Computer Algebra Methods for the Evaluation of Electroweak Contributions to the Muon Anomalous Magnetic Moment

Bachelor thesis

submitted to the
Faculty of Physics
at
Bielefeld University

by

Thomas Luthe

Supervisor & 1st assessor: Prof. Dr. York Schröder
2nd assessor: Dr. Olaf Kaczmarek

Bielefeld, October 5th, 2010

Contents

1 Introduction

In classical electrodynamics the magnetic dipole moment $\vec{\mu}$ of a particle with mass m, charge q and orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$ is given by

$$\vec{\mu}_L = \frac{q}{2m}\vec{L} = Q\mu_0\vec{L} \quad , \tag{1.1}$$

where $\mu_0 = e/2m$, $Q = q/e$ and e is the charge of the positron.
On a quantum level particles may also exhibit a spin $\vec{S}$, which is mathematically equivalent to an angular momentum. The spin operator $\vec{S} = \frac{\vec{\sigma}}{2}$ (where σ_i is the i-th Pauli matrix) then gives rise to an intrinsic magnetic moment

$$\vec{\mu}_S = gQ\mu_0\vec{S} = gQ\mu_0\frac{\vec{\sigma}}{2} \quad . \tag{1.2}$$

The proportionality constant g defined by this equation is the gyromagnetic ratio. The Dirac equation predicts [1] for a lepton l in the classical limit $g_l = 2$. The anomalous magnetic moment a_l is the deviation from this value:

$$a_l = \frac{g_l - 2}{2} \tag{1.3}$$

a_l is of particular interest, because it can be measured as well as predicted from theory to a very high precision. This enables thorough checking of a given theory by searching for deviations between the experimental and theoretical values. The Standard Model (SM) of particle physics predicts for the anomalous magnetic moment of the muon a value of

$$a_\mu^{SM} = 116591790.0(64.6) \quad [1], \tag{1.4}$$

while the current experimental value is

$$a_\mu^{Exp} = 116592080.0(63.0) \quad [1]. \tag{1.5}$$

It is thus one of the most accurately predicted values in physics and a great success for the Standard Model. However there is a discrepancy of 3.2 standard deviations between the two values, which might hint at possible theoretical contributions beyond the Standard Model.
It is important to note that at the achieved level of accuracy the prediction of a_μ relies on many aspects of the SM and is thus testing them all at once. The largest contribution stems from quantum electrodynamics, but one also finds hadronic corrections and contributions from the electroweak sector of the SM. This thesis will outline the calculation of these contributions, focussing on the electroweak force. A computer-based solution will be presented to automate the calculation of parts of the electroweak contribution.

2 Determination of a_μ

2.1 a_μ in quantum field theory

Quantum Electrodynamics (QED) is the quantum field theory describing interactions between charged particles and photons. Its Lagrangian for a spin-$\frac{1}{2}$ field with charge $-e$ such as the muon is given by

$$\begin{aligned}\mathcal{L}^{QED} &= \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi - eJ^\mu A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad [4]\end{aligned} \tag{2.1}$$

where Ψ is a Dirac spinor representing the muon field, $D_\mu = \partial_\mu - ieA_\mu$ the covariant derivative, A_μ the electromagnetic four-potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the field strength tensor and $J^\mu(\mathbf{x}) = \bar{\Psi}(\mathbf{x})\gamma^\mu\Psi(\mathbf{x})$ the electric current. To study the magnetic moment one considers the scattering of a muon with four-momentum $\mathbf{p}$ off the external potential A_μ into the final state with four-momentum $\mathbf{p}' = \mathbf{p} + \mathbf{q}$. The relevant interaction term of the Lagrangian is

$$\mathcal{L}_{int} = -eJ^\mu A_\mu \quad . \tag{2.2}$$

Ignoring terms of higher orders in A_μ the scattering amplitude is given by

$$\mathcal{M} = -ie \left\langle \mu_{\mathbf{p}'} \left| J^\mu(0) \right| \mu_{\mathbf{p}} \right\rangle A_\mu(\mathbf{q}) \quad [2]. \tag{2.3}$$

