it should be emphasized, that the treatment here is quite general and not restricted to Quantum Chromodynamics.

- In the further treatment the pure gauge part of eq.(3.11)

$$\mathcal{L}_{YM} = -\frac{1}{4} F_{\mu\nu,a} F^{\mu\nu}_a, \tag{3.15}$$

  will be identified to be the Yang-Mills Lagrangian.

- Later the term *adjoint representation* will be used. In the adjoint representation the generators have the form $(T_a)_{bc} = -i f_{abc}$ [7].

- Every Lie algebra satisfies the *Jacobi identity* $[T^a,[T^b,T^c]]+[T^b,[T^c,T^a]]+[T^c,[T^a,T^b]] = 0$ [7]. Written in terms of the structure constants this reads

$$f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0 \tag{3.16}$$

3.2. Quantization and Faddeev-Popov Lagrangian

In order to apply the methods from chapter 2 for gauge theories it would be natural to define the generating functional Z in the same way for the action $S_{YM}[A] = \int d^4x \mathcal{L}_{YM}$. Considering the functional integral of this pure gauge part

$$Z = N \int \mathcal{D}A \exp\left(i S_{YM}[A]\right), \tag{3.17}$$

one obtains the differential operator

$$\delta_{ab}\left(\eta^{\mu\nu}\Box - \partial^\mu \partial^\nu\right) \tag{3.18}$$

after partial integration of the fields in the action. This operator is not invertible, because fields of the kind $\partial^\mu \alpha$ belong to the kernel of this operator. But these are simply part of the fields which gauge transform the fields A_μ to some gauge equivalent fields. Thus we integrate over an infinite number of gauge equivalent fields. The solution of the problem is to fix all such redundant fields by a gauge fixing condition

$$G[A] = 0.$$

In the functional integral formalism this can be achieved by inserting a delta functional $\delta[G[A]]$ into the integral. This method is known as the *Faddeev-Popov trick* [11]. Let us label gauge transformed fields by

$$A_\mu^U \equiv U A_\mu U^\dagger + (\partial_\mu U) U^\dagger. \tag{3.19}$$

The trick is to insert a suitable unity [3]

$$1 = \Delta_G^{-1}[A]\Delta_G[A], \tag{3.20}$$

with $\Delta_G[A] = \int \mathcal{D}U \delta[G[A^U]]$ and $\Delta_G^{-1}[A] = \det\left(\frac{\delta G}{\delta\alpha}|_{G=0}\right)$, into the functional integral eq.(3.17), so that [3]

$$Z = \int \mathcal{D}A \det\left(\frac{\delta G}{\delta\alpha}|_{G=0}\right) \int \mathcal{D}U \delta[G[A^U]] e^{iS_{YM}[A]}. \tag{3.21}$$

Due to the fact that the measure $\mathcal{D}U$ is invariant under a transformation $U'' = U'U$, it is easy to show that the functional $\Delta_G[A]$ as well as its inverse $\Delta_G^{-1}[A]$ are gauge invariant. A gauge transformation $A^U \to A$ leaves the measure $\mathcal{D}A$ also invariant. One can therefore write the functional integral as [3]

$$Z = \int \mathcal{D}U \int \mathcal{D}A \det\left(\frac{\delta G}{\delta\alpha}|_{G=0}\right) \delta[G[A]] e^{iS_{YM}[A]}, \tag{3.22}$$

with the integration over the gauge orbits U factored out. The infinite constant $\int \mathcal{D}U$ drops out in the calculation of physical quantities. The delta function can be written as an exponential function, so that the gauge fixing function G is part of the action. Shifting G^a to $G^a + C^a$ with an arbitrary α-independent function C^a and averaging over all these new functions after multiplying Z with a Gaussian weight $\exp(-\frac{i}{2\xi}\int d^4x C_a^2(x))$, results in [3]

$$Z = N \int \mathcal{D}A \det\left(|\frac{\delta G}{\delta\alpha}|_{G=0}\right) \exp\left(i \int d^4x \left(-\frac{1}{4} F_{\mu\nu,a} F_a^{\mu\nu} - \frac{1}{2\xi} G^2\right)\right), \tag{3.23}$$

with a constant N. The determinant $\det(M) \equiv \det\left(|\frac{\delta G}{\delta\alpha}|_{G=0}\right)$ can be written as some integral over anti-commuting Grassmann fields c_a

$$\det(M) = \int \mathcal{D}\bar{c}\mathcal{D}c \exp\left(-i \int d^4x \bar{c}_a(x) M_{ab} c_b(x)\right). \tag{3.24}$$

These new fields c_a are called ghost fields and do only appear in quantum corrections but not in physical states. Nevertheless they are important to maintain gauge symmetry and cannot be neglected in intermediate computations.

Choosing the covariant gauge $G^a[A] = \partial^\mu A_\mu^a$ and considering only infinitesimal gauge transformations, yields [1]

$$\left(\frac{\delta G[A]}{\delta\alpha}\right)_{ab} = \frac{1}{g}\partial_\mu D_a^\mu, \tag{3.25}$$

with $D_{ab}^\mu = \delta_{ab}\partial^\mu + g f_{abc} A_c^\mu$ the covariant derivative in the adjoint representation.

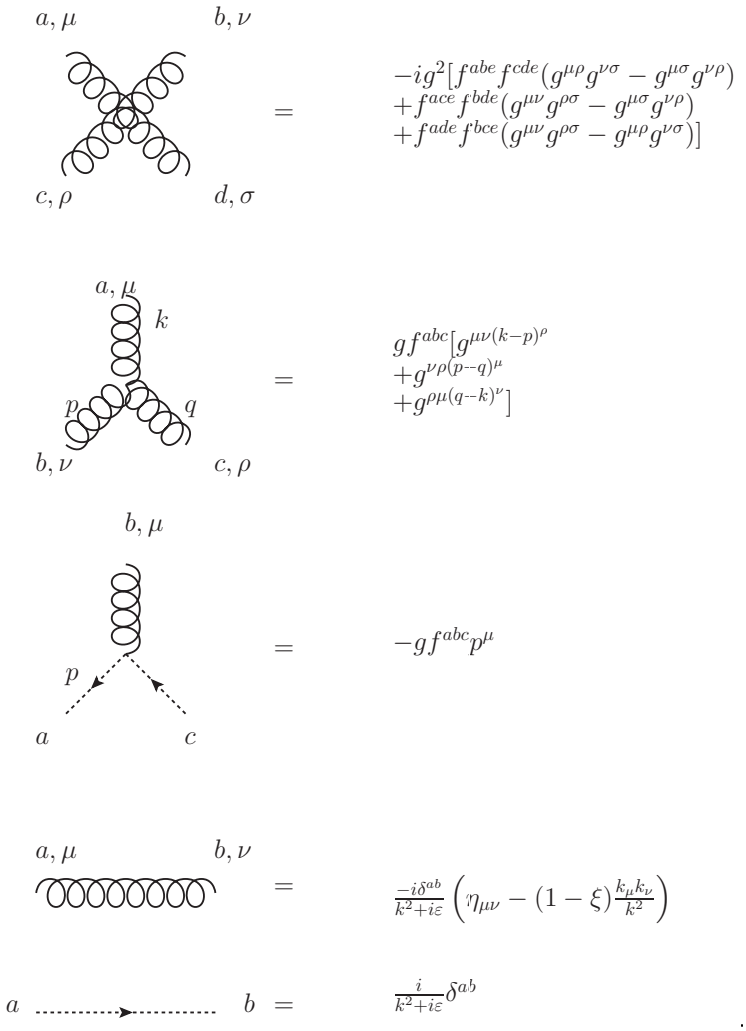


Figure 3.1.: Feynman rules for a gauge fixed Yang-Mills Lagrangian. Curly lines are gauge bosons and dotted lines are ghosts.[1]

Let us summarize what happens after the application of the Faddeev-Popov method. The Lagrangian $\mathcal{L}_{YM}$ gets effectively replaced by

$$\mathcal{L}_{YM} \to \mathcal{L}_{YM} + \mathcal{L}_{gf} + \mathcal{L}_{gh}, \tag{3.26}$$

where $\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial_\mu A^a_\nu)^2$ is the gauge fixing Lagrangian and $\mathcal{L}_{gh} = \bar{c}^a(-\partial^\mu D^{ab}_\mu)c^b$ is the ghost Lagrangian. The quadratic part of the Yang-Mills Lagrangian together with the gauge fixing part provides an invertible differential operator

$$\delta_{ab}\left(\eta^{\mu\nu}\Box + (1-\frac{1}{\xi})\partial^\mu\partial^\nu\right). \tag{3.27}$$

The corresponding propagator and the other Feynman rules are summarized in Fig. 3.1. A more general Lagrangian (c.f eq.(3.11)) also containing scalar matter fields ϕ reads

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\xi}(\partial_\mu A^a_\nu)^2 + (D_\mu\phi)^\dagger(D^\mu\phi) - \frac{1}{2}m^2\phi^\dagger\phi - V(\phi^\dagger\phi) + \bar{c}^a(-\partial^\mu D^{ab}_\mu)c^b. \tag{3.28}$$

3.3. BRST Symmetry

The Lagrangian in eq.(3.28) is of course no longer gauge invariant because it has been gauge fixed. But instead it is invariant under a larger symmetry namely the so called BRST symmetry [12]. To show this, one has to introduce auxiliary bosonic scalar fields B^a and rewrite the Faddeev-Popov Lagrangian in terms of B^a (omitting the scalar field ϕ potential and the mass term) [1] as,

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + (D_\mu\phi)^\dagger(D^\mu\phi) + \frac{\xi}{2}(B^a)^2 + B^a\partial^\mu A^a_\mu + \bar{c}^a(-\partial^\mu D^{ab}_\mu)c^b. \tag{3.29}$$

The original Lagrangian can be restored by integrating out the fields B^a. The Lagrangian in eq.(3.29) is now invariant under the BRST transformations given as,

$$\begin{aligned}
\delta A^a_\mu &= \varepsilon D^{ac}_\mu c^c \\
\delta\phi &= ig\varepsilon c_a T_a\phi \\
\delta c_a &= -\frac{1}{2}g\varepsilon f_{abc}c_b c_c \\
\delta\bar{c}_a &= \varepsilon B_a \\
\delta B_a &= 0,
\end{aligned} \tag{3.30}$$

where ε is an anti-commuting number. The transformations δA^a_μ and $\delta\phi$ are simply gauge transformations with $\alpha^a(x) = g\varepsilon c^a(x)$ and therefore leave the first two terms in eq.(3.29) invariant. The third term in eq.(3.29) is invariant because the fields B^a are invariant. The transformation of the fourth term is canceled due to the specific transformation of the fields $\delta\bar{c}^a$ in the fifth term of the eq.(3.29). The transformation of c^a leads to a term containing a product of two structure constants that vanishes due to the Jacobi identity eq.(3.29) [1] .

BRST invariance leads to so called Ward-identities, which are important for the discussion of renormalization of non-Abelian gauge theories. Furthermore the BRST invariance is important to ensure that physical observables are gauge parameter invariant [1].

3.4. Spontaneous symmetry breaking in gauge theories

Later in this work a gauge boson mass term will be introduced as an infrared cut-off regulator in a resummation scheme. But one has to keep in mind that it is not possible to simply add a mass term

$$\Delta\mathcal{L} = \frac{1}{2} m A_\mu A^\mu \tag{3.31}$$

into the Lagrangian since it breaks the gauge symmetry. But this does not mean that gauge bosons have to be massless once and for all. There is a mechanism called spontaneously symmetry breaking where the massless gauge bosons gain a mass due to an interaction with other particles. In order to see what is meant by spontaneous symmetry breaking consider a Lagrangian with a global $O(N)$ symmetry [1]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^T (\partial^\mu \phi) + \frac{1}{2}\mu^2 \phi^T \phi - \frac{\lambda}{4}\left[\phi^T \phi\right]^2, \tag{3.32}$$

with N real scalar fields $\phi = (\phi_1, ..., \phi_N)$ having a negative mass squared $m^2 = -\mu^2$. These scalar fields are massless and have an interaction term with fields quadratic in the potential

$$V(\phi) = -\frac{1}{2}\mu^2 \phi^T \phi + \frac{\lambda}{4}\left[\phi^T \phi\right]^2. \tag{3.33}$$

This potential has a local maximum at $\phi = 0$ and an infinitely degenerated nontrivial minimum, as long as $\mu^2 > 0$ (see Fig. 3.2). The minimum is given by an orbit with the modulus squared as,

$$\phi_0^T \phi_0 = v^2 = \frac{\mu^2}{\lambda}. \tag{3.34}$$

The modulus $v = \frac{\mu}{\sqrt{\lambda}}$ is called the vacuum expectation value of the fields.

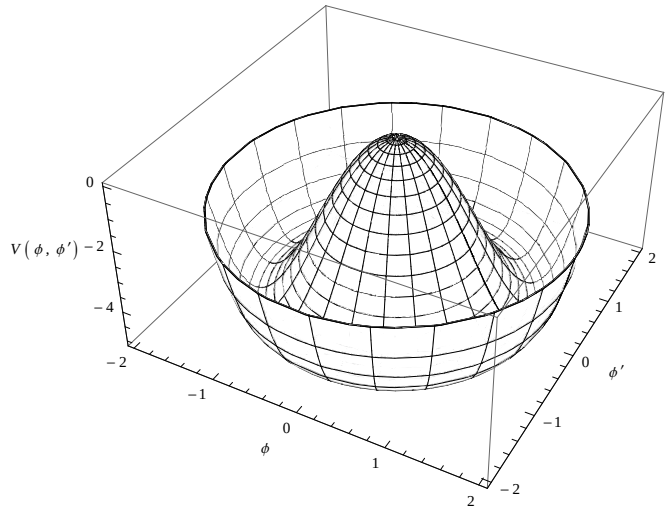


Figure 3.2.: The plot of the *Mexican hat* potential or *Higgs* potential $V(\phi_1, \phi_2) = -\frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) + \frac{\lambda}{4}\left[\phi_1^2 + \phi_2^2\right]^2$. (Plot made with Mathematica)

After choosing one specific point ϕ_0 the fields ϕ can be rewritten in terms of quantum fluctuations χ with $\phi = \phi_0 + \chi$ and the Lagrangian can be expressed in terms of the χ fields. For example if the minimum value has been chosen to be $\phi_0 = (0, ..., 0, v)$ then the shifted field reads

$$\phi(x) = (\chi^1(x), ..., \chi^{N-1}(x), v + \chi^N(x)). \tag{3.35}$$

Writing $\chi^N = \sigma$ for convenience the Lagrangian can be expressed in terms of χ^k and σ by [1]

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu \chi^k)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)(\sigma)^2 - \sqrt{\lambda}\mu(\sigma)^3 \\ &\quad -\sqrt{\lambda}\mu(\chi^k)^2\sigma - \frac{\lambda}{4}(\sigma)^4 - \frac{\lambda}{2}(\chi^k)^2(\sigma)^2 - \frac{\lambda}{4}\left[(\chi^k)^2\right]^2. \end{aligned} \tag{3.36}$$

Now this Lagrangian consists of a massive field σ with mass $m_\sigma = \sqrt{2}\mu$ and $(N-1)$ massless fields $\chi^{k=1,...,N-1}$. The O(N) symmetry in eq.(3.36) is no longer apparent because it is spontaneously broken by choosing one specific point of the minimum orbit. But instead there is an unbroken O(N-1) sub-symmetry in the massless $\chi^{k=1,...,N-1}$ field. These fields are called Goldstone bosons and their appearance is guarantied by the Goldstone theorem [1].

The idea of spontaneous symmetry breaking can easily be extended to non-Abelian local gauge theories. Assuming the Lagrangian be to invariant under a local gauge transformation one has to introduce the covariant derivative defined in eq.(3.6). The covariant kinetic part of the Lagrangian written explicitly in terms of A_μ and ϕ reads

$$\frac{1}{2}(D_\mu \phi^T)(D^\mu \phi) = \partial_\mu \phi^T \partial^\mu \phi + A_\mu \phi^T \partial^\mu \phi - \partial_\mu \phi^T A^\mu \phi - \phi^T A_\mu A^\mu \phi. \tag{3.37}$$

Assuming the fields ϕ to have a non-vanishing vacuum expectation value ϕ_0 and writing them in terms of quantum fluctuations χ, there appears a mass term[2] for the gauge fields

$$\Delta\mathcal{L} = -\frac{1}{2}\phi_0^T A_\mu A^\mu \phi_0 = \frac{1}{2} m_{ab}^2 A_\mu^a A^{\mu,b}, \tag{3.38}$$

with a positive semidefinite mass matrix $m_{ab} = g^2 \phi_0^T T_a T_b \phi_0$ [1]. Additionally there is a quadratic interaction

$$\Delta\mathcal{L} = \frac{1}{2}\left(\phi_0^T A_\mu \partial^\mu \chi - \partial_\mu \chi^T A^\mu \phi_0\right) = \phi_0^T A_\mu \partial^\mu \chi \tag{3.39}$$

between the massless Goldstone bosons and the gauge fields. This term is not very convenient and can be removed in the R_ξ gauge.

3.5. The R_ξ-gauge

Consider the gauge fixing function

$$G = G^a T^a = \frac{1}{ig}\partial_\mu A^\mu - i\xi g\left((T^a\phi_0)^\dagger \chi\right) T^a, \tag{3.40}$$

chosen such that the gauge fixing Lagrangian $\mathcal{L}_{GF} = -\frac{1}{\xi}\mathrm{Tr}\left(G^2\right)$ contains a term $\phi_0^T \partial^\mu A_\mu \chi$ which cancels the inconvenient interaction term eq.(3.39), after partial integration.

Applying the Faddeev-Popov method for this gauge fixing, the quadratic part for the gauge- and Goldstone bosons is [1]

$$\Delta\mathcal{L} = -\frac{1}{2}A_\mu^a\left(\left[-\eta^{\mu\nu}\Box + (1-\frac{1}{\xi})\partial^\mu\partial^\nu\right]\delta^{ab} - \eta^{\mu\nu}(m_A^2)^{ab}\right)A_\nu^b + \frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}\xi^2(m_G^2)^{ij}\chi_i\chi_j, \tag{3.41}$$

with $(m_G^2)^{ij} = \xi g^2(\phi_0^T T_a \phi_0^T T_a)^{ij}$. The interesting aspect of this gauge is that the Goldstone bosons get a gauge dependent mass term. This indicates that the Goldstone bosons are not physical. Therefore they are considered as auxiliary fields. It will also turn out that the ghosts become massive with a gauge dependent mass. For the three dimensional Yang-Mills theory discussed in chapter 4, we will also use an $R_\xi-$gauge with the same motivation to cancel inconvenient terms. Furthermore it is possible to prove the renormalizability of spontaneously broken gauge theories in the R_ξ - gauges [1].

[2] At first sight the sign of the mass term seem to be wrong. But the physical fields appear as the spatial component in this term. The spatial part of the Minkowski metric yields the minus sign.

4. Euclidean three dimensional $SU(N)$ model

Now that all the important quantum field theoretical concepts have been introduced, we are prepared to discuss the model which will be in the focus of this work. The infrared problems in Yang-Mills theories shall be cured by massive gauge boson propagators. The idea is to add a mass term to the free gauge field Lagrangian and to subtract it again in higher orders of perturbation theory. This resummation method should not break gauge invariance. Therefore one has to generate the mass term by means of spontaneous symmetry breaking with an auxiliary field. The resummation will lead to a gap equation for the gauge boson self-energy, whose solution will be an estimate for the dynamically generated gauge boson mass. This model is a generalization of a similar approach by O. Philipsen for $SU(2)$ [19] to an $SU(N)$ theory.

4.1. Theory Lagrangian

Recall the definition of the covariant derivative $D_\mu = 1_{n\times n}\partial_\mu - A_\mu$ with $A_\mu = igA^a_\mu T^a$ (c.f. eq.(3.6)). We start from a three dimensional Euclidean $SU(N)$ gauge field Lagrangian (cf.eq.(3.15))

$$\mathcal{L}_{YM} = -\frac{1}{2g^2}\mathrm{Tr}\left([D_\mu, D_\mu]\right)^2 \tag{4.1}$$

with a coupling g^2 carrying mass dimension $[g^2] = 1$. We apply the spontaneous symmetry breaking, following the approach of [21] and [22], by means of an $SU(N)$ like field

$$\Phi(x) = \frac{m}{g}\exp(\pi(x)),\ \ \pi(x) \equiv i\frac{g}{m}T^a\pi^a(x),\ \ \pi^a(x) \in \mathbb{R}, \tag{4.2}$$

where fields π^a play the role of the dynamic variables and not Φ. The transformation of Φ under $SU(N)$ is defined by $\Phi_\Lambda \equiv U\Phi$ with $U = \exp(\Lambda(x))$ and $\Lambda(x) = igT^a\Lambda^a(x)$. The interaction between Φ and the gauge fields is given as usual by

$$\mathcal{L}_{SSB} = \mathrm{Tr}\left(\left(D_\mu\Phi\right)^\dagger\left(D_\mu\Phi\right)\right), \tag{4.3}$$

where the trace is taken to get a scalar quantity. The gauge bosons gain the mass term

$$\mathcal{L}_{SSB} \ni \Delta\mathcal{L} = -\frac{m^2}{g^2}\mathrm{Tr}(A_\mu A_\mu), \tag{4.4}$$

because of the cyclicity of the trace and the fact that $\Phi^\dagger\Phi = \frac{m^2}{g^2}$. In fact, the appearance of this mass term can also be understood as a consequence of a non trivial vacuum expectation value $\frac{m}{g}$ appearing as the first term of the exponential series of Φ. One can write the fields Φ as $\Phi = \frac{m}{g} + \hat{\Phi}$ where $\hat{\Phi}$ contains the π fields. The crucial point is that self interactions of the $\pi's$

are already contained in the Lagrangian eq.(4.3) due to the nonlinear dependence of Φ on π. The exponential leads to an infinite number of interaction terms which also contain derivatives of the fields. Furthermore the Lagrangian is not of first or second order in the coupling g as usual, but it contains an infinite number of orders in the coupling. Therefore, in perturbation theory, the Lagrangian yields additional vertices in every order. These are generated automatically with `QGRAF` [20] and the corresponding Feynman rules are generated in a `FORM` [29] program. However the theory is renormalizable, because the coupling has a positive mass dimension (compare section 2.4).