The matrix element can be written as

$$\left\langle \mu_{\mathbf{p}'} \left| J^\mu(0) \right| \mu_{\mathbf{p}} \right\rangle = \bar{u}(\mathbf{p}')\Gamma^\mu(\mathbf{p}, \mathbf{p}')u(\mathbf{p}) \tag{2.4}$$

where $\Gamma^\mu(\mathbf{p}, \mathbf{p}')$ is called the vertex function. It can be described in perturbation theory by an expansion in radiative corrections. To the lowest order (no radiative corrections) we have $\Gamma^\mu = \gamma^\mu$. Since the only objects appearing in the relevant Feynman rules are γ^μ, p^μ and p'^μ, and Γ^μ transforms as a four-vector, the general form of Γ^μ can be restricted to contain only linear combinations of the above vectors. Terms including γ^5 vanish for parity reasons [3]. One can then use the ansatz

$$\Gamma^\mu = A\gamma^\mu + B(p^\mu + p'^\mu) + C(p^\mu - p'^\mu) \quad . \tag{2.5}$$

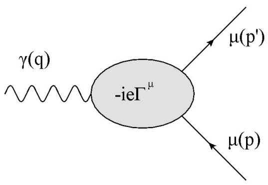
The coefficients A, B and C can include terms of $\not{p}$, $\not{p}'$, $\mathbf{p}^2$, $\mathbf{p}'^2$ and $\mathbf{q}^2$. The first four of these can be expressed in terms of the muon mass m_μ using the Dirac equation and the on-shell condition of the external muon momenta respectively: $\not{p}u(\mathbf{p}) = m_\mu u(\mathbf{p})$, $\bar{u}(\mathbf{p}')\not{p}' = \bar{u}(\mathbf{p}')m_\mu$, $\mathbf{p}^2 = m_\mu^2$ and $\mathbf{p}'^2 = m_\mu^2$.

The coeffients are thus functions of $\mathbf{q}^2$ only.
Using the Ward Identity $q_\mu \Gamma^\mu = 0$ [3] the ansatz can be simplified further. One finds that $\bar{u}(\mathbf{p}') q_\mu \gamma^\mu u(\mathbf{p}) = 0$ and $q_\mu(p^\mu + p'^\mu) = 0$, while $q_\mu(p^\mu - p'^\mu) = 2\mathbf{q}\cdot\mathbf{p} \neq 0$ and thus $C = 0$.
One can further rewrite the ansatz using the Gordon identity

$$\bar{u}(\mathbf{p}')\gamma^\mu u(\mathbf{p}) = \bar{u}(\mathbf{p}')\left[\frac{p^\mu + p'^\mu}{2m_\mu} + \frac{i\sigma^{\mu\nu}q_\nu}{2m_\mu}\right] u(\mathbf{p}) \quad [3] \tag{2.6}$$

where $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$. This identity allows us to use the term $i\sigma^{\mu\nu}q_\nu$ rather than $(p^\mu + p'^\mu)$. The final result can then be written as

$$\Gamma^\mu(\mathbf{p}, \mathbf{p}') = \gamma^\mu F_E(\mathbf{q}^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m_\mu} F_m(\mathbf{q}^2) \quad . \tag{2.7}$$



$$= -ie\bar{u}(\mathbf{p}')\left[\gamma^\mu F_E(\mathbf{q}^2) + \tfrac{i\sigma^{\mu\nu}q_\nu}{2m_\mu} F_m(\mathbf{q}^2)\right] u(\mathbf{p}) \tag{2.8}$$

To find the relation to the magnetic moment one can now take the nonrelativistic limit and compare it to the Born approximation, where the electromagnetic potential is taken to be just the external photon potential. The latter states that the nonrelativistic scattering amplitude is given by

$$f = -\frac{m_\mu}{2\pi}\chi_2^+ \left(-\frac{e}{2m_\mu}\tilde{\vec{A}}(\vec{p} + \vec{p}') + e\tilde{\Phi} - \frac{i}{2}\mu\vec{\sigma}\left[\vec{q} \times \tilde{\vec{A}}\right]\right)\chi_1 \quad [2] \tag{2.9}$$

where $A^\mu = (\Phi, \vec{A})$, $\mu = \frac{|\vec{\mu}|}{|\vec{S}|}$, χ is a Pauli spinor and the tilde denotes a Fourier transform.
The nonrelativistic limit of $\mathcal{M}$ is performed by setting

$$u(\mathbf{p}) = \sqrt{E + m_\mu}\begin{pmatrix} \chi \\ \frac{\vec{p}\cdot\vec{\sigma}}{E + m_\mu}\chi \end{pmatrix} \tag{2.10}$$

where $E = \sqrt{\vec{p}^2 + m_\mu^2}$ and neglecting quadratic and higher order terms in $\mathbf{p}/m_\mu$. The result is [2]:

$$\lim_{|\mathbf{p}| \ll m_\mu} \mathcal{M} = -2iem_\mu\chi_2^+\left[F_E(0)\left(\tilde{\Phi} - \frac{\tilde{\vec{A}}(\vec{p} + \vec{p}')}{2m_\mu}\right) - ie\frac{F_E(0) + F_M(0)}{2m_\mu}\vec{\sigma}\left[\vec{q} \times \tilde{\vec{A}}\right]\right]\chi_1 \tag{2.11}$$

The two results are connected via

$$\lim_{|\mathbf{p}|<<m_\mu} \mathcal{M} = 4\pi i f \quad [2]. \tag{2.12}$$

Comparing (2.11) to (2.9) thus yields

$$F_E(0) = 1,\ \mu = \frac{e}{m}(1 + F_M(0)) \tag{2.13}$$

and when combined with (1.2) and (1.3) finally

$$F_M(0) = a_\mu \quad . \tag{2.14}$$

It should be noted that a_μ does not depend on F_E and thus in its calculation any terms proportional only to γ^μ can be neglected.

2.2 Projection technique

Since a_μ is independent of terms proportional to γ^μ the calculation can be simplified by projecting out these terms. To do this, the vertex function is expanded to linear order in $\mathbf{q}$:

$$\Gamma_\mu(\mathbf{P}, \mathbf{q}) \approx \Gamma_\mu(\mathbf{P}, 0) + q^\nu \frac{\partial}{\partial q^\nu} \Gamma_\mu(\mathbf{P}, \mathbf{q})|_{\mathbf{q}=0} \equiv V_\mu(\mathbf{p}) + q^\nu T_{\nu\mu}(\mathbf{p}) \tag{2.15}$$

where $\mathbf{P} = \mathbf{p} + \mathbf{p}'$. As a final formula for the anomalous magnetic moment one finds [1]

$$\begin{aligned} a_\mu = & \frac{1}{8(d-2)(d-1)m} \mathrm{Tr}\left[(\not{p} + m)\,[\gamma^\mu,\, \gamma^\nu]\,(\not{p} + m) T_{\nu\mu}(\mathbf{p})\right] \\ & + \frac{1}{4(d-1)m^2}\ \mathrm{Tr}\left[\left[m^2\gamma^\mu - (d-1)mp^\mu - d\not{p}p^\mu\right] V_\mu(\mathbf{p})\right]\Big|_{\mathbf{p}^2=m^2} \end{aligned} \tag{2.16}$$

where m is the muon mass and d the space-time dimension. $d = 4 - 2\epsilon$ can be used to provide a dimensional regularisation. This becomes necessary because many of the integrals encountered in the calculations are divergent in 4 dimensions. The limit $\epsilon \to 0$ is then performed at the end of the calculation. The traces provide a readily generalised way to deal with the strings of Gamma matrices that appear in the vertex function and thus make this approach an attractive choice for a computer-based automation of the calculations.

2.3 Experimental determination

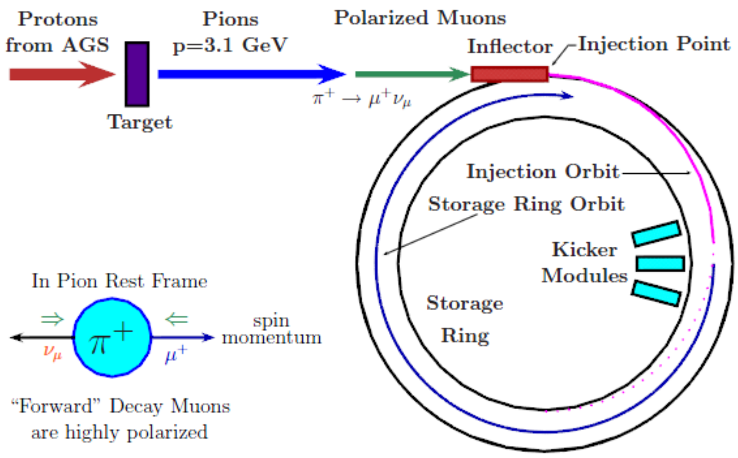


Figure 1: Schematics of the Brookhaven experiment [1]

The following discussion is based on the muon g-2 experiment in Brookhaven as presented in [1], which is illustrated in figure 1.
Protons are accelerated to an energy of 24 GeV and hit a target, producing pions in the process. The pions then decay into muons and neutrinos. Parity violation of the weak interaction responsible for the decay guarantees that the muons carry a spin that is pointed in the direction of their momentum. The polarised muons then enter a circular trajectory in a uniform magnetic field. They orbit with an angular frequency

$$\omega_c = \frac{eB}{m_\mu \gamma} \tag{2.17}$$

where B is the magnetic field strength and γ is the relativistic Lorentz factor. The spin is precessing slightly faster at an angular velocity $\omega_s = \omega_c + \omega_a$, which leads to a shift of 12 arc seconds in the spin axis after each orbit. This effect is due to the Larmor precession caused by a_μ. The difference in velocities is given by

$$\omega_a = a_\mu \frac{eB}{m_\mu} \quad . \tag{2.18}$$

The values for e and m_μ are determined independently, while B and ω_a have to be measured in the experiment. To measure ω_a, one makes use of the fact that the muons eventually decay into either an electron (μ^-) or a