Further analysis of the $\mathcal{L}_{SSB}$ term after writing it in terms of $\hat{\Phi}$, leads to an inconvenient interaction term including $\partial_\mu(\hat{\Phi} - \hat{\Phi}^\dagger)A_\mu$, which can be canceled with an R_ξ like gauge fixing term

$$\mathcal{L}_{GF} = \frac{1}{\xi}\mathrm{Tr}\left(\left(\frac{1}{ig}\partial_\mu A_\mu + i\xi m\mathrm{Tr}\left((\Phi - \Phi^\dagger)T^a\right)T^a\right)^2\right). \tag{4.5}$$

The corresponding ghost part turns out to be

$$\mathcal{L}_{FP} = \mathrm{Tr}\left(2(\partial_\mu\bar{c})\left((\partial_\mu c) - [A_\mu, c]\right) + g\xi m\bar{c}(\Phi^\dagger c + c\Phi)\right), \tag{4.6}$$

where $c \equiv c^a T^a$.

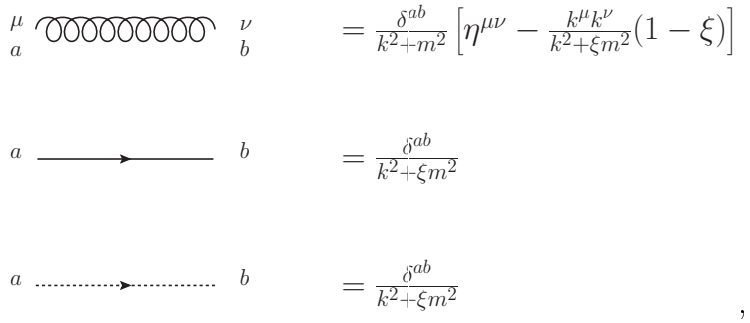
For practical computations it is convenient to add a total derivative

$$\mathcal{L}_{TD} = -\mathrm{Tr}\left(\partial_\mu \frac{m}{g}(\Phi - \Phi^\dagger)A_\mu\right) \tag{4.7}$$

to the Lagrangian, so that one does not need to apply the partial integration for the cancellation of the $\partial_\mu(\hat{\Phi} - \hat{\Phi}^\dagger)A_\mu$ terms. Finally the complete Lagrangian of this model written in terms of $\hat{\Phi}$ reads [21]

$$\begin{aligned}\mathcal{L} \;=\; & \mathrm{Tr}\left(-\frac{1}{2g^2}([D_\mu, D_\nu])^2 + (\partial_\mu\hat{\Phi})^\dagger(\partial_\mu\hat{\Phi}) + \left((\partial_\mu\hat{\Phi})\hat{\Phi}^\dagger - \hat{\Phi}(\partial_\mu\hat{\Phi})^\dagger\right)A_\mu - \frac{m^2}{g^2}A_\mu A_\mu\right. \\ & -\frac{1}{\xi g^2}(\partial_\mu A_\mu)^2 - \frac{\xi m^2}{2}\mathrm{Tr}\left((\hat{\Phi} - \hat{\Phi}^\dagger)T^a\right)(\hat{\Phi} - \hat{\Phi}^\dagger)T^a + 2(\partial_\mu\bar{c})((\partial_\mu c) - [A_\mu, c]) \\ & \left. +g\xi m\bar{c}(\hat{\Phi}^\dagger c + c\hat{\Phi}) + 2\xi m^2\bar{c}c\right).\end{aligned} \tag{4.8}$$

The propagators can be obtained by considering the terms quadratic in the fields A_μ, c and π only. This yields



(4.9)

where solid lines are scalars, dotted lines are ghosts and curly lines are gauge bosons. Up to three-loop there are 15 different vertices presented in Fig. 4.1. An example for the vertex Feynman rules in `FORM` notation can be found in chapter 5.

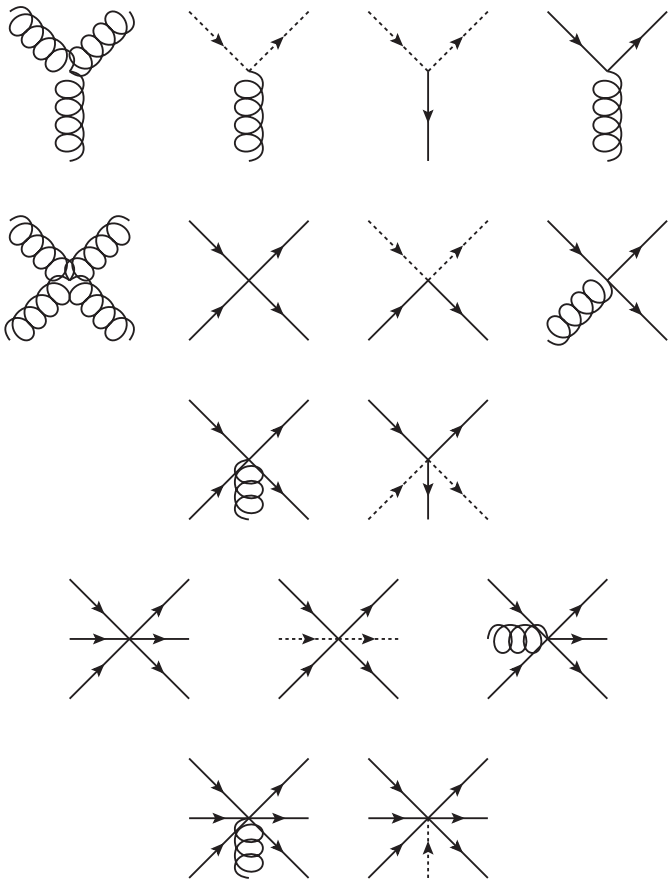


Figure 4.1.: Vertices contributing to three-loop. Curly lines are gauge bosons, dotted lines are ghosts and continuous line are scalars.

4.2. The full massive gauge boson two-point function in the Euclidean $SU(N)$ model

The gauge boson two-point function $\tau_{\mu\nu}^{ab}$ can be calculated using the methods from chapter 2. First we compute the self-energy as a sum of all 1PI graphs and then sum up the corresponding geometric series. In order to simplify this computation, it is useful to isolate the Lorentz and color structure of $\tau_{\mu\nu}^{ab}$ first. How this works out will be explained in the following subsection.

4.2.1. Color structure

The self-energy $\Pi_{\mu\nu}^{ab}$ has only two color indices and thus can only be proportional to δ^{ab}, so that one can write

$$\Pi_{\mu\nu}^{ab} = \delta^{ab}\Pi_{\mu\nu}. \tag{4.10}$$

Although this simple structure is obvious from a theoretical point of view, it is not quite so easy to obtain in a practical calculation. The first step is to project out the color structure by a multiplication with

$$P_{ab} = \frac{1}{N_c^2 - 1}\delta_{ab}, \tag{4.11}$$

so that

$$P_{ab}\Pi_{\mu\nu}^{ab} = \Pi_{\mu\nu}. \tag{4.12}$$

The Feynman rules for this model (see for example section 5.2.3) contain only color tensors like δ^{ab} and f^{abc}, which can be written as traces by

$$\delta^{ab} = \frac{1}{2}\mathrm{Tr}(T^a T^b) \tag{4.13}$$

$$f^{abc} = -2i\mathrm{Tr}([T^a, T^b]T^c). \tag{4.14}$$

After all structure constants have been expressed in terms of traces, any contribution to $\Pi_{\mu\nu}$ contains products like

$$\mathrm{Tr}(T^{a_1}T^{a_2})\mathrm{Tr}(T^{a_1}T^{b_1}T^{a_3})\mathrm{Tr}(T^{a_2}T^{b_2}T^{a_4})\mathrm{Tr}(T^{a_3}T^{a_4})\mathrm{Tr}(T^{b_1}T^{b_2})..., \tag{4.15}$$

where all indices are contracted. The traces can be expressed as sums over the indices by

$$T^{a_1}_{i_1 i_2}T^{a_2}_{i_2 i_1}T^{a_1}_{i_3 i_4}T^{b_1}_{i_4 i_5}T^{a_3}_{i_5 i_3}T^{a_2}_{i_6 i_7}T^{b_2}_{i_7 i_8}T^{a_4}_{i_8 i_6}T^{a_3}_{i_9 i_{10}}T^{a_4}_{i_{11} i_9}T^{b_1}_{i_{12} i_{13}}T^{b_2}_{i_{13} i_{12}}..., \tag{4.16}$$

and with a repeated application of the completeness relation

$$\sum_{a=1}^{N_c^2-1} T^a_{ij}T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{ki}\right) \tag{4.17}$$

the whole color algebraic structure disappears and only the number of colors N_c remains as color related quantity.

4.2.2. Lorentz structure

The two Lorentz indices carried by the self-energy can only come from the tensors $\eta_{\mu\nu}$ and $p_\mu p_\nu$. It is therefore convenient to define linearly independent transverse and longitudinal projectors (following [19])

$$P_{T,\mu\nu} \equiv \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \tag{4.18}$$

$$P_{L,\mu\nu} \equiv \frac{p_\mu p_\nu}{p^2}, \tag{4.19}$$

with the properties

$$P_T^{\mu\sigma} P_{T,\sigma\nu} = P_{T,\nu}^{\mu} \tag{4.20}$$

$$P_L^{\mu\sigma} P_{L,\sigma\nu} = P_{L,\nu}^{\mu} \tag{4.21}$$

$$P_T^{\mu\sigma} P_{L,\sigma\nu} = 0 \tag{4.22}$$

$$P_{T,\mu}^{\mu} = d-1 \tag{4.23}$$

$$P_{L,\mu}^{\mu} = 1, \tag{4.24}$$

where we have used the fact that $\eta^{\mu\nu}\eta_{\mu\nu} = d$ in dimensional regularization. The self-energy can then be expressed in terms of these projectors and some scalar functions Π_T and Π_L as

$$\Pi_{\mu\nu}^{ab}(p,\xi) = \delta^{ab}\Pi_{\mu\nu} = \delta^{ab}\left(P_{T,\mu\nu}\Pi_T(p^2,\xi) + P_{L,\mu\nu}\Pi_L(p^2,\xi)\right), \tag{4.25}$$

where the ξ-dependence has been written in order to indicate that the self-energy is in general a gauge dependent function. Π_T is called the transverse and Π_L the longitudinal part of the self-energy. The projection operators eq.(4.18) and eq.(4.19) can be used to project out either the longitudinal or the transverse part.

The same consideration can be made for the gauge boson propagator leading to the decomposition

$$D_{\mu\nu}^{ab} = \delta^{ab} D_{\mu\nu} = \delta^{ab}\left(\frac{1}{k^2+m^2}P_{T,\mu\nu} + \frac{\xi}{k^2+\xi m^2}P_{L,\mu\nu}\right). \tag{4.26}$$

The summation of all 1PI insertions into the propagator yields the full massive gauge boson propagator

$$\tau_{\mu\nu}^{ab}(k) = \delta^{ab}\left(D_{\mu\nu} + D_{\mu\rho}\Pi^{\rho\sigma}D_{\sigma\nu} + D_{\mu\rho}\Pi^{\rho\sigma}D_{\sigma\tau}\Pi^{\tau\eta}D_{\eta\nu} + ...\right), \tag{4.27}$$

as a geometric series. The computation can be split into two parts by means of the projection operators yielding

$$\tau_{\mu\nu}^{ab}(k) = \delta^{ab}\left(D_T\sum_{n=0}^{\infty}(D_T\Pi_T)^n P_{T,\mu\nu} + D_L\sum_{n=0}^{\infty}(D_L\Pi_L)^n P_{L,\mu\nu}\right), \tag{4.28}$$

where

$$D_T = \frac{1}{k^2 + m^2} \tag{4.29}$$

$$D_L = \frac{1}{k^2 + \xi m^2.} \tag{4.30}$$

The full gauge boson two-point function reads then

$$\tau^{ab}_{\mu\nu}(p) = \delta^{ab} \left(\frac{P_{T,\mu\nu}}{p^2 + m^2 - \Pi_T(k^2,\xi)} + \frac{\xi P_{L,\mu\nu}}{p^2 + \xi m^2 - \xi\Pi_L(k^2,\xi)} \right). \tag{4.31}$$

4.3. Resummation of the model

Now we will study the resummation scheme of this model in more detail. The infrared divergences arising in massless three-dimensional Yang-Mills theories are due to massless gauge boson propagators used in perturbation theory. One possible solution, called *resummation,* is to add a mass term $\mathcal{L}_m$ to the Lagrangian, in order to make propagators massive. The term $\mathcal{L}_m$ will be subtracted again, but in one order higher in the perturbative expansion, so that the gauge boson propagator remains massive. In an infinite perturbative expansion one would get back the original massless theory. In a finite order perturbation theory the undetermined mass parameter coming from $\mathcal{L}_m$, will not vanish though, but it can be interpreted as a higher order contribution to magnetic mass and can be fixed self-consistently by means of a gap equation.

Starting from the massless Yang-Mills Lagrangian in eq.(4.1) and following the approach of [23, 22], the resummation is done by introducing a counting parameter l, rearranging the Yang-Mills Lagrangian to an effective Lagrangian

$$\mathcal{L}_{eff} = \frac{1}{l} \left(\mathcal{L}_{YM}(\sqrt{l}A) + \Delta\mathcal{L}(\sqrt{l}A, \sqrt{l}\pi, \sqrt{l}c) - l\Delta\mathcal{L}(\sqrt{l}A, \sqrt{l}\pi, \sqrt{l}c) \right), \tag{4.32}$$

so that for $l = 1$ the original massless theory is restored. The fields A, π, and c get rescaled to $\sqrt{l}A$, $\sqrt{l}\pi$, and $\sqrt{l}c$, such that l and g^2 always appear together as lg^2. An expansion in l would therefore include the perturbative expansion in g^2. The resummation is then simply done by an l expansion and by setting $l = 1$ in the end of the calculation.

The Lagrangian $\Delta\mathcal{L}$ has to be chosen in a gauge invariant way and there are several ways to do this. Some examples are given in [25, 26, 27, 28]. For example following the so called *procedure A* from [22], one could chose $\Delta\mathcal{L} = \mathcal{L}_{SSB} + \mathcal{L}_{GF} + \mathcal{L}_{FP}$.

However we will use a different approach, namely the so called *procedure B* from [22], where we start from eq.(4.8) and apply the replacement[1]

$$\begin{aligned} g^2 &\to lg^2 \\ m^2 &\to (1-l)m^2. \end{aligned} \tag{4.33}$$

[1]This replacement is not done for the mass m appearing in π, but for the coupling.

After this replacement the new Lagrangian contains terms $\propto l$ and $\propto l^2$ coming from terms $\propto m^2$ and $\propto m^4$ in the old Lagrangian, respectively. Thus one has to extend the Feynman rules for the resummed theory by so called *counter-terms*[2] (c.f [22]).

In order to avoid such terms during the computation we compute the gauge boson transverse self-energy Π_T in the unresummed model from section 4.1 with the Lagrangian defined in eq.(4.8) perturbatively up to three-loop. Afterwards we derive a gap equation for Π_T and apply the resummation scheme defined in eq.(4.33) on the gap equation level. This procedure should be equivalent with procedure B.

The resummed Lagrangian in procedure B has to be BRST-invariant in order to obtain ξ-independent physical results order by order in perturbation theory. The BRST-invariance can be proven by using transformations similar to eq.(3.30), but with fields A, Φ and c rescaled with $\sqrt{l}$. These transformations are[3] [22]

$$\delta_B A_\mu = \omega(\partial_\mu c) + \omega\sqrt{l}[c, A_\mu] \tag{4.34}$$

$$\delta_B \Phi = \omega\sqrt{l} c\Phi \tag{4.35}$$

$$\Rightarrow \delta_B \pi = \omega \sum_{n=0}^{\infty} \frac{B_n l^{n/2}}{n!}(-1)^j \begin{pmatrix} n \\ j \end{pmatrix} \pi^{n-j} c\pi^j + \mathcal{O}(\omega^2) \tag{4.36}$$

$$\delta_B c = \omega\sqrt{l} cc \tag{4.37}$$

$$\delta_B \bar{c} = -\frac{\omega}{\xi}\left((\partial_\mu A_\mu) - (1-l)\xi m^2 \mathrm{Tr}\left((e^{\sqrt{l}\pi} - e^{-\sqrt{l}\pi})T^a\right) T^a/\sqrt{l}\right), \tag{4.38}$$

where B_n are the Bernoulli numbers and ω is an anti-commuting Grassmann number. Furthermore in [22] it was shown that both procedures lead to the same results for the mass gap in one-loop as well as in two-loop.

4.4. Gap equation

The mass parameter in the resummend three dimensional Yang-Mills theory is fixed by a self-consistency equation and the solution of this equation gives an estimate of the mass gap.

The pole of the transverse propagator is believed to be a gauge invariant quantity order by order in perturbation theory for BRST-invariant theories [22, 24]. The longitudinal part can be gauged away in the unitary gauge and is therefore not physical [22, 23]. Following the approach of Eberlein [23], a gap equation can be derived by the natural requirement that the pole of the transverse part of the full propagator

$$\tau_T = \frac{1}{p^2 + m^2 - \Pi_T(p^2)} \tag{4.39}$$

remains at $p^2 = -m^2$. Expanding the self-energy in $p^2 = -m^2 + \varepsilon$ yields [21]

[2] Not to be confused by the renormalization counter term.

[3] Note that in [22] the ghost are defined by $c = igT^a c^a$

$$\tau_T \overset{p^2=-m^2+\varepsilon}{=} \frac{\frac{1}{\left(1-\Pi_T'(-m^2)\right)}}{-\frac{\Pi_T(-m^2)}{\left(1-\Pi_T'(-m^2)\right)} - \varepsilon + \mathcal{O}(\varepsilon^2)}, \tag{4.40}$$

where $\Pi_T'(-m^2) = \partial_{p^2}\Pi_T(p^2)|_{p^2=-m^2}$. The condition that the pole remains in all orders of perturbation theory at $p^2 = -m^2$, results in the *gap equation* [22]

$$\frac{\Pi_T(-m^2)}{\left(1-\Pi_T'(-m^2)\right)} = 0. \tag{4.41}$$

It is easiest to compute the self-energy for the unresummed theory in eq.(4.8) first and then to perform the resummation simply by substituting $m^2 \to (1-l)m^2$ and $g^2 \to lg^2$ in the full propagator as

$$\tau_T = \frac{1}{p^2 + m^2 + (\Pi_T(p^2, (1-l)m^2, lg^2) - lm^2)}, \tag{4.42}$$

leading to the resummed gap equation [21]

$$0 = \left(\frac{\Pi_T(p^2, m^2 - lm^2, lg^2) + lm^2}{1 - \partial_{p^2}\Pi_T(p^2, m^2 - lm^2, lg^2)}\right)_{p^2=-m^2}. \tag{4.43}$$

The L-loop gap equation can be obtained by the l-expansion of eq.(4.43) up to l^L and setting $l = 1$ afterwards. Unfortunately this equation has a rather complicated l-dependence so that an l expansion is not very convenient. But it can be made easier by including the summand[4] $p^2 + m^2$ in the numerator and by applying a mass shift operator $\exp(p^2 l \partial_{m^2})$, leading to [21]

$$0 = \left(e^{p^2 l \partial_{m^2}} \frac{\Pi_T(p^2, m^2, g^2 l) - p^2 - m^2}{1 - \partial_{p^2}\Pi_T(p^2, m^2, lg^2)} + \mathcal{O}(l^{L+1})\right)_{l=1, p^2=-m^2}. \tag{4.44}$$

This equation can now easily be expanded in l and therefore in g^2, using the perturbation expansion

$$\Pi_T(p^2, m^2, g^2 l) = \sum_{n\geq 1} (g^2 l)^n \Pi_T^{(n)}(p^2, m^2) \tag{4.45}$$

for the self-energy[5]. The quantities $\Pi_T^{(n)}(p^2, m^2)$ are the n-loop self-energies in the unresummed theory in eq.(4.8) with g^2 factored out. All information of the resummation is now encoded in the structure of the gap equation.

The gap equation contains a mass and a momentum derivative due to the mass shift operator. Working at the diagram level it is much easier to compute the mass derivative instead of the momentum derivative. For this purpose the dimensional relation [21]

$$\left(p^2\partial_{p^2} + m^2\partial_{m^2} + \frac{4-d}{2} g^2 \partial_{g^2}\right) \Pi_T(p^2, m^2, g^2) = \Pi_T(p^2, m^2, g^2), \tag{4.46}$$

[4] This is allowed, because of the on-shell condition $p^2 = -m^2$

[5] Note that $\Pi_T^{(n)}$ carries the mass dimension $2-n$, whereas Π_T carries the mass dimension 2.

can be used to write all momentum derivatives in terms of mass derivatives.

The expansion of the gap equation in eq.(4.44) up to four-loop, using eq.(4.45) expressed in terms of mass derivatives only, is given as

$$\begin{aligned}
0 &= m^2\left\{1+g^2A\right\} && (4.47)\\
0 &= m^2\left\{1+g^2\frac{4-d}{2}A+g^4\left(\frac{2-d}{2}A^2+B\right)\right\} && (4.48)\\
0 &= m^2\left\{(1+\frac{g^2}{2}\left((4-d)A+C\right)+g^4\left(\frac{(d-4)(d-2)}{4}A^2+(4-d)B+AC\right)\right. && (4.49)\\
&\quad \left.+g^6\left(\frac{(d-2)^2}{4}A^3+\frac{8-3d}{2}AB+D\right)\right\} && (4.50)\\
0 &= m^2\{1+g^2\left(6A(4-d)-6C+E\right)+\mathcal{O}(g^4)\}, && (4.51)
\end{aligned}$$

with abbreviations

$$\Pi_{ab}=\partial^b_{p^2}\partial^a_{m^2}\Pi|_{p^2=-m^2} \tag{4.52}$$

and

$$\begin{aligned}
A &= \frac{\Pi^{(1)}_{00}}{m^2}, && (4.53)\\
B &= \frac{\Pi^{(2)}_{00}}{m^2}+A\Pi^{(1)}_{01}, && (4.54)\\
C &= (d-4)\Pi^{(1)}_{01}-m^2\Pi^{(1)}_{02}, && (4.55)\\
D &= \frac{\Pi^{(3)}_{00}}{m^2}+A\Pi^{(2)}_{01}+\left(\frac{d-4}{2}A+B\right)\Pi^{(1)}_{01} && (4.56)\\
E &= 3(2-d)m^2\Pi^{(1)}_{02}+4m^4\Pi^{(1)}_{03}. && (4.57)
\end{aligned}$$

Assuming the gap equations to be gauge invariant at each order, it is clear that the terms A,...,E should be gauge invariant as well. Indeed this has been proven in [22] for the one- and two-loop equation. As we will show later, this is not the case for higher orders in perturbation theory. In order to understand this problem, it is necessary to have explicit results for the functions Π_{ab}.

Once these functions have been evaluated, eq.(4.47) - eq.(4.51) can be solved for m. This would then give the desired mass gap approximation. The gap equation has the general structure

$$0=m^2\left(1+\frac{g^2}{m}K_1+\frac{g^4}{m^2}K_2+...\right), \tag{4.58}$$

where the dimensionless real numbers K_i can depend on $\frac{\mu^2}{m^2}$ and N_c as well, where μ is the renormalization scale. Setting $m=Kg^2$ and fixing a value for $\frac{\mu}{g^2}$, one can solve this equation at least numerically, leading to an N_c-dependent solution K. For the one- and two-loop result it will even turn out that

$$m = K' g^2 N_c, \tag{4.59}$$

where K' is a N_c-independent number. In order to interpret m as a mass, the solution for K has to be a real non-negative number.

The computation of the self-energy diagrams can be very difficult and needs therefore some computer algebraic techniques. These methods will be discussed in the next chapter.

5. Computational methods in perturbation theory

In this chapter we will present the most important computational methods, algorithms and computer programs used in this work to do the self-energy computation. In the first part we will start with a rather general discussion and classification of Feynman integrals. Then we will explain the so called Laporta algorithm for the reduction of Feynman integrals to a small set of so called master-integrals. Afterwards we will show how to solve these master-integrals numerically. The second part shall give the reader a detailed discussion of the work flow, in particular it will be presented how to obtain scalar Feynman integrals from diagrams.

5.1. Reduction techniques for scalar Feynman integral

5.1.1. Notation

In this general part we will follow and introduce the notation of Laporta [36]

- The number of loops is denoted N_k with loop momenta $k_{i=1,...N_k}$
- Number of external lines is N_e .The external momenta are $p_{i=1,...,N_p}$, where N_p is the number of independent momenta. $N_p = N_e - 1$ if $N_e > 0$.
- The number of internal lines is N_d and the propagators have denominators $D_i = q_i^2 + m_i^2$, where q_i is the momentum and m_i the mass carried by the ith line. q_i is any linear combination of the momenta $\{k_i\}$ and $\{p_i\}$.
- Many calculations in this chapter will be independent of the normalization of the integration measure. Therefore it is useful to hide normalizations like $\frac{1}{(2\pi)^d}$ in a compact notation $\frac{d^d k_l}{(2\pi)^d} \to [d^d k_l]$.

The most general N_k-loop Feynman integral in dimensional regularization can be written as [36]

$$\left(\prod_{l=1}^{N_k} \int [d^d k_l]\right) V_{s\delta}, \tag{5.1}$$

where the integrand

$$V_{s\delta} = \frac{\prod_{i=1}^{N_p} \prod_{j=1}^{N_k} (p_i \cdot k_j)^{\delta_{ij1}} \prod_{i,j=1}^{N_k} (k_i \cdot k_j)^{\delta_{ij2}}}{\prod_{i=1}^{N_d} D_i^{\gamma_i}}, \ \gamma_i \geq 0, \ \delta_{ijl} \geq 0 \tag{5.2}$$

is a product of propagators and scalar products in the numerator. The number of scalar products that can appear in the numerator is $N_{sp} = N_p N_k + N_k(N_k+1)/2$. Since each denominator D_j carries a linear combination of momenta from $\{k_i\}$ and $\{p_i\}$ it is possible to cancel some of the scalar products in the numerator. For every denominator D_j choose a unique scalar product $(p \cdot k)_j$ that appears in the denominator. One can then write [36]

$$\frac{(p \cdot k)_j}{D_j} = \frac{1}{C_j}\left(1 - \frac{D_j - C_j(p \cdot k)_j}{D_j}\right), \tag{5.3}$$

with a constant C_j chosen so that the term $C_j(p \cdot k)_j$ is canceled by the same term appearing in D_j. This cancellation can be done as often as possible in the integrand in eq.(5.2). Finally one ends up with a sum of integrands [36]

$$V'_{ni\alpha\beta} = \frac{\Pi_{j=1}^{N_{sp}-n}(p \cdot k \text{ irred.})_j^{\beta_j}}{\prod_{j=1}^{n} D_{i_j}^{\alpha_j}},\ n \leq N_d,\ \alpha_j, \beta_j \geq 0, \tag{5.4}$$

with $N_{sp} - n$ irreducible scalar products $(p \cdot k \text{ irred.})_j$ which cannot be canceled by some denominator. The index set $i = \{i_1, ..., i_n\}$ labeling the denominators is any subset of $\{1, ..., N_d\}$.

5.1.2. Integral families

For our classification of Feynman integrals it will also be important to understand the notion of integral families. Given N_k loops with N_p-independent external momenta, an integral family is defined by a collection of N_{sp} different denominators $\{D_1, ..., D_{N_{sp}}\}$, so that each of the N_{sp} possible scalar products can be written as a linear combination of inverse propagators and kinematic invariants [39]. Every integral family is therefore characterized by the number of loops and a set of momenta and masses carried by the propagators. Every integral belonging to an integral family can then be written as

$$I(a_1, ..., a_{N_{sp}}) = \left(\prod_{l=1}^{N_k} \int [d^d k_l]\right) \frac{1}{D_1^{a_1} ... D_{N_{sp}}^{a_{sp}}}, \tag{5.5}$$

with indices $a_i \in \mathbb{Z}$. The integrals in eq.(5.4) will be brought to this form in our computation, because many computer programs like `Reduze` [39] work with the concept of integral families.

Example: Consider the one-loop integral

$$\int [d^d k] \frac{2pk}{((k-p)^2 + m^2)^x}. \tag{5.6}$$

The scalar product can be written as $pk = \frac{1}{2}\left((k-p)^2 + m^2 - k^2 - m^2 - p^2\right)$. This suggests to define an integral family by the set of propagators[1] $\{k^2 + m^2, (k-p)^2 + m^2\}$. The integral can therefore be written as

[1] The choice is obviously not unique. One could also have chosen the propagators $\{k^2, (k-p)^2 + m^2\}$. As explained later we will chose the families always in such a way, that propagators with negative powers are massless.

$$I(0, x-1) - I(-1, x) - p^2 I(0, x) \tag{5.7}$$

with

$$I(a, b) = \frac{1}{(k^2 + m^2)^a ((k-p)^2 + m^2)^b}. \tag{5.8}$$

5.1.3. Sectors

Given an integral family with N_{sp} propagators $\{D_1^{-1}, ..., D_{N_{sp}}^{-1}\}$. Each subset $\{D_{i_1}^{-1}, ..., D_{i_n}^{-1}\}$ of $N_d \leq N_{sp}$ propagators defines a sector of the family [39]. Each sector has an identification number

$$ID \equiv \sum_{k=1}^{N_d} 2^{i_k - 1}. \tag{5.9}$$

There are $\begin{pmatrix} N_{sp} \\ N_d \end{pmatrix}$ different sectors for each N_d and $\sum_{N_d=0}^{N_{sp}} \begin{pmatrix} N_{sp} \\ N_d \end{pmatrix} = 2^{N_{sp}}$ sectors for each family. Each integral of a specific sector with N_d propagators has to have positive indices r_i for these propagators with

$$r \equiv \sum_{i=1}^{N_d} r_i \geq N_d. \tag{5.10}$$

All the other $N_{sp} - N_d$ propagators appearing in the integrals have to have powers $-s_i \leq 0$ with

$$s \equiv \sum_{i=1}^{N_{sp}-N_d} s_i \geq 0. \tag{5.11}$$

A *corner integral* of a sector is defined as the integral with $r = N_d$ and $s = 0$. Consider for example the integral family eq.(5.8). The integrals $I(1,0)$, $I(0,1)$ and $I(1,1)$ are the corner integrals of the sectors with ID 1, 2 and 3 respectively.

Each propagator of a family carries a different linear combinations of the momenta $\{k_i\}$ and $\{p_i\}$. But the value of the integrals can be invariant under a *shift* transformation [39]

$$k_i \to \sum_{i=1}^{N_k} M_{ij} k_j + \sum_{j=1}^{N_p} N_{ij} p_j, \, \det(M) = 1. \tag{5.12}$$

The application of this shift to the corner integral of some sector S yields a new set of propagators. If these new propagators form another sector S', then this shift is called *sector relation*. Then with help of the sector relations it is possible to express all integrals of sector S in terms of integrals of the sector S' and subsectors of S', so that the sectors S is eliminated [39].

Considering the integral in eq.(5.8) the shift relation $k \to -k + p$ applied to $I(0,1)$ yields the integral $I(1,0)$.

A *sector symmetry* is a shift which maps a sector S to itself. For example The sector with $ID = 3$ is invariant under the above shift, but individual integrals are not necessarily invariant. For example $I(1,2) \neq I(2,1)$, if the masses in each propagator wouldn't be the same. In the case where both masses are equal, the integral has the *permutation symmetry* $I(1,2) = I(2,1)$ in the indices.

5.1.4. Integration-by-parts

The integration-by-parts (IBP) method was first proposed by Chetyrkin and Tkachov [43] in 1981 and turned out to be a very useful technique which can be applied algorithmically for arbitrary Feynman integrals. The method is as simple as the name suggests. The integral over the total derivative of a function that vanishes at the boundary is zero. Applied to an N_k-loop Feynman integral with N_p-independent external momenta, one can construct $N_k(N_k + N_p)$ equations [36]

$$0 = \left(\prod_{n=1}^{N_k} \int [d^d k_n]\right) \frac{\partial}{\partial k_j^\mu} \left(k_i^\mu V'_{ni\alpha\beta}\right) \tag{5.13}$$

$$0 = \left(\prod_{n=1}^{N_k} \int [d^d k_n]\right) \frac{\partial}{\partial k_j^\mu} \left(p_i^\mu V'_{ni\alpha\beta}\right) \tag{5.14}$$

for each integrand $V'_{ni\alpha\beta}$ defined in eq.(5.4).

Example 1: The best way to understand this method is to start with a simple example. Consider the one-loop tadpole integral (c.f. eq.(2.28))

$$J(x,d,m) = \int [d^d k] \frac{1}{(k^2+m)^x} \tag{5.15}$$

and apply the first relation

$$0 = \int [d^d k] \partial_{k^\mu} \frac{k^\mu}{(k^2+m^2)^x}. \tag{5.16}$$

The application of the derivative leads to a sum of integrals

$$0 = \int [d^d k] \left(\frac{d}{(k^2+m^2)^x} - \frac{2xk^2}{(k^2+m^2)^{x+1}}\right), \tag{5.17}$$

with a reducible scalar product k^2 in the numerator. After cancellation one ends up with an infinite system of equations

$$(\frac{d}{2} - x)J(x,d,m) + xm^2 J(x+1,d,m) = 0, \;\; \forall x > 0. \tag{5.18}$$

The important observations is that these equations can be applied iteratively, so that for any value x the integral $J(x,d,m)$ can be expressed simply in terms of a single integral $J(1,d,m)$ with[2]

$$J(x+1,d,m) = \frac{(\frac{d}{2}-x)\cdot ... \cdot (\frac{d}{2}-1)}{(m^2)^x\,\Gamma(x+1)} J(1,d,m). \tag{5.19}$$

The recursion stops at $J(1,d,m)$ because there is a $\frac{1}{0}$ in the recursion for $x=0$.

Considering integrals with higher powers x as more difficult than integrals with lower powers, a master-integral is the simplest integral that remains after the repeated application of an integration-by-parts relation. In general the aim is to express a class of integrals in terms of a small set of *simpler* master-integrals. There are two possible ways to achieve this. One way is to find a general solution of the infinite system of integration-by-parts relations. A general solution is a set of combined IBP's that can be applied recursively, so that in each recursion step the sum of indices of denominators gets lowered [36]. This idea relies basically on an *ordering* principle that will be discussed in the next section in more detail.

Example 2: To see how a solution can be found in a more advanced example consider the most general one-loop two-point integral

$$I(s_1,s_2,m_1,m_2) = \int [d^dk_1] \frac{1}{(k_1^2+m_1^2)^{s_1}((k_1-p)^2+m_2^2)^{s_2}},\ s_1,s_2>0 \tag{5.20}$$

with p not necessarily on-shell. The first IBP relation

$$0 = \int [d^dk_1] \frac{\partial}{\partial k_1^\mu} \frac{k_1^\mu}{(k_1^2+m_1^2)^{s_1}((k_1-p)^2+m_2^2)^{s_2}} \tag{5.21}$$

leads to the equation

$$\begin{aligned} 0 \;=\; & (d-2s_1-s_2)I(s_1,s_2,m_1,m_2) + 2s_1m_1^2 I(s_1+1,s_2,m_1,m_2) \\ & -s_2 I(s_1-1,s_2+1,m_1,m_2) + (m_1^2+m_2^2+p^2)s_2 I(s_1,s_2+1,m_1,m_2), \end{aligned} \tag{5.22}$$

and the second relation

$$0 = \int [d^dk_1] \frac{\partial}{\partial k_1^\mu} \frac{p^\mu}{(k_1^2+m_1^2)^{s_1}((k_1-p)^2+m_2^2)^{s_2}}, \tag{5.23}$$

leads to the equation

[2] In this case this relation is not a surprise since the solution of $I(x)$ given in (2.28) is known to be a ratio of Gamma functions. The equation (5.18) rewritten to

$$I(x+1) = \frac{1}{m^2}\frac{(x-\frac{d}{2})}{x} I(x),$$

is indeed the functional equation of a ratio of Gamma functions.

$$\begin{aligned} 0 &= (s_2 - s_1) I(s_1, s_2, m_1, m_2) + s_1 I(s_1 + 1, s_2 - 1, m_1, m_2) - s_2 I(s_1 - 1, s_2 + 1, m_1, m_2) \\ &+ s_1(m_1^2 - m_2^2 - p^2) I(s_1 + 1, s_2, m_1, m_2) + s_2(m_1^2 - m_2^2 + p^2) I(s_1, s_2 + 1, m_1, m_2), \end{aligned} \tag{5.24}$$

where in the calculation in between all scalar products appearing due the differentiation have been canceled. In order to find a general solution for this system of equations it is useful to write these relations as operator equations by defining raising an lowering operators $\mathbf{1}^{\pm}$ and $\mathbf{2}^{\pm}$ so that for example $\mathbf{1}^{+}\mathbf{2}^{-} I(s_1, s_2, m_1, m_2) = I(s_1 + 1, s_2 - 1, m_1, m_2)$. Then just subtract both equations and solve one of them to $I(s_1, s_2 + 1, m_1, m_2)$ to get the equation

$$2 s_2 m_1^2 \mathbf{2}^{+} I(s_1, s_2) = \left(s_1 \mathbf{1}^{+}\mathbf{2}^{-} - s_1(p^2 + m_2^2 + m_1^2)\mathbf{1}^{+} - (d - 2s_2 - s_1) \right) I(s_1, s_2), \tag{5.25}$$

which relates an integral with a sum of indices $s_1 + s_2 + 1$ to integrals with a sum of indices $s_1 + s_2 + 1$ or $s_1 + s_2$. Plugging this equation into eq.(5.22) leads then to the relation

$$\begin{aligned} s_1(m_1^2 - 2m_1 m_2 + m_2^2 + p^2)(m_1^2 + 2m_1 m_2 + m_2^2 + p^2)\mathbf{1}^{+} I(s_1, s_2, m_1, m_2) &= \\ + \left(-2 s_2 m_2^2 \mathbf{1}^{-}\mathbf{2}^{+} + s_1(p^2 + m_2^2 + m_1^2)\mathbf{1}^{+}\mathbf{2}^{-} \right) I(s_1, s_2, m_1, m_2) & \\ + \left(d(-p^2 + m_1^2 - m_1^2) + 2 s_2(p^2 + m_1^2) + s_1(p^2 - 3m_2^2 + m_1^2) \right) I(s_1, s_2, m_1, m_2), & \end{aligned} \tag{5.26}$$

which also relates an integral with a sum of indices $s_1 + s_2 + 1$ to integrals with a sum of indices $s_1 + s_2$. The combination of eq.(5.25) and eq.(5.26) can therefore be considered as a complete solution of the system. With an iterative application of these identity one can express any integral $I(s_1, s_2, m_1, m_2)$ in terms of $I(1, 1, m_1, m_2)$ and the known tadpole integrals $I(s_1, 0, m_1, m_2)$ and $I(0, s_2, m_1, m_2)$. This is visualized in Fig. 5.1 for the integral $I(3, 2)$. The iteration stops, when we set the condition for the indices s_i to be non-negative.

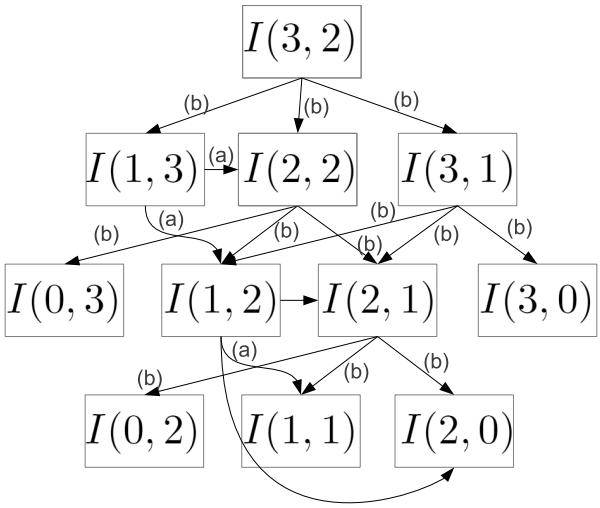


Figure 5.1.: Application of the recurrence relation to the integral $I(3,2)$. (a)=eq.(5.25) and (b)=eq.(5.26). The integral $I(1,3)$ obtained from $I(3,2)$ cannot be further simplified with eq.(5.26) without getting negative indices. Therefore one has to apply eq.(5.25) to it.

In general it can become rather complicated to find a complete solution for the infinite system of equations derived from eq.(5.13) and eq.(5.14), because the number of relations and the number of indices grow rapidly with increasing number of loops and legs.

However for two-loop self-energy integrals there has been found a solution for the general system of equations obtained from IBP for arbitrary masses by Tarasov [30]. There is also a `Mathematica` [31]application called `TARCER` [32], where this solution has been implemented. Furthermore there are general solutions for three-loop massless two-point integrals found by Chetyrkin and Tkachov [43] and for three-loop massive two-point integrals found by Broadhurst [34] realized in the `FORM` packages `MINCER` [33] and `MATAD` [35] respectively.

For all other cases where it is too difficult to find such a general solution it is good to find only a sufficiently large finite solution of the system of integration-by-parts identities. This can be done with the Laporta algorithm, which will be described below.