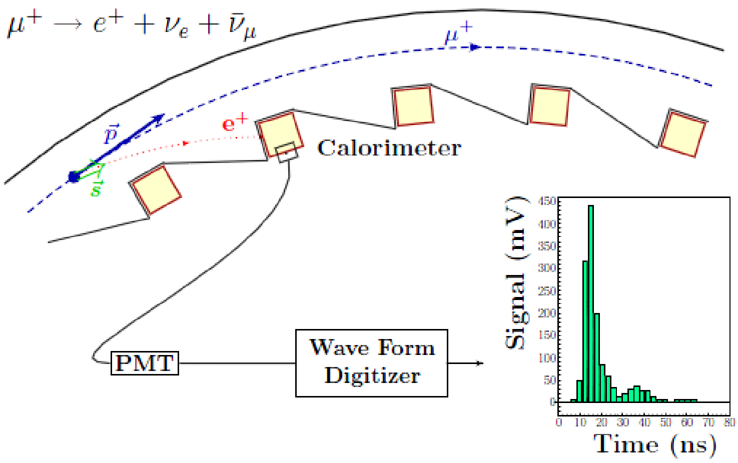


Figure 2: Muon decay and positron detection [1]

positron (μ^+) and two neutrinos. Parity violation once again ensures that the electrons/positrons are emitted in the direction of the muon spin. The energy of the electrons/positrons is then measured with a set of calorimeters as can be seen in figure 2. This allows inference of the spin direction and thus ω_a. The current experimental values (see [1]) for a_{μ^+} and a_{μ^-} respectively are

$$\begin{aligned} a_{\mu^+} &= 11659204(7) \times 10^{-10} \\ a_{\mu^-} &= 11659214(8) \times 10^{-10} \end{aligned} \tag{2.19}$$

giving the combined value of

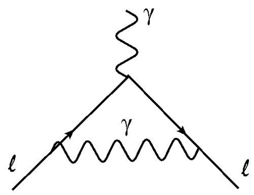
$$a_\mu = 11659208.0(6.3) \times 10^{-10}\,. \tag{2.20}$$

3 Contributions to a_μ

The current Standard Model prediction for muon anomalous magnetic moment is $a_\mu = 116591790(65) \times 10^{-11}$ [1]. This value is composed of many contributions. The vertex function Γ^μ is a sum over all possible Feynman diagrams that share the same external lines. The diagrams can then be grouped together into classes according to their constituents.

3.1 QED

The QED contribution contains all Feynman diagrams that are comprised of only charged leptons and photons. It can be expanded in the fine-structure constant $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$ in perturbation theory, where every loop adds an additional factor α. Hence higher order loop diagrams yield a smaller contribution and can be neglected up to a certain precision. There is only one first order (one loop) diagram:



The result was first calculated by Schwinger in 1948 [5]:

$$a^{QED,1} = \frac{\alpha}{2\pi} \tag{3.1}$$

A derivation of this term will be given in section 5. Since it does not depend on any mass, it is equally valid for all leptons. The numeric value is $a^{(QED,1)} = 116140973.289(43)$ [1], which is by far the largest contribution to a_μ.
Starting from two-loop diagrams, closed fermion loops appear, which lead to a dependence of a_μ on mass ratios. Light electron vacuum polarisations give rise to potentially large logarithms such as $\ln(\frac{m_\mu}{m_e})$, while heavy tau loops are suppressed by ratios such as $\frac{m_\mu^2}{m_\tau^2}$ [1]. The mass-independent universal terms have been analytically evaluated for two and three loops. Analytical results for 4-loop diagrams have been found only in a few cases. Other 4-loop diagrams have been evaluated using estimates and approximations. 5-loop diagrams are not yet well explored, but a first estimate exists. Altogether the current theoretical QED contribution is

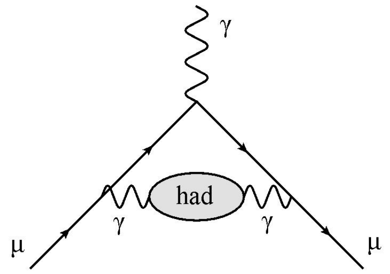
$$a_\mu^{QED} = 116584718.104(.148) \times 10^{-11} \quad [1]. \tag{3.2}$$

3.2 Hadronic contributions

Diagrams with hadronic loops appear at the level of 2-loop diagrams and above. They are governed by Quantum Chromodynamics (QCD) and cannot be treated in a perturbative way because the strong coupling constant is large at low energies [1]. Moreover, quark masses are not well-defined like the lepton masses. These problems prevent a strictly analytical approach. The hadronic contribution can be further divided into two categories:

3.2.1 Hadronic Vacuum Polarisation

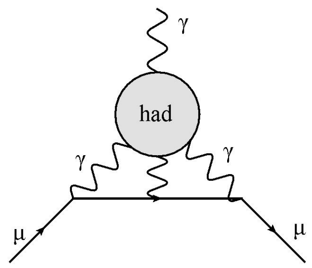
The leading hadronic contribution is given by:



The hadronic loop is described by a self-energy function Π, which can be related to the cross-section ($e^+e^- \to$ hadrons) via the dispersion relation [1]. The cross-section is known from experiments and can thus be used to obtain an estimate on the hadronic vacuum polarisation contribution. The current value is

$$a_\mu^{(HadVP)} = 6802.7(52.6) \times 10^{-11} \quad [1]. \tag{3.3}$$

3.2.2 Hadronic Light-by-Light Scattering



At the 3-loop level hadronic loops with four photons attached start to appear. This so-called light-by-light scattering is particularly difficult to evaluate. This stems from the fact that three of the four photons are virtual and their momenta have to be integrated over. Since no experimental data is available for this situation, the approach outlined in section 3.2.1 cannot be used. There is however an estimate involving chiral perturbation theory that is explained in [1]. According to it the contribution to the anomaluos magnetic moment is

$$a_\mu^{(HadLbL)} = 116.0(39.0) \times 10^{-11}. \tag{3.4}$$

3.3 Electroweak contributions

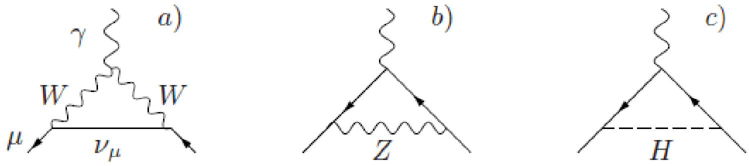


Figure 3: Leading order weak diagrams [1]

The leading order weak contributions shown in figure 3 can be evaluated analytically. In the limit $\left(\frac{m_\mu}{m_{W/Z}}\right)^2 \approx 0$ the results for the first two diagrams are given by

$$\begin{aligned} a_\mu^{(EW,1)}(W) &= \frac{\sqrt{2}G_F m_\mu^2}{16\pi^2}\frac{10}{3} \approx 388.70(0)\times 10^{-11} \quad [1] \\ a_\mu^{(EW,1)}(Z) &= \frac{\sqrt{2}G_F m_\mu^2}{16\pi^2}\frac{(1-4sin^2(\Theta_W))^2-5}{3} \approx -193.89(2)\times 10^{-11} \end{aligned} \tag{3.5}$$

where G_F is the Fermi coupling constant and Θ_W is the weak mixing angle. A detailed calculation for the diagram involving the Z boson will be provided in section 5. The contribution from the Higgs exchange depends on the not yet determined mass m_H:

$$\begin{aligned} a_\mu^{(EW,1)}(H) &= \frac{\sqrt{2}G_F m_\mu^2}{4\pi^2}\int_0^1 dx \frac{(2-x)x^2}{x^2+(1-x)\frac{m_H^2}{m_\mu^2}} \\ &\approx \frac{\sqrt{2}G_F m_\mu^2}{4\pi^2}\begin{cases} \frac{m_\mu^2}{m_H^2}\ln\left(\frac{m_H^2}{m_\mu^2}\right) & \text{for } m_H \gg m_\mu \\ \frac{3}{2} & \text{for } m_H \ll m_\mu \end{cases} \quad [1] \end{aligned} \tag{3.6}$$