5.1.5. Laporta algorithm

The idea of the Laporta algorithm is to construct a finite system of integration-by-parts identities, sufficiently large to express all desired integrals in terms of *master-integrals*. Let us first introduce some further notation following [36]. Consider the integral of the type $V'_{in\alpha\beta}$ defined in eq.(5.4). They can be classified by the number of denominators n and by the numbers M_p and M_d defined

by[3]

$$M_p \equiv \sum_{j=1}^{N_{sp}-n} \beta_j,\ M_d \equiv \sum_{j=1}^{n} (\alpha_j - 1). \tag{5.27}$$

Let furthermore $\begin{bmatrix} n; & M_p \\ & M_d \end{bmatrix}$ be the set of all integrands $V'_{in\alpha\beta}$ with n denominators and powers $\alpha = (\alpha_1, ..., \alpha_n)$ and $\beta = (\beta_1, ..., \beta_{N_{sp}-n})$ satisfying eq.(5.27) and let $\begin{bmatrix} n; & 0...a_i \\ & 0...b_i \end{bmatrix}$ be the set of all integrands with $0 \leq M_p \leq a_i$ and $0 \leq M_d \leq b_i$. The index i at a_i and b_i is related to the index set $i = \{i_1, ..., i_n\}$ and is therefore a label for the sectors.

In order to reduce an integral $V'_{iN_d\alpha\beta}$ belonging to a sector with N_d propagators one generates systematically all IBP relations for all integrals belonging to sufficiently large chosen sets $\begin{bmatrix} n; & 0...a_i \\ & 0...b_i \end{bmatrix}$ for all sectors with $n \leq N_d$ propagators. This yields a finite system of equations which can be solved in a Gauss algorithmic way. Each equation is solved to the *most difficult* integral appearing in each relation and is then plugged into the other equations in order to eliminate all difficult integrals. The difficulty of an integral is determined by an *order of priority* defined below. Finally the system of IBP relations has been brought to a form, where all equations relate *difficult* integrals in terms of remaining less difficult integral. These remaining integrals are the *master-integrals.*

The algorithms to construct and solve the finite system of equations reads explicitly:

[3]Note that earlier in the context of integral families the values r in eq.(5.10) and s in eq.(5.11) have been introduced. In fact $r = M_d - n$ and $s = M_p$. Here we keep the notation of Laporta in order to formulate his algorithm in the original form. It is good to know both notations, because some programs use the first and some programs the second one.

Algorithm 5.1 The Laporta algorithm

`for`($n = N_k, ..., N_d$)
 `for`(all combinations of n denominators $\{D_{i_1}, ..., D_{i_n}\} \subseteq \{D_1, ..., D_{N_d}\}$)
  `for`($M_d = 0, ..., b_i$)
   `for`($M_p = 0, ..., a_i$)
    `for`(all $W(n, i, \alpha, \beta) \in \left[n;\ \begin{matrix} M_p \\ M_d \end{matrix} \right]$)
     `for`(x=1,...,$N_k(N_k + N_p)$)
     `do`

1. Generate xth identity for $W(n, i, \alpha, \beta)$
2. Let $\left(\prod_{l=1}^{N_k} \int [dk_l]\right) \sum_j c_j W_j = 0$ be this identity
3. Plug in all already known integrals from earlier steps into this identity to get a *new identity* $\left(\prod_{l=1}^{N_k} \int [dk_l]\right) \sum_j c'_j W'_j = 0$.

   `if`(*new identity* linearly independent to generated system): Solve it to the most difficult integral $W'_l(n', i', \alpha', \beta') \in \left[n';\ \begin{matrix} M'_p \\ M'_d \end{matrix} \right]$ according to an *order of priority* (see below), then add and substitute the equation for W'_l to the system.

The order of priority The order of priority chosen for the three-loop self-energy computation is

1. the greatest n'
2. the greatest M'_p
3. the greatest M'_d
4. the greatest i'_1,..., the greatest i'_n
5. the greatest α'_1,..., the greatest α'_n
6. the greatest β'_1,..., the greatest β'_n.

This ordering determines the set of master-integrals, which remain after the reduction. A different ordering leads to different master-integrals. We have modified this ordering in `Reduze` in order to get integrals without negative exponents. In Laporta's paper [36] the second and the third lines are switched.

As explained above the numbers a and b determine the size of the constructed system. The values a and b have to be carefully chosen, so that the system is large enough to reduce all

integrals to master-integrals. There is no general rule for the choice of a and b, but there is a *golden rule* given by Laporta. The set [36]

$$G_{ab} = \bigcup_{n=N_k}^{N_d} \left[n : \begin{array}{c} 0...a \\ 0...b \end{array} \right] \tag{5.28}$$

is the set of *seed-integrals*, integrals which are used to generate identities for sectors with N_d propagators. Laporta's *golden rule*[4] is that all integrals belonging to G_{ab} can be reduced to master-integrals using the identities generated from G_{ab}. The number of seed-integrals for the set G_{ab} is [36]

$$N_{\text{seeds}}(G_{ab}) = \sum_{n=N_k}^{N_d} \left(\begin{array}{c} N_d \\ n \end{array} \right) \left(\begin{array}{c} N_{sp} - n + a \\ a \end{array} \right) \left(\begin{array}{c} n + b \\ b \end{array} \right). \tag{5.29}$$

The number of equations constructed from G_{ab} is [36]

$$N_{ide}(G_{ab}) = N_k(N_k + N_p)N_{\text{seeds}}(G_{ab}). \tag{5.30}$$

These numbers will help to understand the difficulties that can arise in higher loops computations.

This algorithm has been realized in several programs like `AIR` [38], `FIRE` [41], `Reduze` [39] and `LiteRed` [42]. For the reduction of the self-energy integrals in this work we have used `Reduze`.

5.1.6. Calculation of master-integrals with difference equations

When all integrals have been reduced to master-integrals one definitely likes to find a solution of these integrals as well. In principle it would be nice to find a general d-dependent result for the integrals, but this is too complicated in most cases. In practice one is actually only interested in the numerical coefficients in the ε expansion for some specific space time dimension d. In the best case one can find these coefficients analytically in terms of numbers like π, $\zeta(n)$ or any other function that can be calculated to an arbitrary precision. But in principle it is sufficient to find these coefficients numerically to a very high precision. The expansion in ε and the computation of the numerical coefficients can be done by means of difference equations, which will be explained below. This section relies basically on chapters 4-6 from [36] .

5.1.6.1. Difference equations and master-integrals

A difference equation of order R is an equation for a function $U(x)$ of the form [36]

$$\sum_{j=0}^{R} p_j(x)U(x+j) = F(x), \tag{5.31}$$

[4] In some few cases it can happen that this golden rule is not sufficient. Raising the values for a and/or b by 1 should in the most cases be enough to reduce all integrals to master-integrals.

where $F(x)$ is a known function and p_j are polynomials depending on x, d and eventually on some mass scales. Suppose this equation can be found by means of IBP relations[5] of a Feynman integral

$$U(x) = \prod_{j=1}^{N_L} \left(\int [d^d k_j] \right) \frac{1}{D_1^x D_2 ... D_{N_d}}, \tag{5.32}$$

with one propagator raised by an index x. If it is possible to find a solution for eq.(5.31), then one has found a solution for the Feynman integral in eq.(5.32). In general it is not possible to find an exact solution for such a difference equation. But a numerical solution can be found by means of a factorial series ansatz. A factorial series is of the form [36]

$$U(x) = \mu^x \sum_{s=0}^{\infty} a_s \frac{\Gamma(x+1)}{\Gamma(x+1-K+s)}, \tag{5.33}$$

with constants μ and K which have to be determined for a specific solution. Like in the theory of ordinary differential equation the general solution $U(x)$ of an inhomogeneous difference equations is a sum

$$U(x) = U^{HO}(x) + U^{IH}(x) \tag{5.34}$$

of a particular solution U^{IH} of the inhomogeneous equation and the general solution $U^{HO}(x)$ of the homogeneous equation

$$\sum_{j=0}^{R} p_j(x) U(x+j) = 0. \tag{5.35}$$

The homogeneous solution can be written as a linear combination

$$U^{HO}(x) = \sum_{i=1}^{R} \omega_i(x) U_i^{HO}(x) \tag{5.36}$$

of independent solution U_i^{HO} and periodic function $\omega_j(x) = \omega_j(x+1)$. Thus $w(x)$ is constant for integer x and we will write $w(x) = \eta$ in this case. The factorial series ansatz will lead to R different solutions according to R different pairs (μ_i, K_j) for the homogeneous equation. The values μ_{IH} and K_{IH} for the inhomogeneous solution are determined by the inhomogeneity $F(x)$. In the following we will give an application oriented explanation, how to obtain a numerical solution for difference equations and how to get the numerical ε-expansion for the master-integrals. The following steps summarize what has to be done in principle:

[5] There is an algorithm given in Laporta's paper [36] in section 3.2 for the construction of a system of difference equations from IBP. This algorithm is basically a modified version of the Laporta algorithm presented in this work. One important modification is, that one index in the IBP's is a symbol x and that there are two additional conditions in the order of priority. We won't go into detail here, because this is not part of this work.

1. Find recurrence relations for the coefficients $a_s^{(i,j)}$ for R different pairs (μ_i, K_j) according the homogeneous solutions and for a_s^{IH} for pairs (μ_{IH}, K_{IH}) belonging to the inhomogeneous solution.

2. Compute the large x-behavior of the integral in eq.(5.32) (see section 5.1.6.3). This will lead to a factorial series with some specific μ_0 and K_0. For this series the first coefficients $a_0, a_1, ...$ can be computed with a Taylor expansion. Finally the comparison of this factorial series with the general solution $U(x) = \sum_{i=1}^{R} \eta_i U_i^{HO} + U^{IH}$ of the difference equation (for integer x) leads to the determination of the unknown constants η_i in the linear combination.

3. For a difference equation of order R compute the coefficients $a_s^{(i,j)}$ and a_s^{IH} iteratively for R different values $x_{max}, ..., (x_{max} - R) > K$. For values $x \leq K$ the series diverges. In general one cannot find a closed solution for the recurrence relations. Thus one has to stop the computation at some $a_{s_{max}}$ with sufficient high s_{max}. The coefficients will be d-dependent. In order to finish the computation in a reasonable amount of time it is necessary to expand the recurrence relation in ε in every iteration step.

4. Sum up the factorial series for the values $x_{max}, ..., (x_{max} - R)$ to obtain the solutions $U(x_{max}), ...U(x_{max} - R)$.

5. Plug the solutions $U(x_{max}), ..., U(x_{max} - R)$ into the difference equation and solve the equation to $U(1)$ iteratively. Here it is also advisable to do an ε expansion in every iteration step. This part of the calculation is called push down.

Some comments:

- Step 2 works only for integrals, where external momenta are Euclidean and their scalar products $p_i \cdot p_j$ form a semidefinite non-negative matrix and the masses are not zero. Since fully massive vacuum bubbles have no external momenta this method works very well for them. For the self-energy integrals in the Euclidean 3d Yang-Mills $SU(N)$ model this condition is not fulfilled since $p^2 = -1$. Therefore one has to apply another method proposed by Laporta [36] in section 5.2 and 5.3. We have not yet applied this method for the three-loop master-integrals because it is rather difficult to automatize. Instead we will present the application of this method to a tadpole integral class as example here.

- The numerical result is very sensitive to the choice of x_{max} and s_{max}. The series converges very well for large x_{max} so that only a comparably small s_{max} is needed to get a lot of digits for $U(x_{max})$. But there are some practical problems. Firstly the coefficients a_s can become very huge ($\sim 10^{1000}$) for large s, whereas the factor $\frac{\Gamma(x+1)}{\Gamma(x+1-K+s)}$ in the factorial series becomes very small ($\sim 10^{-(1000)(1+\varepsilon)}$). This can lead to very bad numerical problems due to round-off errors. It is therefore advisable to solve the recurrence relation for $b_s = a_s/\Gamma(x+1-K+s)$ and to sum up the factorial series $\Gamma(x+1)\sum_{s=0}^{s_{max}} b_s$ instead.

- The good convergence behavior of the series for large x_{max} is spoiled by round-off errors leading to huge loss of digits during the push down. The higher x_{max} the more digits get lost during the pushdown. It is therefore important to compute the values

$U(x_{max}), ..., U(x_{max} - R)$ to a very high precision by choosing a high enough s_{max}. One has to play with these parameters in order to get an optimal result. In the computation in this work values of the order $x_{max} \sim 10^2$ and $s_{max} \sim 10^3$ turned out to be good. For details see the example computation in the end of this section.

5.1.6.2. Step 1: How to obtain the recurrence relations

The derivation of the recurrence relation can be done in an automatized way by using operators $\boldsymbol{\pi}$ and $\boldsymbol{\rho}$.

Operators $\boldsymbol{\pi}$ and $\boldsymbol{\rho}$ The operators $\boldsymbol{\pi}$ and $\boldsymbol{\rho}$ are defined by [36]

$$\boldsymbol{\rho}^m U(x) \equiv \frac{\Gamma(x+1)}{\Gamma(x-m+1)} U(x-m),\ \boldsymbol{\rho}^m 1 \equiv \boldsymbol{\rho}^m \equiv \frac{\Gamma(x+1)}{\Gamma(x-m+1)}, \tag{5.37}$$

and

$$\boldsymbol{\pi} U(x) = x(U(x) - U(x-1)). \tag{5.38}$$

It is easy to show that the operators have the properties

$$\boldsymbol{\rho}^m \boldsymbol{\rho}^n U(x) = \boldsymbol{\rho}^{m+n} U(x), \tag{5.39}$$

$$[\boldsymbol{\pi}, \boldsymbol{\rho}] U(x) = \boldsymbol{\rho} U(x) \tag{5.40}$$

$$xU(x) = (\boldsymbol{\rho} + \boldsymbol{\pi}) U(x). \tag{5.41}$$

Homogeneous solution The homogeneous solution can be obtained by applying the following steps to the difference equation [36]:

1. Write down the homogeneous difference equation

$$\sum_{j=0}^{R} p_j(x) U^{HO}(x+j) = 0. \tag{5.42}$$

2. Shift the variable $x \to x - R$

$$\sum_{j=0}^{R} q_j(x) U^{HO}(x-j) = 0, \tag{5.43}$$

with new polynomials $q_{R-i}(x) = p_i(x-R)$, so that their highest argument is x.

3. Make the substitution $U^{HO}(x) = \mu^x V^{HO}(x)$ and divide the μ's so that the lowest power in μ is 0. This yields

$$\sum_{j=0}^{R} q_j(x)\mu^{R-j}V^{HO}(x-j) = 0. \tag{5.44}$$

4. Multiply the equation with $\prod_{k=0}^{R-1}(x-k)$ and use eq.(5.37), rewritten as $\left(\prod_{j=0}^{N}(x-j)\right)V(x-N-1) = \boldsymbol{\rho}^{N+1}V(x)$, to obtain an operator equation

$$\left(\sum_{l=0}^{R}\phi_l(x,\mu)\boldsymbol{\rho}^l\right)V^{HO}(x) = 0, \tag{5.45}$$

where $\phi_l(x,\mu)$ is a polynomial obtained from sorting the equation by powers in $\boldsymbol{\rho}$.

5. Now substitute $x \to \boldsymbol{\pi} + \boldsymbol{\rho}$ according to eq.(5.41) and commute all $\boldsymbol{\pi}$ operators to the left of $\boldsymbol{\rho}$ using eq.(5.40). This yields the *first canonical form*

$$\left(\sum_{l=0}^{m+1} f_l(\pi,\mu)\boldsymbol{\rho}^l\right)V^{HO}(x) = 0, \tag{5.46}$$

where $m \geq R-1$ is any integer depending on the x-dependence of the difference equation. It turns out that the polynomial f_{m+1} is always $\boldsymbol{\pi}$-independent.

6. Solve the *characteristic equation*

$$f_{m+1}(\mu) = 0 \tag{5.47}$$

leading to $\lambda \leq R$ distinct nonzero solutions $\mu_{i=1,\ldots,\lambda}$. Choose any of these solutions and plug it into eq.(5.46).

7. Make the factorial series ansatz

$$V^{HO}(x) = \sum_{s=0}^{\infty} a_s \rho^{K-s}, \tag{5.48}$$

apply the commutator in eq.(5.40) to bring all $\boldsymbol{\pi}'s$ to the right and use the fact that $\boldsymbol{\pi}$ acting on a constant is zero. This is effectively equal to the substitution $\pi\boldsymbol{\rho}^s \to s\boldsymbol{\rho}^s$. Finally this leads to a recurrence relations for the coefficients a_s

$$\sum_{n=0}^{m} a_{s-n}^{(i)} f_{m-n}(K+m-s,\mu_i) = 0 \;\; \forall\, s \geq m. \tag{5.49}$$

8. Solve the *indicial equation*

$$f_m(K+m,\mu_i) = 0 \tag{5.50}$$

with solutions $K_{i1,\ldots,}K_{i\nu_i}$ where ν_i is the multiplicity of μ_i, so that there are $\sum_{i=1}^{\lambda}\nu_i = R$ solutions. Choose one K_{ij} and plug it into eq.(5.49) to obtain a recurrence relation for coefficients $a_s^{(i,j)}$ for a specific pair (μ_i, K_{ij}).

9. Repeat steps 6-8 for all other possible pairs (μ, K) to get all the other recurrence relations

Inhomogeneous solution The recurrence relation for the inhomogeneous difference equation can be obtained very similarly [36]. There are only some slight modifications. For the equation

$$\sum_{j=0}^{R} p_j(x) U(x+j) = F(x), \tag{5.51}$$

apply steps 1 - 5 and assume that the function $F'(x) = F(x-R)$ has a *known* μ and K-dependence due to known factorial series expansion[6] $F'(x) = \mu_{IH}^x \sum_s c_s \rho^{K_{IH}-s}$ with known c_s. This is true because the inhomogeneity is given as the solution of another difference equation which has already been solved[7]. After applying step 7 the parameters μ and K on the the left hand side coming from the ansatz $U^{IH} = \mu^x \sum_s a_s \rho^{s-K}$ have to be equal to μ_{IH} and K_{IH}. This leads to the recurrence relations

$$\sum_{n=0}^{m} a_{s-n} f_{m-n}(K_{IH} - s, \mu_{IH}) = c_s,\ s \geq m. \tag{5.52}$$

with known c_S.

5.1.6.3. Step 2: Determination of arbitrary coefficients from large x-behavior

Still following [36] start from an N_k-loop integral

$$U(x) = \int [d^d k_1] \frac{1}{\left(k_1^2 + m_1^2\right)^x} g(k_1), \tag{5.53}$$

where g is an $(N_k - 1)$-loop integral of the type

$$g(k_1) = \left(\prod_{l=2}^{N_k} [d^d k_l] \right) \frac{\prod_{j=1}^{N_{sp}-N_d} (p \cdot k \,\text{irred})_j}{D_2 D_3 ... D_{N_d}}, \tag{5.54}$$

with a semidefinite non-negative matrix $p_i \cdot p_j$ and non-vanishing masses in each propagator. Lets assume here that $[d^d k_1] = \frac{d^d k_1}{2\pi^{d/2}}$. Like the one-loop tadpole integral in eq.(2.28), $U(x)$ can be written in d-dimensional spherical coordinates

$$U(x) = \frac{1}{\Gamma(d/2)} \int_0^\infty \frac{dk_1^2 \left(k_1^2\right)^{d/2-1}}{\left(k_1^2 + m_1^2\right)^x} f(k_1^2), \tag{5.55}$$

with $f(k_1^2) = \frac{1}{\Omega_d} \int d\Omega_d(\hat{k}_1) g(k_1)$. Substitute $k_1^2 = m_1^2 \frac{u}{1-u}$ and write $\tilde{f}(u) = f(m_1^2 u/(1-u))$ to obtain the integral

$$U(x) = \frac{\left(m_1^2\right)^{d/2-x}}{\Gamma(d/2)} \int_0^1 du u^{d/2-1} (1-x)^{x-1-d/2} \tilde{f}(u). \tag{5.56}$$

[6] In general it can be a sum $F(x) = \sum_l \mu_{IH,l}^x T_l(x)$ with different values $\mu_{IH,l}$ and $K_{IH,l}$. But for simplicity and according to the examples presented here it is sufficient to consider this case only.

[7] In general the difference equation is part of a triangular system of difference equations

Due to the properties of the external momenta and masses, the large x-behavior of the integral $U(x)$ is dominated by the small k_1^2 behavior of f [36]. For small k_1^2 (and so small u) the function $\tilde{f}$ can be written as an expansion in u with

$$\tilde{f}(u) = u^\alpha (1-u)^{d/2+1} \sum_{s=0}^{\infty} b_s u^s, \tag{5.57}$$

where $\alpha \geq 0$ vanishes in the case of no scalar products in the numerator of $U(x)$ and the factor $(1-u)^{d/2+1}$ is chosen for convenience. It is then possible to express the integral in eq.(5.53) indeed in terms of an infinite sum of Euler Beta functions and therefore as a factorial series

$$U(x) = \mu_0^x \sum_{s=0}^{\infty} a_s \frac{\Gamma(x+1)}{\Gamma(x+1-K_0+s)}, \tag{5.58}$$

with

$$\mu_0 = \frac{1}{m_1^2},\ K_0 = -d/2 - \alpha,\ a_s = b_s m_1^d \Gamma(s+d/2+\alpha)/\Gamma(d/2). \tag{5.59}$$

The coefficients a_s in eq.(5.58) are simply determined by b_s in eq.(5.57), which is the Taylor expansion of $f(u)$. Thus the large $x-$behavior of the integral $U(x)$ and the resulting coefficients a_s are determined by the one loop less and one propagator less integrals g. In general the integral g is itself a solution of a difference equation, whose large x-behavior has to be determined before.

Finally we can compare the above expansion of $U(x)$ with the factorial series obtained by the operator approach for difference equations. By equating both expansions

$$\left(\frac{1}{m_1^2}\right)^x \sum_{s=0}^{\infty} a_s \boldsymbol{\rho}^{-D/2-\alpha-s} = \sum_{j=1}^{R} \eta_j \mu_j^x \sum_{s=0}^{\infty} \hat{a}_{js} \boldsymbol{\rho}^{K_j - s} + \sum_l \left(\mu_l^{IH}\right)^x \sum_{s=0}^{\infty} a_{ls}^{IH} \boldsymbol{\rho}^{K_l^{IH}-s}, \tag{5.60}$$

we obtain the unknown constants η_j. By comparison it is clear that only solutions with $\mu_j = \frac{1}{m_1^2}$ can contribute to the whole solution. The numbers K_j of the homogeneous solution have to satisfy the condition [36]

$$K_j + d/2 + \alpha = \text{integer} \leq 0 \tag{5.61}$$

and for the inhomogeneous solution they have to satisfy eq.(5.61) or [36]

$$0 < K_j + d/2 + \alpha = \text{integer} \leq \max_l K_l^{IH} + d/2 + \alpha. \tag{5.62}$$

5.1.6.4. Application: Massive sunset-type vacuum integrals

Consider the fully massive n-loop sunset integrals

$$S(n,d,x) = \left[\int \frac{d^d p}{(1+p^2)^2}\right]^{-n} \left(\prod_{j=1}^{n} \int \frac{d^d k_j}{1+k_j^2}\right) \frac{1}{[1+(k_1+...+k_n)^2]^x}, \tag{5.63}$$

where $S(1,d,x) = \Gamma(x+1-d/2)/\Gamma(x+1)/\Gamma(2-d/2)$ is the analytic solution of the well known one-loop case. In order to have measure independent results, the integral is normalized by the one-loop tadpole integral with one index raised, so that $S(1,d,1) = 1$.

In general one can derive with IBP's the difference equations of the kind (see appendix A)

$$\sum_{j=0}^{r_n} p_{n,j}(d,x)\frac{\Gamma(x+j)}{\Gamma(x+1)} S(n,d,x+j) = c_n \Gamma(n+1) S(1,d,x). \tag{5.64}$$

with polynomials $p_{n,j}(d,x)$ (see appendix A). The nice property of this class is, that the homogeneity is always given by the one-loop tadpole integral. For the sunset integral class one can derive the large x-behavior very generally. For $n \geq 2$ the one loop less integral can be written as

$$g(k_1) = 2\pi^{d/2}\left[\int \frac{d^d p}{(1+p^2)^2}\right]^{-n}\left(\prod_{j=2}^{n}\int \frac{d^d k_j}{1+k_j^2}\right)\frac{1}{1+(k_1+k_2+...+k_n)^2}. \tag{5.65}$$

Let ϕ be the angle between k_1 and $q = k_2 + ... + k_n$. Working in spherical coordinates the $|k_1|^2$ expansion for $n \geq 2$ is simply

$$\begin{aligned} g(k_1) \approx 2\pi^{d/2}&\left[\int \frac{d^d p}{(1+p^2)^2}\right]^{-n}\left(\prod_{j=2}^{n}\int \frac{d^d k_j}{1+k_j^2}\right)\left[\frac{1}{q^2+1} - \frac{2q\cos\phi}{(q^2+1)^2}|k_1|\right. \\ &\left. + \left(\frac{-2}{(q^2+1)^2} + \frac{8q^2\cos^2\phi}{(q^2+1)^3}\right)|k_1|^2 + \mathcal{O}(|k_1|^3)\right]. \end{aligned} \tag{5.66}$$

Integrating first over the angular part with $\int d\Omega_d(\hat{k}_1) = \int d\Omega_{d-1}\int_0^\pi d\phi \sin^{d-2}\phi$ and using

$$\int_0^\pi d\phi \sin^{d-2}\phi\cos\phi = 0,\ \frac{\Omega_{d-1}}{\Omega_d}\int_0^\pi d\phi\sin^{d-2}\phi\cos^2\phi = d \tag{5.67}$$

yields

$$\begin{aligned} f(k_1^2) \approx 2\pi^{d/2}&\left[\int \frac{d^d p}{(1+p^2)^2}\right]^{-n}\left(\prod_{j=2}^{n}\int \frac{d^d k_j}{1+k_j^2}\right)\left[\frac{1}{(q^2+1)} + \left(\frac{4-d}{d}\frac{1}{(q^2+1)^2} - \frac{4}{d}\frac{1}{(q^2+1)^3}\right)k_1^2\right] \\ =&\frac{1}{\Gamma(2-d/2)}\left[S(n-1,d,1) + \left(\frac{4-d}{d}S(n-1,d,2) - \frac{4}{d}S(n-1,d,3)\right)k_1^2\right] \end{aligned} \tag{5.68}$$

so that for $n \geq 2$

$$a_0 = \frac{1}{\Gamma(2-d/2)}S(n-1,d,1) \tag{5.69}$$

$$a_1 = \frac{1}{\Gamma(2-d/2)}\left(\frac{4-d}{d}S(n-1,d,2) - \frac{4}{d}S(n-1,d,3)\right). \tag{5.70}$$

The large x-behavior is indeed determined by the one loop less solution of the same sunset integral class. For the integrals up to four-loop a_1 is not needed, but at five-loop a_1 is not determined by the inhomogeneous equation. In the one-loop case $g = 2\pi^{d/2}\left[\int \frac{d^dp}{(1+p^2)^2}\right]^{-1}$ and thus $a_0 = \frac{1}{\Gamma(2-d/2)}$. Furthermore from mass $m = 1$ and $\alpha = 0$ (no scalar products in the numerator) it is clear that $\mu_0 = 1$ and $K_0 = -\frac{d}{2}$.

Example: Derivation of the one-loop recurrence relation Although the analytic solution of the one-loop tadpole integral is well known, it is useful to derive the numerical solution by means of difference equations as well. Firstly the operator approach can be applied exemplary very well and secondly the recurrence relation is needed for the inhomogeneous part of the the two-loop calculation. The one-loop integral

$$S(1,d,x) = \left[\int \frac{d^dp}{(1+p^2)^2}\right]^{-1} \int \frac{d^dk_1}{\left(1+k_1^2\right)^{x+1}}, \tag{5.71}$$

has the homogeneous first order difference equation [37]

$$(d-2x-2)S(1,d,x) + 2(x+1)S(1,d,x) = 0. \tag{5.72}$$

Applying the first three steps of section 5.1.6.2 with $S(1,d,x) = \mu^x V(x)$ yields

$$(d-2x)V(x-1) + 2x\mu V(x) = 0. \tag{5.73}$$

The fourth step leads then to the operator equation

$$\left((d-2x)\boldsymbol{\rho} + 2x^2\mu\right)V(x) = 0. \tag{5.74}$$

The first canonical form obtained by step 5 reads

$$\left(2(\mu-1)\boldsymbol{\rho}^2 + (d-2\mu)\boldsymbol{\rho} + (-2+4\mu)\boldsymbol{\pi}\boldsymbol{\rho} + 2\mu\boldsymbol{\pi}^2\right)V(x) = 0. \tag{5.75}$$

The solution of the characteristic equation $2\mu - 2 = 0$ according to step 6 is $\mu = 1$. The canonical form simplifies then to

$$\left((d-2)\,\boldsymbol{\rho} + 2\boldsymbol{\pi}\boldsymbol{\rho} + 2\boldsymbol{\pi}^2\right)V(x) = 0. \tag{5.76}$$

The factorial series ansatz $V(x) = \sum_s c_s \boldsymbol{\rho}^{K-s}$ in step 8 leads then to the equation

$$c_{s-1}(2 - 4s + 2s^2 + 4K - 4Ks + 2K^2) + c_s(-2s + d + 2K); \tag{5.77}$$

For $s = 0$ the indicial equation yields $K = -\frac{d}{2}$. Finally one ends up with the recurrence relation

$$0 = c_{s-1}(-1 + d/2 + s)^2 - sc_s. \tag{5.78}$$

Example: two-loop recurrence relations and exclusion principle The two-loop difference equation reads [37]

$$(-2+d-x)S(n,d,x)+(-3+d-2x)S(n,d,x+1)+3(x+1)S(n,d,x+2) = -2S(1,d,x). \quad (5.79)$$

It is of second order and inhomogeneous. Therefore there should be two homogeneous and one inhomogeneous solutions. Using the operator approach for the homogeneous equations leads to the characteristic equation $-1-2\mu+3\mu^2=0$ with solutions

$$\mu_1 = 1,\ \mu_2 = -\frac{1}{3}. \quad (5.80)$$

The indicial equation is the same for for μ_1 and μ_2 and reads $-2+2d+4K=0$ with solution

$$K_1 = K_2 = \frac{1}{2} - \frac{d}{2}. \quad (5.81)$$

From the exclusion principle defined in eq.(5.61) with $K_0 = -d/2$ and $\mu_0 = 1$ it is clear that no homogeneous solution does contribute. The operator approach yields the inhomogeneous recurrence relation

$$\begin{aligned} 0 &= -a_{s-2}\frac{3}{8}(-4+d+2s)(-2+d+2s)^2 \\ &\quad +\frac{1}{4}a_{s-1}(-2+d+2s)(-6+5d+14s) + a_s(-2-4s) + 2c_s. \end{aligned} \quad (5.82)$$

The coefficients a_s are then completely determined by c_s, which are fixed by the large x-behavior of the one-loop sunset integral.

Results up to five-loop The difference equations and the corresponding values for K and μ can be found in appendix A. For all difference equations from two- to five-loop it was possible to set all coefficients $\eta_i = 0$, although it was not forced by the exclusion principle eq.(5.61) and eq.5.62. Then only the inhomogeneous part contributes to the solution, but with the first coefficients a_0 and a_1 in eq.(5.52) undetermined. For these coefficients one can simply use eq.(5.69) and eq.(5.70) to fix the solutions of the inhomogeneous recurrence relations.

Due to the normalization of the integration measure the first finite part of the integrals at n-loop is at ε^n. Here we present the ε expansion in $\varepsilon = \frac{4-d}{2}$ and $\varepsilon = \frac{3-d}{2}$ for the sunset class up to five-loop.

At two-loop we used $x_{max} = 400$ and $s_{max} = 3000$ leading to more than 300 correct digits. The first ten digits at two-loop are

$$\begin{aligned} S(2,4-2\varepsilon,1) &= -1.5000000000 - 4.5000000000\varepsilon - 6.9841391419\varepsilon^2 - 18.0087816235\varepsilon^3 \\ &\quad - 27.9942235636\varepsilon^4 - 72.0037865980\varepsilon^5 + \mathcal{O}(\varepsilon^6) \quad (5.83) \\ S(2,3-2\varepsilon,1) &= 1.0000000000\varepsilon^{-1} + 0.3781395675 - 0.1136827414\varepsilon \\ &\quad + 0.1332134197\varepsilon^2 - 0.1277903581\varepsilon^3 + 0.1264142054\varepsilon^4 \quad (5.84) \\ &\quad - 0.1256874950\varepsilon^5 + \mathcal{O}(\varepsilon^6). \quad (5.85) \end{aligned}$$

At three-loop we used $x_{max} = 230$ and $s_{max} = 3000$ yielding about 140 correct digits. The first 10 digits are

$$\begin{aligned} S(3,4-2\varepsilon,1) &= 2.0000000000 + 7.6666666667\varepsilon + 17.5000000000\varepsilon^2 \\ &\quad + 22.9166666667\varepsilon^3 + 21.2517910513\varepsilon^4 - 184.2300051053\varepsilon^5 + \mathcal{O}(\varepsilon^6) \qquad (5.86) \\ S(3,3-2\varepsilon,1) &= -8.0000000000\varepsilon^{-1} - 41.8192902221 - 306.5667083612\varepsilon \\ &\quad - 1505.2007494737\varepsilon^2 - 11039.3611000292\varepsilon^3 - 54185.4445181265\varepsilon^4 \qquad (5.87) \\ &\quad - 397417.9975979478\varepsilon^5 + \mathcal{O}(\varepsilon^6) \qquad (5.88) \end{aligned}$$

At four-loop we used $x_{max} = 210$ and $s_{max} = 3300$ and obtained a result with about 80 correct digits. The first 10 digits are

$$\begin{aligned} S(4,4-2\varepsilon,1) &= -2.50000000000 - 11.666666667\varepsilon - 31.7013888889\varepsilon^2 \\ &\quad - 67.5289351852\varepsilon^3 - 140.2205432875\varepsilon^4 - 573.5347004607\varepsilon^5 + \mathcal{O}(\varepsilon^6) \qquad (5.89) \\ S(4,4-2\varepsilon,1) &= 45.0000000000\varepsilon^{-1} + 356.7418536252 + 3163.2279808546\varepsilon \\ &\quad + 18912.3750509141\varepsilon^2 + 138631.6834120886\varepsilon^3 + 777993.9266004253\varepsilon^4 \qquad (5.90) \\ &\quad + 5.3868190464 \cdot 10^6\varepsilon^5 + \mathcal{O}(\varepsilon^6) \qquad (5.91) \end{aligned}$$

At five-loop we used x_{max} =250 and s_{max}=5000 leading to about 70 correct digits. The first 10 digits are

$$\begin{aligned} S(5,4-2\varepsilon,1) &= +3.0000000000 + 16.5000000000\varepsilon + 51.9583333333\varepsilon^2 + 125.6715277778\varepsilon^3 \\ &\quad + 259.9875578704\varepsilon^4 + 347.3551162195\varepsilon^5 + \mathcal{O}(\varepsilon^6) \qquad (5.92) \\ S(5,3-2\varepsilon,1) &= -224.0000000000\varepsilon^{-1} - 2265.0653101210 - 23860.0497548140\varepsilon \\ &\quad - 169712.3419671662\varepsilon^2 - 1.4490646307 \cdot 10^6\varepsilon^3 - 9.6353063852 \cdot 10^6\varepsilon^4 \qquad (5.93) \\ &\quad - 8.5028112332 \cdot 10^7\varepsilon^5 + \mathcal{O}(\varepsilon^6) \qquad (5.94) \end{aligned}$$

These results have been confirmed by a comparison with the analytic prefactors of the divergent parts in 4 dimensions [45]

$$S(2,4-2\varepsilon,1) = -\frac{3}{2}\left[1 + 3\varepsilon + \mathcal{O}(\varepsilon^2)\right] \qquad (5.95)$$

$$S(3,4-2\varepsilon,1) = +\frac{4}{2}\left[1 + \frac{23}{2\cdot 3}\varepsilon + \frac{5\cdot 7}{2^2} + \mathcal{O}(\varepsilon^3)\right] \qquad (5.96)$$

$$S(4,4-2\varepsilon,1) = -\frac{5}{2}\left[1 + \frac{2\cdot 7}{3}\varepsilon + \frac{11\cdot 83}{2^3 3^2}\varepsilon^2 + \frac{7\cdot 1667}{2^4 3^3}\varepsilon^3 + \mathcal{O}(\varepsilon^4)\right] \qquad (5.97)$$

$$S(5,4-2\varepsilon,1) = +\frac{6}{2}\left[1 + \frac{11}{2}\varepsilon + \frac{29\cdot 43}{2^3 3^2}\varepsilon^2 + \frac{37\cdot 67\cdot 73}{2^5 3^3 5}\varepsilon^3 + \frac{197\cdot 4561}{2^7 3^4} + \mathcal{O}(\varepsilon^5)\right]. \qquad (5.98)$$

The three-dimensional results up to four-loop agree with the results of [46]

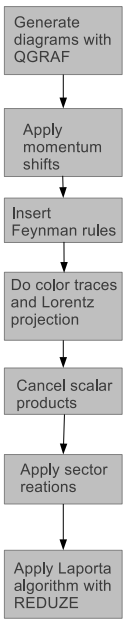


Figure 5.2.: The work-flow: From diagrams generated by QGRAF to scalar-integrals

5.2. The computation of Π_T: From diagrams to integrals

The discussions in section 5.1 were related to scalar integrals only. In this section we will explain how to attain Feynman integrals from Feynman diagrams. This will exemplary be done on the basis of the self-energy contributions to Π_T and its derivatives which are needed for the three-loop gap equation. The work flow is summarized in Fig. 5.2.

5.2.1. QGRAF

The diagrams contributing to the gauge boson's self-energy $\Pi^{ab}_{\mu\nu}$ at a certain loop order are generated by `QGRAF` [20]. In this program a physical theory can be specified simply as a symbolic list of propagators and vertices. For instance the three dimensional $SU(N)$ Yang Mills theory containing vertices with (sc)alar particles, (gl)uons and (gh)osts, is simply implemented by

```
** ** 2pt-functions
[gh,hg,-,notadpole]
[gl,gl,+,notadpole]
[sc,sc,+,notadpole]
** 3pt-functions
[hg,gl,gh]
[hg,sc,gh]
[gl,gl,gl]
```

```
[gl,sc,sc]
**and so on for higher vertices
```

For example the output file for a crossed ladder three-loop diagram looks like (c.f. first diagram in Fig. 5.3)

```
(-1)*ext(gl(-1,kq),gl(-2,-kq))*
prop(gh(1,-k1),hg(2,k1))*
prop(gh(3,-k1+kq),hg(4,k1-kq))*
prop(sc(5,-k2),sc(6,k2))*
prop(sc(7,k2+kq),sc(8,-k2-kq))*
prop(gh(9,-k3),hg(10,k3))*
prop(gl(11,-k1+k3),gl(12,k1-k3))*
prop(gh(13,-k2-k3),hg(14,k2+k3))*
prop(sc(15,k1-k2-k3-kq),sc(16,-k1+k2+k3+kq))*
vrtx(hg(4,k1-kq),gl(-1,kq),gh(1,-k1))*
vrtx(gl(-2,-kq),sc(5,-k2),sc(7,k2+kq))*
vrtx(hg(2,k1),gl(11,-k1+k3),gh(9,-k3))*
vrtx(hg(14,k2+k3),sc(15,k1-k2-k3-kq),gh(3,-k1+kq))*
vrtx(hg(10,k3),sc(6,k2),gh(13,-k2-k3))*
vrtx(gl(12,k1-k3),sc(8,-k2-kq),sc(16,-k1+k2+k3+kq))
```

This is simply a symbolic list of propagators and vertices, with the right symmetry factors and signs. The fields in the propagators and vertices are labeled with their momenta and with some numbers. Negative numbers indicate external and positive numbers internal particles. Propagators are connected to vertices which carry the same number. QGRAF does not provide any Lorentz and color structure. This has to be added separately in a FORM [29] program.

At one-loop there are four diagrams, at two-loop 38 diagrams and at three-loop there are 895 diagrams contributing to $\Pi^{ab}_{\mu\nu}$. Two examples for three-loop diagrams can be found in Fig. 5.3.

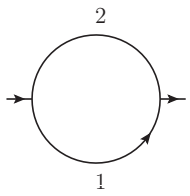
Figure 5.3.: Two three-loop diagrams. The first diagram belongs to the *crossed ladder* topology. Diagrams of this kind vanish due to the color sum. The second one is a typical example for a *ladder* topology.

5.2.2. Momentum convention for self-energies

There are $N_k(N_k+3)/2$ scalar products for N_k-loop self-energies according to section 5.11. Therefore a one-loop integral family has to have two different propagators, a two-loop family has to have five propagators and a three-loop family has to have nine propagators. The propagators of the QGRAF output carry an arbitrary momentum assignment, so that momentum is conserved

at every vertex. For further calculations it is useful to shift the momenta in all diagrams at a certain loop order to list of conveniently chosen momenta. Every momentum of this list can then be associated to a line of a Feynman graph. For self-energies up to three-loop there are four topologies which exhaust all possible momentum assignments. Let M_{N_k} be the set of chosen momenta of a N_k-loop self-energy family. The nth entry of this set is defined to be the momentum carried by the nth line of the associated graph.