Taking current bounds for m_H into account the contribution is found to be negligible at $a_\mu^{(EW,1)}(H) \leq 5\times 10^{-14}$ [1].
The 2-loop level contributions once again feature hadronic loops, but have been calculated to be $a_\mu^{(2,EW)} = -42.08(1.80)\times 10^{-11}$ [1]. Only few 3-loop diagrams have been evaluated. The total electroweak contribution up to second order to the anomalous magnetic moment is

$$a_\mu^{EW} = 153.2(1.8)\times 10^{-11} \quad [1]. \tag{3.7}$$

3.4 Errors and limitations

Contribution	Error	ppm of contribution
QED	0.2	0.0017
Hadronic vacuum polarisation	52.6	7700
Hadronic light-by-light	39.0	340000
Electroweak interaction	1.8	12000
Theory total	64.6	0.55
Experiment	63.0	0.54

Table 1: Errors in units of 10^{-11} [1]

As can be seen from table 1, the by far largest error in the theoretical prediction is due to the hadronic contributions. Some progress on the understanding of these contributions is expected from better experimental data on e^+e^--annihilations and a new theoretical approach for the diagrams involving pions [1].
The QED contribution is well understood and its error negligible for the combined theoretical value. An even higher precision is prevented by the uncertainty in the value of the fine-structure constant α and the still incomplete 5-loop calculations [1].
While the relative error of the electroweak contribution is sizeable due to hadronic loops, its absolute error is not the limiting factor for the theoretical precision because contributions are suppressed by the mass ratios $\frac{m_\mu}{m_W}$, $\frac{m_\mu}{m_Z}$ or $\frac{m_\mu}{m_H}$.

Experimental precision is limited primarily by the challenge of creating and measuring a homogeneous magnetic field. While its error is currently very close to the theoretical one, several improvements on experiments are conceivable that will increase precision by approximately one order of magnitude in the near future [1].

4 Useful formulae

4.1 Momentum integrals

Common integrals are of the form

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{\left(\mathbf{k}^2\right)^m}{\left(\mathbf{k}^2 - A\right)^n}$$

where $A > 0$. To avoid the poles in the denominator one can perform a Wick rotation that transforms the zero component of the four-vector $k^0 \to ik^0$. The Minkwoski metric then becomes Euclidean and the dot product $\mathbf{k}^2$ becomes $-\mathbf{k}^2$.

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{\left(\mathbf{k}^2\right)^m}{\left(\mathbf{k}^2 - A\right)^n} = (-1)^{m+n}\, i \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{\left(\mathbf{k}^2\right)^m}{\left(\mathbf{k}^2 + A\right)^n}$$

The integral is now well-defined if $d + 2m < 2n$. The next step is to scale the integral to make it independent of A:

$$(-1)^{m+n}\, iA^{d/2+m-n} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{\left(\mathbf{k}^2\right)^m}{\left(\mathbf{k}^2 + 1\right)^n}$$

After changing to d-dimensional spherical coordinates and evaluating the angular integrals one finds

$$(-1)^{m+n}\, iA^{d/2+m-n} \frac{1}{(2\pi)^d} \frac{2\pi^{d/2}}{\Gamma\left(d/2\right)} \int_0^\infty \frac{dk\, k^{d-1+2m}}{\left(k^2+1\right)^n} \quad .$$

The remaining one-dimensional integral can be solved analytically and yields

$$\int_0^\infty \frac{dk\, k^{d-1+2m}}{\left(k^2+1\right)^n} = \frac{\Gamma\left(d/2+m\right)\Gamma\left(n-m-d/2\right)}{2\Gamma\left(n\right)} \quad .$$

For the whole momentum integral one then finds:

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{\left(\mathbf{k}^2\right)^m}{\left(\mathbf{k}^2 - A\right)^n} = (-1)^{m+n}\, iA^{d/2+m-n} \frac{\Gamma\left(d/2+m\right)\Gamma\left(n-m-d/2\right)}{2^d\pi^{d/2}\Gamma\left(d/2\right)\Gamma\left(n\right)} \tag{4.1}$$

4.2 Tensor integrals

One also encounters integrals of the form

$$\int d^d\mathbf{k}\, k^\mu k^\nu f(\mathbf{k}^2) \quad .$$

Lorentz invariance dictates that the integral is proportional to $\eta^{\mu\nu}$:

$$\int d^d\mathbf{k}\, k^\mu k^\nu f(\mathbf{k}^2) = \eta^{\mu\nu} I$$

One can then argue as follows

$$\eta_{\mu\nu} \int d^d\mathbf{k}\, k^\mu k^\nu f(\mathbf{k}^2) = \eta_{\mu\nu}\eta^{\mu\nu} I$$

$$\Rightarrow \int d^d\mathbf{k}\, \mathbf{k}^2 f(\mathbf{k}^2) = dI$$

to find that

$$\int d^d\mathbf{k}\, k^\mu k^\nu f(\mathbf{k}^2) = \frac{\eta^{\mu\nu}}{d} \int d^d\mathbf{k}\, \mathbf{k}^2 f(\mathbf{k}^2) \quad . \tag{4.2}$$