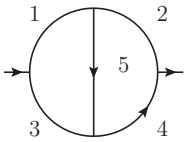
At one-Loop there are at most two linearly independent momenta belonging to the set M_1 in Fig. 5.4.



$$M_1 = \{k_1,\, k_1 - k_q\}$$

Figure 5.4.: One-loop topology and the momenta

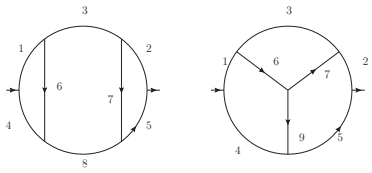
In the two-Loop case there are five independent momenta belonging to the set M_2 associated to each line in Fig. 5.5.



$$M_2 = \{k_1,\, k_2,\, k_1 - k_q,\, k_2 - k_q,\, k_1 - k_2\}$$

Figure 5.5.: Two-loop topology and the momenta

At three-Loop things get a bit more complicated. It is not possible to map all momenta onto one single diagram. But there are at most nine momenta belonging to M_3 appearing in an integral. These can be mapped onto the two topologies in Fig. 5.6.



$$M_3 = \{k_1,\, k_2,\, k_3,\, k_1 - k_q,\, k_2 - k_q,\, k_1 - k_3,\, k_2 - k_3,\, k_3 - k_q,\, k_1 - k_2\}$$

Figure 5.6.: Three-loop topologies and the momenta

The momentum list has always been chosen in such a way, that there are at most two different momenta in each linear combination in the propagator. This makes it later easier to cancel the scalar products. The momentum shifts are done in a given`FORM` setup.

5.2.3. Feynman Rules

After all diagrams got the right momentum shift one has to explain the computer what the symbols `vrtx` and `prop` actually mean in an explicit mathematical sense. So one has to plug in the Feynman Rules. The Feynman rules are generated automatically by a given `FORM` program and the symbols `vrtx` and `prop` get then replaced by these rules. For example the `FORM` output for the 4-vertex with one gauge boson and three scalar reads

```
al vrtx(gl(a1?,m1?,p1?),sc(a2?,p2?),sc(a3?,p3?),sc(a4?,p4?))=
+tr(a1,a2,a3,a4)*(p3(m1))*(-si*sg^2*sM^-1)
+tr(a1,a2,a4,a3)*(p4(m1))*(-si*sg^2*sM^-1)
+tr(a1,a3,a2,a4)*(p2(m1))*(-si*sg^2*sM^-1)
+tr(a1,a3,a4,a2)*(p4(m1))*(-si*sg^2*sM^-1)
+tr(a1,a4,a2,a3)*(p2(m1))*(-si*sg^2*sM^-1)
+tr(a1,a4,a3,a2)*(p3(m1))*(-si*sg^2*sM^-1)
```

where `tr(a1,a2,a3,a4)` means $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$ and `p3(m1)` is a momentum with Minkowski index `m1`. The other symbols `si,sg,sM` are the imaginary unit, the couping and the mass respectively.

5.2.4. Lorentz and Color projection

All self-energy diagrams in the three dimensional Euclidean Yang-Mills theory carry a lot of Lorentz and color structure. The computer algebra language `FORM` is specialized to deal with Lorentz and Dirac structure. For example it knows the 4-vector formalism and sum convention even in d dimensions. So one can simply apply the projector $\frac{1}{d-1}P_{T,\mu\nu}$ defined in (4.18) to obtain Π_T.

Since `FORM` can deal with symbolic manipulations very well, one can simply manipulate the color traces appearing in the Feynman rules with the completeness relation written in eq.(4.17).

Finally only Π_T remains as a linear combination of scalar self-energy integrals of the type of eq.(5.2) with rational functions in d and ξ as coefficients.

The color and Lorentz projection and the application of the completeness relation are done in a given `FORM` program.

5.2.5. Self-energy scalar integrals

In the case of the self-energy integrals ($N_p = 1$) it is very simple to write scalar products in terms of a sum of squared momenta by means of the polarization identities

$$k_i \cdot k_j = \frac{1}{2}\left(k_i^2 + k_j^2 - (k_i - k_j)^2\right) \tag{5.99}$$

$$k_i \cdot k_q = \frac{1}{2}\left(k_i^2 + k_q^2 - (k_i - k_q)^2\right). \tag{5.100}$$

These momenta belong to the list of linear combinations in M_{N_k} for the four topologies in Fig. 5.4 - Fig. 5.6. Thus each scalar product can be written as a sum of massless propagators with negative power.

Parallel to the computation of $\hat{\Pi}_{00}^{(n)}(p^2, m)$ we applied the mass and momentum derivatives to get$\Pi_{ab}^{(n)}$ for $a, b > 0$, defined in eq.(4.52), which are needed for the gap equation as well. Afterwards we worked with on-shell external momentum $p^2 = -m^2$ and used dimensionless quantities $\hat{\Pi}_{ab}^{(n)}(d, \xi, N_c)$ defined by [21]

$$\Pi_{ab}^{(n)} = \left(m^2\right)^{1-a-b} \left[\frac{g^2 N_c J(1, d, m)}{m^2(1-d)}\right]^n \hat{\Pi}_{ab}^{(n)}(d, \xi, N_c), \tag{5.101}$$

where $J(1, d, m)$ is the one-loop tadpole integral defined in eq.(2.28). The resulting functions $\hat{\Pi}_{ab}^{(n)}(d, \xi, N_c)$ are then a sum of rational functions in d and ξ times dimensionless on-shell integrals

$$\hat{I}(a_1, ..., a_{N_{sp}}, sm_1, ..., sm_{N_{sp}}) \equiv \left(\frac{1}{J(1, d, 1)}\right)^{N_k} \left(\prod_{n=1}^{N_k} \int \frac{d^d k_n}{(2\pi)^d}\right) \prod_{j=1}^{N_{sp}} \frac{1}{(q_j^2 + sm_j)^{a_j}}|_{p^2=-1}, \tag{5.102}$$

where $sm_i \in \{0, 1, \xi\}$,$a_i \in \mathbb{Z}$ and $q_i \in M_{N_k}$. Propagators with negative indices a_j always carry mass $sm_j = 0$ and propagators with positive indices carry mass 1 or ξ. Later these integrals will also be represented as Feynman graphs. A two-loop example for this graphical notation is given by

$$= \hat{I}(1, 1, 1, 1, 1, \xi, 1, \xi, 1, \xi), \tag{5.103}$$

where thin line carry $sm = 1$ and thick lines carry $sm = \xi$.

We apply the sector shifts to every integral, so that as much integrals as possible of them are replaced by integrals with a lower sector ID. Finally we are left with large number of scalar integrals of the type of eq.(5.102) with many combination of positive and negative indices a_i

and masses sm_j. At three-loop we end up with around 30000 of these integrals and we used the Laporta algorithm implemented in `Reduze` to reduce this large set of integrals to master-integrals. This is part of the next section.

5.2.6. Three-loop reduction with Reduze 2.0.9

`Reduze 2` is a program written in C++ for distributed Feynman integral reduction [39]. It supports `MPI` and computer algebraic tools like `GiNac` [47] and `Fermat` [48]. Let us briefly discuss the settings for this reduction.

At first one has to specify the kinematics in a file `kinematics.yaml`. This file contains the external momenta and kinematic invariants like masses and Mandelstam variables [40]. For example for two-point integrals our file reads

```
kinematics :
incoming_momenta: [p1]
outgoing_momenta: [p2]
momentum_conservation: [p1,p2]
kinematic_invariants:
- [ xi, 0]
scalarproduct_rules:
- [ [ p1,p1 ], 1 ]
```

Note that the external momentum in `Reduze` has to be set to 1 and not to -1. `Reduze` works intrinsically Minkowskian. In the end of the reduction each integral can be multiplied with $(-1)^{r+s}$ by setting the option `toggle_metric_convention` in order to get an Euclidean result.

The integral families have to be listed in a file called `integralfamilies.yaml`. The loop momenta, propagators, and possible permutation symmetries in the indices, have to be specified in this file [40]. According to conventions for the self-energy integrals a typical definition of a integral family may look like:

```
integralfamilies:
- name: fm222222010
loop_momenta: [k1,k2,k3]
propagators:
- [ "k1", "xi" ]
- [ "k2", "xi" ]
- [ "k3", "xi" ]
- [ "k1-q1", "xi" ]
- [ "k2-q1", "xi" ]
- [ "k1-k3", "xi" ]
- [ "k2-k3", 0 ]
- [ "k3-q1", 1 ]
- [ "k1-k2", 0 ]
```

```
permutation_symmetries:
- [ [ 1, 4 ], [ 2, 5 ], [ 3, 8 ] ]
- [ [ 1, 3 ], [ 4, 8 ], [ 7, 9 ] ]
- [ [ 1, 8 ], [ 2, 5 ], [ 3, 4 ], [ 7, 9 ] ]
```

The family name `fm222222010` was chosen according to the mass arrangement $\{\xi,\xi,\xi,\xi,\xi,\xi,0,1,0\}$, where the 2 means ξ. Altogether there appear 390 different mass assignments in the $\hat{\Pi}^{(3)}_{00}$ calculation. For each of them an integral family has been defined in `Reduze`. Due to the fact that zero mass propagators have always negative indices, some of these families contain only integrals that are actually subsectors of other families. But `Reduze` is able to identify these as subsectors, so that no more reductions than necessary have to be done.

The integrals that shall be reduced, are listed in a file `myintegrals` with the format [40]

```
familyname t ID r s E1 E2 ...E9,
```

where `t` is the number of denominators, `ID` is the sector identification number, `r` is the sum of all powers of the denominators, `s` is the sum of all powers of scalar products and `E1,...,E9` are the exponents of the propagators. For example an integral could look like

```
fm211002100 5 103 6 2 1 1 2 0 -1 1 1 0 -1 .
```

The reduction has to be organized in a job file. The file `job_reduction.yaml` could look like [40]

```
max_parallel_jobs: 4
jobs:
- setup_sector_mappings:
conditional: true
- reduce_sectors:
conditional: false
sector_selection:
select_recursively: [ [ fm100011100, 113 ], [ fm121200000, 15 ] ]
alternative_input_directory:
"reductions_r8_s4"
identities:
ibp:
- { r: [t, 8], s: [0, 6] }
lorentz: # may help to reduce certain integrals at border of seed range
- { r: [t, 8], s: [0, 6] }
sector_symmetries: # sometimes important, doesn't harm
- { r: [t, t], s: [0, 1] }
- select_reductions:
```

```
input_file: "myintegrals"
output_file: "myintegrals.tmp"
- reduce_files:
equation_files: ["myintegrals.tmp"]
output_file: "myintegrals.sol"
- export:
input_file: "myintegrals.sol"
output_file: "myintegrals.sol.inc"
output_format: "form"
toggle_metric_convention: "yes"
```

The job `setup_sector_mappings` applies the sectors shifts discussed in section 5.1.3. With `select_recursively` one can specify the families and their sectors which will be considered for the reduction. In the example above the sector `113` of the family `fm100011100` and the sector `15` of the family `fm121200000` will be reduced. These are already the largest sectors, which can appear in these families, due to the fact that zero masses have negative propagators.

The job `reduce_sectors` produces all identities for all seed-integral with specified values r and s. In the above example all identities are generated for $r \in [t, 8]$ and $s \in [0, 6]$.

`Reduze` provides integration-by-parts identities, Lorentz identities and sector symmetries. For an integral with external momenta $p_1, ..., I_{N_p}$, Lorentz identities [44] can be derived from the antisymmetric matrix

$$\sum_{n=1}^{N_p} \left(p_{n,\nu} \frac{\partial}{\partial p_n^\mu} - p_{n,\mu} \frac{\partial}{\partial p_n^\nu} \right) I(p_1, ..., p_{N_p}) = 0, \tag{5.104}$$

with $N_p(N_p - 1)/2$ independent components, by contracting eq.(5.104) with $p_{i,\nu} p_{j,\nu}$. In the case for two-point function, where only one external momentum exists, these identities are trivial though.

Let us analyze the complexity of the three-loop reduction. In table 6.1 there are some extreme integrals with the highest values for r and s appearing in the calculation. According to Laporta's *golden rule* we listed the number of integration-by-parts identities N_{ide} and the number of seeds, which would be needed for the reduction of each integral.

Integral	family	N_d	r	$a = s$	$b = r - N_d$	N_{seeds}	N_{ide}
$\hat{I}(1,1,1,1,1,1,1,-3,1)$	fm111111101	8	8	3	0	11 228	134 736
$\hat{I}(1,1,-1,1,1,1,1,-2,1)$	fm110111101	7	7	3	0	5 785	69 420
$\hat{I}(1,1,1,1,1,-2,-2,1,-1,)$	fm222220010	6	6	5	0	13 832	165 984
$\hat{I}(1,1,-1,-1,-2,1,1,1,-2)$	fm220002220	5	5	6	0	11 760	141 120
$\hat{I}(3,1,1,1,0,1,0,-2,-1)$	fm121101000	5	7	3	2	13 335	160 020
$\hat{I}(2,1,0,-4,0,1,0,1,-2)$	fm210002020	4	5	6	1	17 094	205 128
$\hat{I}(2,1,1,2,0,0,0,-2,-1)$	fm111100000	4	6	3	2	8 652	103 824
$\hat{I}(1,1,1,0,0,0,-2,-1,-2)$	fm221000000	3	3	4	0	210	2 520
$\hat{I}(2,2,1,0,-2,-1,0,0,0)$	fm111000000	3	5	3	2	840	10 080

Table 5.1.: Some integrals with extreme values for r and s. The values b is mostly very low and does never exceed $b = 2$, whereas the value a becomes even 6. The notation is $\hat{I}(s_1,...s_9) = \hat{I}(s_1,...,s_9,sm_1,...,sm_9)$ (compare eq.(5.102)), so that masses are omitted in the integrals argument.

The maximal values that appear are $r = 8$ and $s = 6$. The simplest setup for the reduction would be to list all 390 families in `select_recursively` and to produce all identities for seeds with $r \leq 8$ and $s \leq 6$ for all these families. In table 6.2 is listed how many seeds and identities have to be generated for this setup for a sector with N_d propagators.

N_d	$b = 8 - N_d$	N_{seed}	N_{ide}
8	0	98 427	1 181 124
7	1	241 010	2 892 120
6	2	317 562	3 810 744
5	3	277 410	3 328 920
4	4	161 700	1 940 400
3	5	51 744	620 928

Table 5.2.: Number of seeds and identities for $r = 8$ and $s = 6$ fixed for all sectors for a single family

Comparing table 5.1 and 5.2 it becomes clear, that in such a setup much more identities have to be generated. And in fact it is very time and memory consuming to generate all identities for all 390 families and their subsectors for the maximum values $r = 8$ and $s = 6$. Especially the negative exponents are problematic, because they lead to large terms in the numerator. In many cases the generation of identities is rather fast ($\leq$ 6000 seconds CPU-time), but for many $t = 5$ and $t = 6$ sectors it takes around 50 000 seconds CPU-time[8]. The reduction can then even take more than 1 million seconds CPU-time for $t = 5$ and $t = 6$ sectors. Furthermore 48 GB ram are not enough for this setup, which leads to a lot of crashes during the reduction.

On the other hand it turned out to be a rather fast and much less memory consuming method

[8]We worked on a system with Intel® Xeon® processors (2.8 GHz and 12MB cache) with 48 GB RAM.

to treat every integral family separately. For every family determine the highest values for r and s and generate the corresponding seeds and identities. Using such a setup it takes at most 24 hours to reduce a complete integral family to master-integrals. In most cases it took about two hours to reduce an integral family. Altogether it took 2 weeks to reduce all families to master-integral using this alternative setup. The disadvantage of this setup is that `Reduze` does not chose the same master-integral basis for all reduction. After all reductions had been plugged into $\hat{\Pi}^{(3)}_{00}$ it was necessary to reduce all appearing master-integrals to the same basis. But such a reduction ($r \leq 8$ and $s = 0$) is very fast. The resulting three-loop master-integral can be found in appendix B.

As a check the reduction was additionally done in another setup, where all identities are generated for all 390 families for seed integrals with $r \leq 8$ and $s \leq 5$. After the reduction has been finished, the remaining integrals with $s = 6$ have been reduced separately and brought to the same basis with the $s \leq 5$ reduction. This method yielded a different result and it is not yet clear why. This second result has much larger polynomials and also some further master-integrals (also listed in appendix B). One could argue, that this reduction lead only to an disadvantageous basis and could further be simplified. As a check the remaining master-integrals have been further reduced with $r \leq 9$ and $s \leq 1$ leading to no significant difference.

As long as the reason for the difference between both results is not found there is no reliable result for $\hat{\Pi}^{(3)}_{00}$.

`Reduze` is actually optimized for two-loop calculations and indeed the reductions for the one and two-loop self-energies and their derivatives were successful. To be specific we applied `Reduze` for $\hat{\Pi}^{(1)}_{00}$, $\hat{\Pi}^{(1)}_{01}$, $\hat{\Pi}^{(1)}_{02}$, $\hat{\Pi}^{(1)}_{03}$, $\hat{\Pi}^{(1)}_{10}$, $\hat{\Pi}^{(1)}_{11}$, $\hat{\Pi}^{(1)}_{12}$, $\hat{\Pi}^{(2)}_{00}$ and $\hat{\Pi}^{(2)}_{01}$ and checked them by using the dimensional relation eq.(4.46). For the results see appendix B. The ability to manage three-loop reductions as well has to be seen as an additional feature, with some restrictions though.

There are some very important options which are useful for very complicated computations.

- When the option `conditional` is set to `true`, then the corresponding job will be skipped, if it has already been done. This will also happen if the job has been done for some other settings. This option is useful if the program has crashed for some reason.
- The option `alternative_input_directory` can be used to use some old reduction. This may improve the reduction speed.
- The option `max_parallel_jobs` restricts the number of reductions at the same time. This may be useful to avoid memory problems.

6. Solutions of the gap equations and gauge dependence

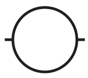
The main focus of this section is to analyze the gauge dependence of the gap equation and to give a review of the one- and two-loop results. The individual results of the $\Pi_{ab}^{(n)}$ in terms of master-integrals and the solutions of the one- and two-loop master-integrals can be found in appendix B. In the following integrals are represented as Feynman graphs. Normal lines carry mass 1 and thick lines carry mass ξ (c.f. eq.(5.103)).

6.1. One-loop gap equation and solutions

Let's recall the one-loop gap equation (c.f. eq.(4.47))

$$0 = m^2 \left\{1 + g^2 A\right\}, \tag{6.1}$$

where $A = \left[\frac{N_c J(1,d,m)}{m^2(1-d)}\right] \hat{\Pi}_{00}^{(1)}$. In terms of master-integrals eq.(B.1) this equation reads

$$0 = m^2 \left(1 + \left[\frac{g^2 N_c J(1,d,m)}{m(1-d)}\right] \left(\frac{9}{8}(4d-5)\right.\right.$$  $$+ \frac{1}{4}(2d-3)(2d-5)$$  $$\left.\left.\right)\right). \tag{6.2}$$

Using the ε expansion (appendix B.1.2) for $d = 3 - 2\varepsilon$ leads to the equation

$$0 = m^2 \left(1 - \frac{g^2 N_c}{8\pi m}\left(+\frac{63}{16}\ln 3 - \frac{3}{4}\right)\right), \tag{6.3}$$

which yields a trivial solution $m = 0$ and a nontrivial solution

$$m_{1-\text{loop}} = \frac{g^2 N_c}{8\pi}\left(+\frac{63}{16}\ln 3 - \frac{3}{4}\right) \approx 0.142276 g^2 N_c. \tag{6.4}$$

This manifestly gauge invariant result was first derived by [19] for a $SU(2)$ theory and later confirmed in [22] for this generalized $SU(N)$ model.

6.2. Two-loop gap equation and solutions

Let's recall the two-loop gap equation (c.f. eq.(4.48))

$$0 = m^2 \left\{1 + g^2 \frac{4-d}{2} A + g^4 \left(\frac{2-d}{2} A^2 + B\right)\right\}. \tag{6.5}$$

The g^4 part of this equation consists of the ξ-independent part A and the part $B = \left(\frac{NJ(1,d,m)}{m^2(1-d)}\right)^2 \left(\hat{\Pi}_{00}^{(2)} + \hat{\Pi}_{00}\hat{\Pi}_{01}^{(1)}\right)$ (c.f. eq.(4.54)). If the gap equation is gauge invariant, B has to be invariant as well. The result

$$\begin{aligned}
\hat{\Pi}_{00}^{(2)} + \hat{\Pi}_{00}\hat{\Pi}_{01}^{(1)} \;=\; & -\frac{(-2+d)(-3+2d)(301-665d+538d^2-188d^3+24d^4)}{32(-4+3d)}\Big(\;\underline{\bigcirc}\;\Big)^2 \\
& -\frac{9(-2+d)(-165+258d-114d^2+8d^3)}{32}\;\bigcirc\!\!-\!\!\underline{\bigcirc} \\
& +\frac{3(-295+797d-616d^2+144d^3)}{128}\Big(\bigcirc\!\!-\Big)^2 \\
& -\frac{(61752-166654d+165227d^2-70632d^3+10800d^4)}{192(-4+3d)}\;\ominus \\
& -\frac{3(-2+d)(-405+656d-312d^2+32d^3)}{64}\;\oslash \\
& -\frac{3(-760+1241d-712d^2+144d^3)}{64}\;\bigcirc\!\!\!\!\diagup \\
& +\frac{3(d-1)(-245+176d)}{64}\;\oslash
\end{aligned} \tag{6.6}$$

in terms of master-integrals is indeed manifestly gauge invariant. Using the three dimensional ε expansion for the master-integrals from appendix B.1.2 and B.2.2 yields the divergent result

$$\hat{\Pi}_{00}^{(2)} + \hat{\Pi}_{00}^{(1)}\hat{\Pi}_{01}^{(1)} = \frac{3}{20\varepsilon} - 4.25206 + \mathcal{O}(\varepsilon).$$

Using eq.(B.30), leads to the renormalization scale and scheme dependent gap equation

$$\begin{aligned}
0 =& m^2 + m\left(\frac{g^2N_c}{8\pi}\right)\frac{1}{2}\left(\frac{3}{4} - \frac{63}{16}\ln(3)\right) \\
& + \left(\frac{g^2N_c}{8\pi}\right)^2\left(\frac{3}{20\varepsilon} - 4.25206 + \frac{9}{2560}\left(-176 - 840\ln(3) + 2205\ln(3)^2\right) + \frac{3}{10}\ln\left(\frac{\bar{\mu}^2}{4m^2}\right)\right).
\end{aligned} \tag{6.7}$$

Working in the MS renormalization scheme, setting $m = Kg^2N_c$ and choosing $\frac{\bar{\mu}}{g^2N_C} = 0.1$ leads to the two solutions

$$K_1 = 0.1692 \tag{6.8}$$

$$K_2 = 4.4 \cdot 10^{-9}, \tag{6.9}$$

in agreement with [22]. Furthermore setting the renormalization scale $\mu^2 = m^2 = K^2 g^4 N_c^2$ yields the polynomial equation

$$0 = K^2 - 0.071138K - 0.0151659 \tag{6.10}$$

with a positive solution

$$K_1 = 0.1634 \tag{6.11}$$

and an unphysical negative solution

$$K_2 = -0.0926, \tag{6.12}$$

also in agreement with [22]. The scale dependence of m is very weak, for details see [22].

The two-loop calculation for the magnetic mass was also performed for $N_c = 2$ by [23] with a gauge dependent result due to some missing counter-terms. Furthermore it is in good agreement with lattice simulations performed by Karsch et al. for an $SU(2)$ theory with a result $m_{\text{lattice}} = 0.35(1)g^2$ [49]

6.3. Three-loop gap equation and gauge dependence in higher orders

When we continue the gauge invariance argument for the gap equation in higher orders[1], it is easy to see that the term $C = \left[\frac{N_c J(1,d,m)}{m^2(1-d)}\right]\left((d-4)\hat{\Pi}_{01}^{(1)} - \hat{\Pi}_{02}^{(1)}\right)$ in the g^2 of the three-loop gap equation (c.f. eq.(4.49)) has to be gauge invariant as well. Unfortunately the result

$$\begin{aligned}(d-4)\hat{\Pi}_{01}^{(1)} - \hat{\Pi}_{02}^{(1)} &= \frac{1}{4}(28 - 57d + 39d^2 - 11d^3 + d^4 + 2\xi)\\ &+ \frac{1}{2}(-1 + d - \xi)\\ &+ \frac{1}{4}(-43 + 62d - 29d^2 + 4d^3)\\ &+ \frac{1}{2}(4 - 4d + 4\xi - \xi^2)\\ &+ \frac{1}{4}(-1 + d)\end{aligned} \tag{6.13}$$

is manifestly ξ-dependent. Not only the polynomial prefactors of the master-integrals are gauge dependent, but also the master-integrals themselves. Furthermore there is no chance to find any gauge invariant linear combination of $\hat{\Pi}_{01}$ and $\hat{\Pi}_{02}$ using ξ-independent coefficients, because the

[1] The following discussion is partly from [51]

ξ-dependent integral  appears in $\hat{\Pi}^{(1)}_{02}$ but not in $\hat{\Pi}^{(1)}_{01}$.

Additionally one could try to find a linear combination of mass and momentum derivatives

$$\alpha_1\hat{\Pi}^{(1)}_{11} + \alpha_2\hat{\Pi}^{(1)}_{02} + \alpha_3\hat{\Pi}^{(1)}_{10} + \alpha_4\hat{\Pi}^{(1)}_{01} + \alpha_5\hat{\Pi}^{(1)}_{00} \tag{6.14}$$

where the ξ-independent coefficients α_1,...,α_5 have to be determined. Using a generalized dimensional relation in eq.(4.46) for the one-loop contribution

$$\hat{\Pi}^{(1)}_{1,n-1} = \hat{\Pi}^{(1)}_{0n} + \frac{2n-d}{2}\hat{\Pi}^{(1)}_{0,n-1} \tag{6.15}$$

leads to

$$(\alpha_1+\alpha_2)\hat{\Pi}^{(1)}_{02} + \left(\alpha_1\frac{4-d}{2} + \alpha_3 + \alpha_4\right)\hat{\Pi}^{(1)}_{01} + \left(\alpha_3\frac{2-d}{2} + \alpha_5\right)\hat{\Pi}^{(1)}_{00}. \tag{6.16}$$

This expression can only be ξ-independent if $\alpha_1 = -a_2$ holds, such that the ξ-dependent master-integral  is canceled. However this is not given by the gap equation. This problem continues in higher orders including higher derivatives. Assuming the g^2 part of the four-loop gap equation (c.f. eq.(4.51)) to be gauge invariant leads to the conclusion that $E - 6C$ must be gauge invariant. Since both $\hat{\Pi}_{02}$ and $\hat{\Pi}_{03}$ contain the integral  there should be a combination

$$\hat{\Pi}_{02}(\beta_1 d + \beta_2) + \hat{\Pi}_{03}, \tag{6.17}$$

so that  cancels out. But it is not possible to keep β_1 and β_2 ξ-independent then. So there can't be any gauge invariant combinations including $\hat{\Pi}_{02}$ and $\hat{\Pi}_{03}$. The possibility to express the gap equation in terms of massive derivatives only and the inevitable appearance of $\hat{\Pi}_{02}$ and $\hat{\Pi}_{03}$ in the gap equation in higher orders implies the gap equation can't be gauge invariant. In order to make sure that the derivatives have been calculated correctly the dimensional relations in eq.(6.15) have been checked for all $\Pi^{(1)}_{0n}$'s up to $n = 3$.

The gauge parameter dependence becomes also unavoidable in the g^6 part of the three-loop gap equation. Many three-loop master-integrals[2] for $\hat{\Pi}^{(3)}_{00}$ (see appendix B) contain the gauge parameter ξ. For example

$$I(1,0,0,0,1,1,1,0,0,1,0,0,0,1,2,1,0,0) = \text{  }.$$

There is obviously no possibility for such integrals to be canceled by other integrals coming from $\hat{\Pi}^{(n\leq 2)}_{ab}$.

[2]As stated in the last chapter the three-loop reduction is not yet reliable. But both reduction contain such gauge parameter dependent integrals. It is therefore expectable that these integrals are really part of the true reduction.

7. Conclusion

The one- and two-loop computations have successfully been repeated and the reductions for $\hat{\Pi}_{03}^{(1)}$, $\hat{\Pi}_{12}^{(1)}$ and $\hat{\Pi}_{01}^{(2)}$ lead to reliable new results. The gauge dependence of the three-loop gap equation is an unsolved issue. One possible reason could be, that the pole of the transverse gauge boson propagator is actually not a gauge invariant quantity and therefore unphysical. Maybe a different definition for the gauge boson mass has to be used. The dimensional relation check (see eq.(6.15)) indicates no error in the computation and the fact that procedure A and B lead to the same result at one- and two-loop confirm that the computation is correct.

The three-loop master-integrals for the three-loop self-energy in both reductions are themselves gauge-invariant. It would be an interesting check whether or not these integrals appear also in procedure A of [22], where the self-energies are computed in a resummed theory with counter terms $\propto l$. But at first it is important to find a reliable reduction for the three-loop self-energy. Thus an important future project is to find a bug in the three-loop computation.

Another future project could be to analyze the gauge invariance of this theory by computing the pressure p, which should be gauge invariant.

The gauge dependent two-loop result in [23] showed only a weak ξ-dependence and has also given a reasonable numerical result. After all three-loop master-integrals have been evaluated, one could check whether or not the three-loop result is also weakly dependent on the gauge parameter. Then it could be possible to get further estimates of the convergence of the perturbative expansion.

The method to solve master-integrals by difference equation has successfully applied to sunset vacuum bubbles. In the future it should be applicable in an automatized way also for the three-loop self-energy integrals.

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Finally I'd like to thank again Hiromichi, Sayantan, Thomas and Viktor for proofreading my thesis.

Erklärung

Hiermit versichere ich, die vorliegende Masterarbeit selbstständig und nur unter Benutzung der im Literaturverzeichnis angegebenen Quellen erstellt zu haben.

Marc Sangel

A. Details to the solution of sunsets integrals

We used the r-th order difference equations for the $S(n,d,x)$

$$\sum_{j=0}^{r_n} p_{n,j}(d,x)\frac{\Gamma(x+j)}{\Gamma(x+1)}S(n,d,x+j) = c_n\Gamma(n+1)S(1,d,x) \tag{A.1}$$

with polynomials p and constants c, which read [37]

$n=1:\ r_1=1,\ p_{1,0}=x(-2+d-2x),\ p_{1,1}=2(x+1),\ c_1=0.$

$n=2:\ r_2=2,\ p_{2,0}=x(-2+d-x),\ p_{2,1}=-3+d-2x,\ p_{2,2}=3,\ c_2=-2.$

$n=3:\ r_3=2,\ p_{3,0}=x(-6+3d-2x)(-2+d-x),$
$p_{3,1}=2(24-17d+3d^2+27x-10dx+7x^2),\ p_{3,2}=16(-3+d-x),\ c_3=12.$

$n=4:\ r_4=4,\ p_{4,0}=x(4-2d+x)(2-d+x)(6-3d+2x),$
$p_{4,1}=360-399d+147d^2-18d^3+526x-405dx+78d^2x+234x^2-93dx^2+32x^3,$
$p_{4,2}=-144+129d-27d^2-2x+21dx+20x^2,\ p_{4,3}=9(-42+9d-16x),\ p_{4,4}=90,\ c_4=2.$

$n=5:\ r_5=4,\ p_{5,0}=x(-6+3d-2x)(-10+5d-2x)(-2+d-x)(-4+2d-x),$
$p_{5,1}=2(24-17d+3d^2+24x-10dx+4x^2)(120-98d+20d^2+87x-36dx+15x^2),$
$p_{5,2}=-4(-1248+932d-196d^2+8d^3-2130x+1180dx-148d^2x-975x^2+294dx^2-129x^3),$
$p_{5,3}=-128(150-83d+11d^2+85x-26dx+11x^2),\ p_{5,4}=-768(-6+2d-x),\ c_5=4.$

For μ and K we obtained for the homogeneous equations

$n=1:\ \mu=1,\ K=-d/2.$

$n=2:\ \mu_1=1,\ \mu_2=-1/3,\ K_1=K_2=(1-d)/2.$

$n=3:\ \mu_1=1,\ \mu_2=-1/8.\ K_1=-d/2,\ K_2=1-d$

$n=4:\ \mu_1=1,\ \mu_2=1,\ \mu_3=-1/3,\ \mu_4=-1/15,\ K_1=-d/2,\ K_2=1-d.$

$n=5:\ \mu_1=1,\ \mu_2=1,\ \mu_3=-1/8,\ \mu_4=-1/24,\ K_1=-d/2,\ K_2=(-2-d)/2.$

The recurrence relations for the inhomogeneous part (using $K=-d/2$ and $\mu=1$)

$n=1: +a_{s-1}(2-4s+2s^2-2d+2ds+1/2d^2)+a_s(-2s)=0.$

$$
\begin{aligned}
n=2 \quad : \quad & +a_{s-2}(6-15s+12s^2-3s^3-15/2d+12ds-9/2ds^2+3d^2-9/4d^2s-3/8d^3) \\
& +a_{s-1}(3-10s+7s^2-4d+6ds+5/4d^2) \\
& +a_s(-2-4s) \\
& +2c_s = 0
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
n=3 \quad : \quad & -a_{s-3}\cdot 4\cdot((-4+d+2s)^2(3d^2+d(-26+8s)+4(12-7s+s^2)) \\
& +a_{s-2}(41d^3+d^2(-438+238s)+4d(384-404s+103s^2)+8(-220+332s-161s^2+25s^3)) \\
& +a_{s-1}(-80+144s-52s^2+45d-55ds-11/2d^2) \\
& +a_s(18s) \\
& -12c_{s-1} = 0
\end{aligned} \tag{A.3}
$$

$$
\begin{aligned}
n=4 \quad : \quad & +a_{s-5}(259200-613440s+610920s^2-331920s^3+106290s^4 \\
& -20070s^5+2070s^6-90s^7-306720d+610920ds-497880ds^2+212580ds^3 \\
& -50175ds^4+6210ds^5-315ds^6+152730d^2-248940d^2s+159435d^2s^2 \\
& -50175d^2s^3+15525/2d^2s^4-945/2d^2s^5-41490d^3+53145d^3s \\
& -50175/2d^3s^2+5175d^3s^3-1575/4d^3s+53145/8d^4-50175/8d^4s^4 \\
& +15525/8d^4s^2-1575/8d^4s^3-10035/16d^5+3105/8d^5s-945/16d^5s^2 \\
& +1035/32d^6-315/32d^6s-45/64d^7) \\
& +a_{s-4}(2989152-7454736s+7663448s^2-4144332s^3+1238660s^4 \\
& -192996s^5+12164s^6-3951912d+8405592ds-7099386ds^2 \\
& +2969728ds^3-613634ds^4+49956ds^5+2117590d^2-3687969d^2s \\
& +2394650d^2s^2-685530d^2s^3+72827d^2s^4-1175127/2d^3 \\
& +787196d^3s-349313d^3s^2+51206d^3s^3+354941/4d^4-326685/4d^4s \\
& +74439/4d^4s^2-55021/8d^5+13153/4d^5s+3389/16d^6) \\
& +a_{s-3}(463296-1223232s+1286928s^2-669456s^3+170256s^4 \\
& -16656s^5-521128d+1181424ds-1003618ds^2+375910ds^3-51948ds^4 \\
& +220740d^2-411418d^2s+254003d^2s^2-51514d^2s^3-85149/2d^3 \\
& +121333/2d^3s-20881d^3s^2+13681/4d^4-6269/2d^4s-60d^5) \\
& +a_{s-2}(33456-109632s+134728s^2-70896s^3+12824s^4-26716d \\
& +81778ds-84274ds^2+26988ds^3+6088d^2-18736d^2s+13699d^2s^2-213/2d^3 \\
& +1275d^3s-261/4d^4) \\
& +a_{s-1}(-1840s+9408s^2-5264s^3-306ds-5604ds^2-36d^2s) \\
& +a(s)(900s+900s^2) \\
& -48c_{s-1}=0. \qquad \text{(A.4)}
\end{aligned}
$$

$$
\begin{aligned}
n=5 \quad : \quad & +a_{s-6}(132710400-246620160s+197922816s^2-89571840s^3 \\
& +24994560s^4-4402176s^5+477696s^6-29184s^7+768s^8-156487680d \\
& +255430656ds-176649984ds^2+67095552ds^3-15115776ds^4 \\
& +2019840ds^5-148224ds^6+4608ds^7+78234624d^2-109471104d^2s \\
& +63151488d^2s^2-19226112d^2s^3+3258240d^2s^4-291456d^2s^5 \\
& +10752d^2s^6-21769536d^3+25327104d^3s-11668224d^3s^2+2661120d^3s^3 \\
& -300480d^3s^4+13440d^3s^5+3700464d^4-3430848d^4s+1181280d^4s^2 \\
& -179040d^4s^3+10080d^4s^4-394464d^5+272928d^5s-62352d^5s^2 \\
& +4704d^5s^3+25800d^6-11832d^6s+1344d^6s^2-948d^7+216d^7s+15d^8) \\
& +a_{s-5}(83681280-150030336s+114100736s^2-47682944s^3+11816192s^4 \\
& -1734656s^5+139520s^6-4736s^7-91327488d+141739520ds-90905280ds^2 \\
& +30829184ds^3-5828736ds^4+582272ds^5-24000ds^6+41330816d^2 \\
& -53757600d^2s+27769024d^2s^2-7119488d^2s^3+905728d^2s^4-45728d^2s^5 \\
& -10112848d^3+10555232d^3s-4104448d^3s^2+704576d^3s^3-45040d^3s^4 \\
& +1450496d^4-1137120d^4s+295312d^4s^2-25400d^4s^3-122296d^5 \\
& +63944d^5s-8308d^5s^2+5624d^6-1470d^6s-109d^7) \\
& +a_{s-4}(23695488-41207232s+29640824s^2-11272212s^3+2386100s^4 \\
& -265980s^5+12164s^6-23090208d+34217864ds-20189934ds^2+5923824ds^3 \\
& -863414ds^4+49956ds^5+8958566d^2-10825167d^2s+4888202d^2s^2 \\
& -976838d^2s^3+72827d^2s^4-3550557/2d^3+1639440d^3s-502931d^3s^2 \\
& +51206d^3s^3+756065/4d^4-475563/4d^4s \\
& +74439/4d^4s^2-81327/8d^5+13153/4d^5s+3389/16d^6) \\
& +a_{s-3}(3829824-6569760s+4483392s^2-1517040s^3+253536s^4-16656s^5 \\
& -3134028d+4524182ds-2443036ds^2+583702ds^3-51948ds^4+937675d^2 \\
& -1073966d^2s+408545d^2s^2-51514d^2s^3-124122d^3+204857/2d^3s \\
& -20881d^3s^2+26219/4d^4-6269/2d^4s-60d^5) \\
& +a_{s-2}(361536-643072s+424360s^2-122192s^3+12824s^4-219756d \\
& +331290ds-165238ds^2+26988ds^3+38523d^2 \\
& -46134d^2s+13699d^2s^2-2763/2d^3+1275d^3s-261/4d^4) \\
& +a_{s-1}(16512-36448s+25200s^2-5264s^3-5298d \\
& +10902ds-5604ds^2+36d^2-36d^2s) \\
& +a_s(-900s+900s^2) \\
& -480c_{s-2}=0. \qquad \text{(A.5)}
\end{aligned}
$$

B. Results for the 3 dimensional $SU(N)$ theory

B.1 One-loop

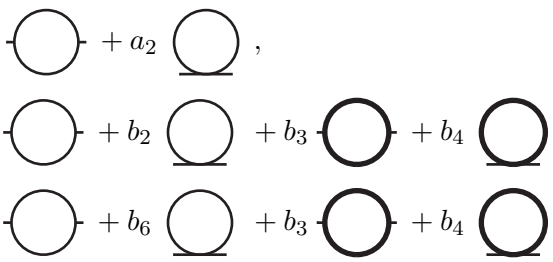
B.1.1 One-loop reduction

Recall that thick lines carry mass $\sqrt{\xi}$ and thin lines carry mass 1. The One-loop self-energy and its first derivatives with respect to the mass and to the momentum are

$$\hat{\Pi}^{(1)}_{00} = a_1 \quad + a_2 \quad , \tag{B.1}$$

$$\hat{\Pi}^{(1)}_{10} = b_1 \quad + b_2 \quad + b_3 \quad + b_4 \quad \tag{B.2}$$

$$\hat{\Pi}^{(1)}_{01} = b_5 \quad + b_6 \quad + b_3 \quad + b_4 \quad , \tag{B.3}$$