The same principles applied to a more general integral with an even number n indices yield

$$\int d^d\mathbf{k}\, k^{\mu_1} k^{\mu_2} \cdots k^{\mu_n} f(\mathbf{k}^2) = \frac{\Lambda^{\mu_1\mu_2\cdots\mu_n}}{a_n(d)} \int d^d\mathbf{k} \left(\mathbf{k}^2\right)^{\frac{n}{2}} f(\mathbf{k}^2)$$

where $\Lambda^{\mu_1\mu_2\cdots\mu_n}$ is the totally symmetric sum consisting only of products of Minkowski metrics $\eta^{\mu_i\mu_j}$ and $a_n(d)$ is a constant of proportionality that is given by

$$a_n(d) = \eta_{\mu_1\mu_2} \cdots \eta_{\mu_{n-1}\mu_n} \Lambda^{\mu_1\mu_2\cdots\mu_n} \quad .$$

Since there are $(n-1)!!$ possible permutations for n indices in $\frac{n}{2}$ Minkowski metrics, $a_n(d)$ consists of $(n-1)!!$ terms, $(n-3)!!$ of which contain $\eta^{\mu_{n-1}\mu_n}$. Upon contraction with $\eta_{\mu_{n-1}\mu_n}$ the latter gain a factor of d, while the remaining $(n-1)!! - (n-3)!! = (n-2)((n-3)!!)$ terms turn into permutations of the other $n-2$ indices. Since there are only $(n-3)!!$ such permutations and the sum has to be symmetric, each of the terms gains a factor $n-2$. We thus find

$$a_n(d) = (d+n-2)a_{n-2}(d)$$

and using $a_2(d) = d$ then yields

$$a_n(d) = \prod_{i=0}^{\frac{n}{2}-1} (d+2i) \quad .$$

As a master formula we find

$$\int d^d\mathbf{k}\, k^{\mu_1} k^{\mu_2} \cdots k^{\mu_n} f(\mathbf{k}^2) = \frac{\Lambda^{\mu_1\mu_2\cdots\mu_n}}{\prod_{i=0}^{\frac{n}{2}-1}(d+2i)} \int d^d\mathbf{k} \left(\mathbf{k}^2\right)^{\frac{n}{2}} f(\mathbf{k}^2) \quad . \tag{4.3}$$

4.3 Chisholm identity

It is often necessary to contract Gamma matrices within strings of Gamma matrices. This can be done in d dimensions with the Chisholm identity

$$\begin{aligned} &\gamma^\mu \gamma^{\nu_1} \cdots \gamma^{\nu_n} \gamma_\mu \\ &= \begin{cases} 2\gamma^{\nu_n}\gamma^{\nu_1} \cdots \gamma^{\nu_{n-1}} + 2\gamma^{\nu_{n-1}} \cdots \gamma^{\nu_1}\gamma^{\nu_n} + (d-4)\gamma^{\nu_1} \cdots \gamma^{\nu_n} & \text{if n even} \\ -2\gamma^{\nu_n} \cdots \gamma^{\nu_1} - (d-4)\gamma^{\nu_1} \cdots \gamma^{\nu_n} & \text{if n odd} \end{cases} \end{aligned} \tag{4.4}$$

Starting at the relation $\gamma^\mu \gamma_\mu = d$ the identity can be easily confirmed via complete induction. We find

$$\gamma^\mu \gamma^{\nu_1} \cdots \gamma^{\nu_n} \gamma_\mu = -\gamma^\mu \gamma^{\nu_1} \cdots \gamma^{\nu_{n-1}} \gamma_\mu \gamma^{\nu_n} + 2\gamma^{\nu_n} \gamma^{\nu_1} \cdots \gamma^{\nu_{n-1}}$$

Assuming eq. 4.4 is valid for $n-1$ then shows that the same relations hold also for n.