These results have been calculated in [22] and confirmed via a `Reduze` reduction. The corresponding d-dependent prefactors reads

$$a_{1,2} = \{\frac{9}{8}(4d-5), \frac{1}{4}(2d-3)(2d-5)\} \tag{B.4}$$

$$b_{1,2,3,4} = \{\frac{3}{16}(d-2)(4d-5), \frac{1}{8}(12d^2-31d+18), \frac{1}{4}(1-4\xi), \frac{1}{2}(3-2d)\} \tag{B.5}$$

$$b_{5,6} = \{\frac{3}{4}(d-2)(4d-5), \frac{1}{2}(d^3-3d^2+4d-3)\}. \tag{B.6}$$

The gap equation for higher l expansions contains also the higher mass and momentum derivatives. $\hat{\Pi}^{(1)}_{11}$ and $\hat{\Pi}^{(1)}_{02}$ have been calculated in [21] . $\hat{\Pi}^{(1)}_{03}$ and $\hat{\Pi}^{(1)}_{12}$ are new results. The results are

$$\hat{\Pi}^{(1)}_{11} = b_7 \; + b_8 \; + b_9 \; + b_{10} \; + b_{11} \; \tag{B.7}$$

$$\hat{\Pi}^{(1)}_{02} = b_{12} \; + b_{13} \; + b_{14} \; + b_{15} \; + b_{11} \; \tag{B.8}$$

$$\hat{\Pi}^{(1)}_{03} = b_{16} \; + b_{17} \; + b_{18} \; + b_{19} \; + b_{20} \; \tag{B.9}$$

$$\hat{\Pi}^{(1)}_{12} = b_{21} \; + b_{22} \; + b_{23} \; + b_{24} \; + b_{25} \; , \tag{B.10}$$

with d and ξ-dependent prefactors

$$\begin{aligned} b_{7,8,9,10,11} &= \{\frac{1}{8}(4d^3-29d^2+62d-34), \frac{1}{4}(d-2)(4d^2-15d+8)-\xi/2, -\frac{1}{8}(d+2)-\frac{\xi}{2}(d-4), \\ &\quad -\frac{1}{4}(d-2)(2d-5)+\frac{\xi}{2}, 2(d-1)+\frac{\xi}{2}(\xi-4)\} \end{aligned} \tag{B.11}$$

$$\begin{aligned} b_{12,13,14,15} &= \{\frac{1}{4}(8d^3-58d^2+124d-77), \frac{1}{4}(d^4-3d^3-7d^{2+19d-4})-\frac{\xi}{2}, \\ &\quad -\frac{3}{4}-\xi(d-4), \frac{1}{2}(2d^2-10d+11)+\frac{\xi}{2}\} \end{aligned} \tag{B.12}$$

$$b_{16} = \frac{1}{6}(429 - 803d + 472d^2 - 106d^3 + 8d^4) \tag{B.13}$$

$$b_{17} = \frac{1}{24(-4+\xi)}(-145d^3(-4+\xi) - 11d^4(-4+\xi) + 3d^5(-4+\xi) + d^2(-3396 + 831\xi)) \tag{B.14}$$

$$b_{18} = -\frac{3}{2(-1+4\xi)}(2 + (-19+3d)\xi + 2(-5+d)^2\xi^2) \tag{B.15}$$

$$b_{19} = \frac{3}{4\xi(-4+\xi)(-1+4\xi)}(16 + 64\xi - 554\xi^2 + 234\xi^3 - 24\xi^4 - d^3\xi(-4+\xi)(-1+4\xi)) + d^2(8 + 8\xi - 163\xi^2 + 42\xi^3) + 4d(-6 - 9\xi + 137\xi^2 - 43\xi^3 + 2\xi^4) \tag{B.16}$$

$$b_{20} = \frac{3}{2(-4+\xi)}(-52 + 4d^2(-3+\xi) - 28\xi + 23\xi^2 - 3\xi^3 + d(64 - 8\xi - 7\xi^2 + \xi^3)) \tag{B.17}$$

$$b_{21} = \frac{1}{24}(330 - 749d + 472d^2 - 106d^3 + 8d^4) \tag{B.18}$$

$$b_{22} = \frac{1}{4(-4+\xi)}(-736 - 89d^3(-4+\xi) + 8d^4(-4+\xi) + 22\xi + 36\xi^2 + 36d^2(-37+9\xi)) + d(1852 - 376\xi - 15\xi^2) \tag{B.19}$$

$$b_{23} = \frac{1}{8(-1+4\xi)}(-6 + 60\xi - 216\xi^2 - 4d^2\xi(1+2\xi) + d(-3 + 16\xi + 80\xi^2)) \tag{B.20}$$

$$b_{24} = \frac{1}{4(-4+\xi)(-1+4\xi)}(-d^3\xi(-4+\xi)(-1+4\xi) + d^2(24 - 64\xi - 115\xi^2 + 38\xi^3)) \tag{B.21}$$

$$b_{25} = \frac{1}{4(-4+\xi)}(4d^2(-14+5\xi) - 6(36 + 16\xi - 15\xi^2 + 2\xi^3) + d(272 - 36\xi - 34\xi^2 + 5\xi^3)). \tag{B.22}$$

As a nontrivial check we verified the generalized dimensional relation

$$\hat{\Pi}^{(1)}_{1,n-1} = \hat{\Pi}^{(1)}_{0n} + \frac{2n-d}{2}\hat{\Pi}^{(1)}_{0,n-1}, \tag{B.23}$$

following from eq.(4.46) for this results. The coefficients b_{16} ,...,b_{25} have partly poles in $\xi = 0$ $\xi = 4$ and $\xi = \frac{1}{4}$. This is not a problem, because all ξ-dependent integrals with such a value for ξ get further reduced to tadpole diagrams, so that the pole in the complete result disappears.

B.1.2 Expansion of the one-loop master-integrals

Let's recall the definition of the dimensionless one-loop integrals

$$\hat{I}(a_1, a_2, sm_1, sm_2) \equiv \frac{1}{J(x,d,1)} \int \frac{d^d k_1}{(2\pi)^d} \prod_{j=1}^{2} \frac{1}{(q_j^2 + sm_j)^{a_j}}|_{p^2=-1}, \tag{B.24}$$

where $(q_1, q_2) = (k_1, k_1 - p)$. The master-integrals for $\Pi^{(1)}$and its derivatives are [22]


$$= \hat{I}(1,1,1,1) \overset{d=3-2\varepsilon}{=} -\frac{\ln 3}{2} + \mathcal{O}(\varepsilon) \tag{B.25}$$


$$= \hat{I}(1,1,\xi,\xi) \overset{d=3-2\varepsilon}{=} -\frac{1}{2}\ln\frac{\sqrt{\xi}+1}{\sqrt{\xi}-1} + \mathcal{O}(\varepsilon) \tag{B.26}$$


$$= \hat{I}(1,1,1,\xi) \overset{d=3-2\varepsilon}{=} -\frac{1}{2}\ln\frac{\sqrt{\xi}+2}{\sqrt{\xi}} + \mathcal{O}(\varepsilon) \tag{B.27}$$


$$= \hat{I}(1,0,1,0) \overset{d=3-2\varepsilon}{=} 1 \tag{B.28}$$


$$= \hat{I}(1,0,\xi,0) \overset{d=3-2\varepsilon}{=} \xi^{(d-2)/2} \tag{B.29}$$

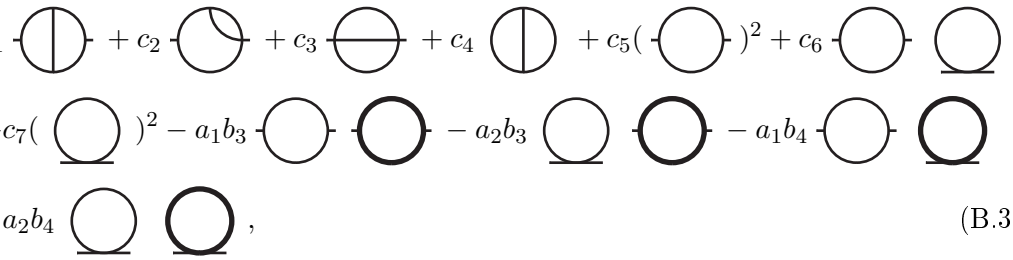
Furthermore the mass dependent one-loop integral has the ε-expansion [22]

$$J(x, 3-2\varepsilon, m) = -\frac{m}{4\pi}\left(\frac{\bar{\mu}}{2m}\right)^{2\varepsilon}(1 + 2\varepsilon + \mathcal{O}(\varepsilon^2)). \tag{B.30}$$

B.2 Two-loop

B.2.1 Two-loop reduction

For the three-loop gap equation the first mass derivative of the two-loop self-energy is needed. Recall that thick lines carry mass $\sqrt{\xi}$ and thin lines carry mass 1. A line with black dot has an index raised. The two-loop self-energy in terms of master-integrals reads [22]

$\hat{\Pi}^{(2)}_{00}$ =

(B.31)

with coefficients

$$c_1 = \frac{3}{64}(d-1)(176d-245) \tag{B.32}$$

$$c_2 = -\frac{3}{64}(144d^3-712d^2+1241d-760) \tag{B.33}$$

$$c_3 = -\frac{10800d^4-70632d^3+165227d^2-166654d+61752}{192(3d-4)} \tag{B.34}$$

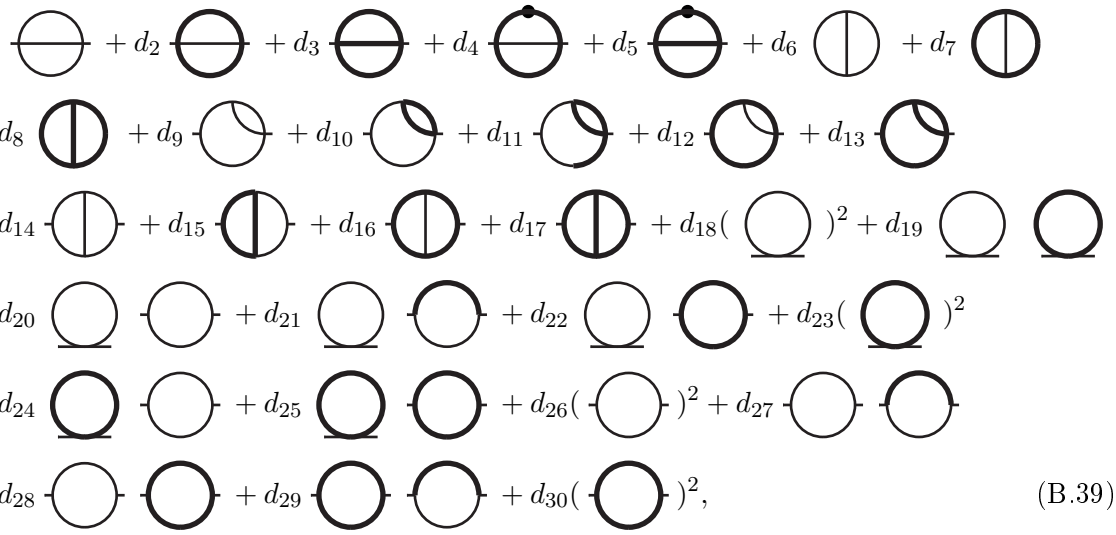
$$c_4 = -\frac{3}{64}(d-2)(32d^3-312d^2+656d-405) \tag{B.35}$$

$$c_5 = \frac{3}{128}(32d^2-148d+155) \tag{B.36}$$

$$c_6 = -\frac{3}{16}(16d^4-188d^3+668d^2-940d+465) \tag{B.37}$$

$$c_7 = -\frac{1}{32}\frac{2d-3}{3d-4}(24d^5-164d^4+452d^3-680d^2+597d-242) \tag{B.38}$$

Our result for the first derivative reads

$\hat{\Pi}_{01}^{(2)} =$ 

(B.39)

This result is also consistent with the dimensional relation (c.f. eq.(4.46)). The complete result with all coefficients d_j has been uploaded at [50] .

B.2.2 Two-loop master-integrals

Let's recall the definition of the dimensionless two-loop integrals

$$\hat{I}(a_1,...,a_5,sm_1,...,sm_5) \equiv \left(\frac{1}{J(x,d,1)}\right)^2 \left(\prod_{n=1}^{2}\int\frac{d^dk_n}{(2\pi)^d}\right)\prod_{j=1}^{5}\frac{1}{(q_j^2+sm_j)^{a_j}}|_{p^2=-1}, \tag{B.40}$$

where $(q_1,...,q_5) = (k_1,\, k_2,\, k_1-p,\, k_2-p,\, k_1-k_2)$.

The gauge invariant two-loop master-integrals for $\Pi_T^{(2)}$ are [22]


$$= \hat{I}(1,1,1,1,1,1,1,1,1,1) \overset{d=3-2\varepsilon}{=} \frac{f(1/3)-f(7/9)}{\sqrt{2}} + \mathcal{O}(\varepsilon) \tag{B.41}$$


$$= \hat{I}(1,1,1,0,1,1,1,1,0,1) \overset{d=3-2\varepsilon}{=} \frac{1}{8}\left[\ln^2 3 - \frac{\pi^2}{6} + 6\mathrm{Li}_2(1/3) - 2\mathrm{Li}_2(1/9)\right] + \mathcal{O}(\varepsilon) \tag{B.42}$$


$$= \hat{I}(0,1,1,0,1,0,1,1,0,1) \overset{d=3-2\varepsilon}{=} \frac{1}{4\varepsilon} + \left(\frac{1}{2} - 2\ln 2\right) + \mathcal{O}(\varepsilon) \tag{B.43}$$


$$= \hat{I}(1,1,0,0,1,1,1,0,0,1) \overset{d=3-2\varepsilon}{=} \frac{1}{4\varepsilon}\left(\frac{1}{2} + 2\ln\frac{3}{2}\right) + \mathcal{O}(\varepsilon), \tag{B.44}$$

where $f(x) = \Im\mathrm{Li}_2(x + i\sqrt{1-x^2})$.

B.3 Three-loop

B.3.1 Three-loop reduction

Let's recall the definition of the dimensionless three-loop integrals

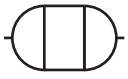
$$\hat{I}(a_1,...,a_9,sm_1,...,sm_9) \equiv \left(\frac{1}{J(x,d,1)}\right)^3 \left(\prod_{n=1}^{3}\int \frac{d^dk_n}{(2\pi)^d}\right) \prod_{j=1}^{9} \frac{1}{(q_j^2+sm_j)^{a_j}}|_{p^2=-1}, \tag{B.45}$$

where $(q_1,...,q_5) = (k_1,\, k_2,\, k_3,\, k_1-p,\, k_2-p,\, k_1-k_3,\, k_2-k_3,\, k_3-p,\, k_1-k_2)$. The reduction of $\hat{\Pi}_{00}^{(3)}$ contains the following gauge invariant master-integrals.

t=8, ID=255

$$\hat{I}(1,1,1,1,1,1,1,0,1,1,1,1,1,1,1,1,0,1) =$$ 

t=8, ID=383

$$\hat{I}(1,1,1,1,1,1,1,1,0,1,1,1,1,1,1,1,1,0) =$$ 

t=7, ID=127

$$\hat{I}(1,1,1,1,1,1,1,0,0,1,1,1,1,1,1,1,0,0) =$$ 

$$\hat{I}(2,1,1,1,1,1,1,0,0,1,1,1,1,1,1,1,0,0) =$$ 

t=7, ID=319

$$\hat{I}(1,1,1,1,1,1,0,0,1,1,1,1,1,1,1,0,0,1) =$$ 

t=7, ID=381

$$\hat{I}(1,0,1,1,1,1,1,0,1,1,0,1,1,1,1,1,0,1) =$$ 

$$\hat{I}(1,0,1,2,1,1,1,0,1,1,0,1,1,1,1,1,0,1) =$$ 

t=7, ID=379

$$\hat{I}(1,1,0,1,1,1,1,0,1,1,1,0,1,1,1,1,0,1) =$$ 

t=7, ID=367

$$\hat{I}(1,1,1,1,0,1,1,0,1,1,1,1,1,0,1,1,0,1) =$$ 

t=6, ID=380

$$\hat{I}(0,0,1,1,1,1,1,0,1,0,0,1,1,1,1,1,0,1) =$$ 

$$\hat{I}(0,0,1,2,1,1,1,0,1,0,0,1,1,1,1,1,0,1) =$$ 

t=6, ID=317

$$\hat{I}(1,0,1,1,1,1,0,0,1,1,0,1,1,1,1,0,0,1) =$$ 

t=6, ID=303

$$\hat{I}(1,1,1,1,0,1,0,0,1,1,1,1,1,0,1,0,0,1) =$$ 

t=6, ID=377

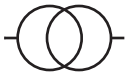
$$\hat{I}(1,0,0,1,1,1,1,0,1,1,0,0,1,1,1,1,0,1) =$$ 

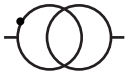
t=6, ID=125

$$\hat{I}(1,0,1,1,1,1,1,0,0,1,0,1,1,1,1,1,0,0) =$$ 

$$\hat{I}(1,0,1,2,1,1,1,0,0,1,0,1,1,1,1,1,0,0) =$$ 

t=6, ID=123

$$\hat{I}(1,1,0,1,1,1,1,0,0,1,1,0,1,1,1,1,0,0) =$$ 

$$\hat{I}(2,1,0,1,1,1,1,0,0,1,1,0,1,1,1,1,0,0) =$$ 

t=6, ID=371

$$\hat{I}(1,1,1,0,0,1,1,0,1,1,1,1,0,0,1,1,0,1) =$$ 

t=5, ID=376

$$\hat{I}(0,0,0,1,1,1,1,0,1,0,0,0,1,1,1,1,0,1) =$$ 

t=5, ID=316

$$\hat{I}(0,0,1,1,1,1,0,0,1,0,0,1,1,1,1,0,0,1) =$$ 

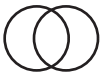
t=5, ID=121

$$\hat{I}(1,0,0,1,1,1,1,0,0,1,0,0,1,1,1,1,0,0) =$$ 

t=5, ID=124

$$\hat{I}(0,0,1,1,1,1,1,0,0,0,0,1,1,1,1,1,0,0) =$$ 

t=4, ID=120

$$\hat{I}(0,0,0,1,1,1,1,0,0,0,0,0,1,1,1,1,0,0) =$$ 

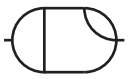
t=4, ID=113

$$\hat{I}(1,0,0,0,1,1,1,0,0,1,0,0,0,1,1,1,0,0) =$$ 

$$\hat{I}(2,0,0,0,1,1,1,0,0,1,0,0,0,1,1,1,0,0) =$$ 

$$\hat{I}(3,0,0,0,1,1,1,0,0,1,0,0,0,1,1,1,0,0) =$$ 

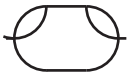
The second reduction contains additionally the following gauge invariant master-integrals.
t=7, ID=239

$$\hat{I}(1,1,1,1,0,1,1,1,0,1,1,1,1,0,1,1,1,0) =$$ 

t=6, ID=238

$$\hat{I}(0,1,1,1,0,1,1,1,0,0,1,1,1,0,1,1,1,0) =$$ 

t=6, ID=363

$$\hat{I}(1,1,0,1,0,1,1,0,1,1,1,0,1,0,1,1,0,1) =$$ 

t=6, ID=231

$$\hat{I}(1,1,1,0,0,1,1,1,0,1,1,1,0,0,1,1,1,0) =$$ 

t=5, ID=362

$$\hat{I}(0,1,0,1,0,1,1,0,1,0,1,0,1,0,1,1,0,1) =$$ 

Furthermore there are the following ξ-dependent master-integrals
t=5, ID=316

$$\hat{I}(0,0,1,1,1,1,0,0,1,0,0,2,2,1,2,0,0,2)$$
$$\hat{I}(0,0,1,2,1,1,0,0,1,0,0,2,2,1,2,0,0,2)$$
$$\hat{I}(0,0,2,1,1,1,0,0,1,0,0,2,2,1,2,0,0,2)$$

t=5, ID=121

$$\hat{I}(1,0,0,1,1,1,1,0,0,2,0,0,2,1,2,2,0,0)$$
$$\hat{I}(1,0,0,1,2,1,1,0,0,1,0,0,1,1,1,2,0,0)$$
$$\hat{I}(1,0,0,1,2,1,1,0,0,2,0,0,2,1,2,2,0,0)$$
$$\hat{I}(1,0,0,2,1,1,1,0,0,1,0,0,1,1,1,2,0,0)$$
$$\hat{I}(1,0,0,2,1,1,1,0,0,2,0,0,2,1,2,2,0,0)$$
$$\hat{I}(1,0,0,1,1,1,1,0,0,1,0,0,1,1,1,2,0,0)$$

t=5, ID=124

$$\hat{I}(0,0,1,1,1,1,0,0,1,0,0,2,1,1,1,0,0,1)$$
$$\hat{I}(0,0,1,1,1,1,0,0,1,0,0,2,1,2,1,0,0,2)$$
$$\hat{I}(0,0,1,1,1,2,0,0,1,0,0,2,1,1,1,0,0,1)$$
$$\hat{I}(0,0,1,1,1,2,0,0,1,0,0,2,1,2,1,0,0,2)$$
$$\hat{I}(0,0,1,2,1,1,0,0,1,0,0,2,1,1,1,0,0,1)$$
$$\hat{I}(0,0,1,2,1,1,0,0,1,0,0,2,1,2,1,0,0,2)$$
$$\hat{I}(0,0,2,1,1,1,0,0,1,0,0,2,1,1,1,0,0,1)$$
$$\hat{I}(0,0,2,1,1,1,0,0,1,0,0,2,1,2,1,0,0,2)$$

t=4, ID=113

$$\hat{I}(1,0,0,0,1,1,1,0,0,1,0,0,0,1,2,1,0,0)$$
$$\hat{I}(1,0,0,0,1,1,2,0,0,1,0,0,0,1,2,1,0,0)$$
$$\hat{I}(1,0,0,0,1,2,1,0,0,1,0,0,0,1,2,1,0,0)$$
$$\hat{I}(1,0,0,0,2,1,1,0,0,1,0,0,0,1,2,1,0,0)$$
$$\hat{I}(2,0,0,0,1,1,1,0,0,1,0,0,0,1,2,1,0,0)$$
$$\hat{I}(1,0,0,0,1,1,1,0,0,2,0,0,0,2,1,2,0,0)$$
$$\hat{I}(1,0,0,0,1,1,2,0,0,2,0,0,0,2,1,2,0,0)$$
$$\hat{I}(1,0,0,0,1,2,1,0,0,2,0,0,0,2,1,2,0,0)$$
$$\hat{I}(1,0,0,0,2,1,1,0,0,2,0,0,0,2,1,2,0,0)$$
$$\hat{I}(2,0,0,0,1,1,1,0,0,2,0,0,0,2,1,2,0,0)$$

t=4, ID=120

$$\hat{I}(0,0,0,1,1,1,1,0,0,0,0,0,1,1,2,1,0,0)$$
$$\hat{I}(0,0,0,1,1,1,1,0,0,0,0,0,1,2,2,2,0,0)$$

The second reduction contains also the following additional gauge dependent integrals
t=4, ID=113

$$\hat{I}(1,0,0,0,1,2,1,0,0,2,0,0,0,2,2,2,0,0)$$
$$\hat{I}(1,0,0,0,2,1,1,0,0,2,0,0,0,2,2,2,0,0)$$

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