4.4 Diracology

Since the vertex function is sandwiched between the Dirac spinors $\bar{u}(\mathbf{p}')$ and $u(\mathbf{p})$ (see eq. 2.8) and u obeys the Dirac equation we can use the relations

$$\begin{aligned} \not{p} u(\mathbf{p}) &= m_\mu u(\mathbf{p}) \\ \bar{u}(\mathbf{p}')\not{p}' &= \bar{u}(\mathbf{p}') m_\mu \\ \bar{u}(\mathbf{p}')\not{p} &= \bar{u}(\mathbf{p}')(m_\mu - \not{q}) \\ \bar{u}(\mathbf{p}')\not{q} &= \bar{u}(\mathbf{p}')(m_\mu - \not{p}) \quad . \end{aligned} \tag{4.5}$$

4.5 Feynman parametrisation

A vertex function usually contains integrals over more than one propagator. The denominator then is a product of terms such as $(\mathbf{k}^2 - m^2)$. To avoid terms with powers of $\mathbf{k}$ greater than 2 one can perform a Feynman parametrisation:

$$\frac{1}{A_1^{m_1} A_2^{m_2} \ldots A_n^{m_n}} = \int_0^1 dx_1 \ldots dx_n \delta\left(1 - \sum x_i\right) \frac{\prod x_i^{m_i - 1}}{\left(\sum x_i A_i\right)^{\sum m_i}} \frac{\Gamma\left(\sum m_i\right)}{\prod \Gamma\left(m_i\right)} \tag{4.6}$$

5 Explicit calculation of a one-loop diagram

Equipped with the formulae of section 4 we can now proceed with a detailed evaluation of a one-loop diagram:

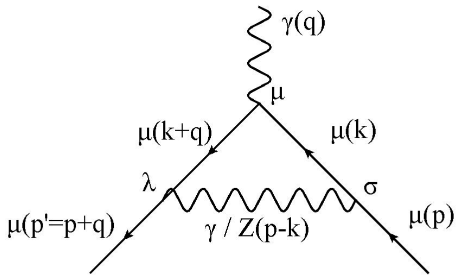


Figure 4: one-loop diagram with photon/Z boson exchange

Using the Feynman rules (see appendix A) for the diagram we find

$$\begin{aligned}
&\bar{u}(\mathbf{p}+\mathbf{q}) \int \frac{d^4\mathbf{k}}{(2\pi)^4} \frac{ie}{c_w s_w} \gamma^\lambda \left(c_1 + c_2\gamma^5\right) \frac{i\left(\not{k} + \not{q} + m_\mu\right)}{(\mathbf{k}+\mathbf{q})^2 - m_\mu^2} (-ie\gamma^\mu) \times \\
&\times \frac{i\left(\not{k} + m_\mu\right)}{\mathbf{k}^2 - m_\mu^2} \frac{ie}{c_w s_w} \gamma^\sigma \left(c_1 + c_2\gamma^5\right) \frac{-i\eta_{\lambda\sigma}}{(\mathbf{p}-\mathbf{k})^2 - m_Z^2} u(\mathbf{p}) \\
&= \bar{u}(\mathbf{p}+\mathbf{q}) \frac{-e^3}{c_w^2 s_w^2} \times \\
&\times \int \frac{d^4\mathbf{k}}{(2\pi)^4} \frac{\gamma^\lambda \left(c_1 + c_2\gamma^5\right) \left(\not{k} + \not{q} + m_\mu\right) \gamma^\mu \left(\not{k} + m_\mu\right) \gamma_\lambda \left(c_1 + c_2\gamma^5\right)}{\left((\mathbf{k}+\mathbf{q})^2 - m_\mu^2\right)\left(\mathbf{k}^2 - m_\mu^2\right)\left((\mathbf{p}-\mathbf{k})^2 - m_Z^2\right)} u(\mathbf{p})
\end{aligned} \tag{5.1}$$

where the parameters c_1, c_2, c_w, c_s and m_Z will allow us to recover the result for either exchange particle. After using the Feynman parametrisation (4.6) and evaluating one of the integrals with the help of the delta function the momentum integral becomes

$$\begin{aligned}
&2\int_0^1 dx dz \Theta(1-x-z) \times \\
&\times \int \frac{d^4\mathbf{k}}{(2\pi)^4} \frac{\gamma^\lambda \left(c_1 + c_2\gamma^5\right) \left(\not{k} + \not{q} + m_\mu\right) \gamma^\mu \left(\not{k} + m_\mu\right) \gamma_\lambda \left(c_1 + c_2\gamma^5\right)}{\left(\mathbf{k}^2 + 2\mathbf{k}\cdot(x\mathbf{q} - z\mathbf{p}) + x\mathbf{q}^2 - m_\mu^2 + z m_\mu^2 + z(\mathbf{p}^2 - m_Z^2)\right)^3}.
\end{aligned} \tag{5.2}$$

Since the external momenta $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{p}+\mathbf{q}$ are on shell, one finds the relations $\mathbf{p}^2 = m_\mu^2$, $\mathbf{q}^2 = 0$ and $\mathbf{p}\cdot\mathbf{q} = 0$. Using these, the denominator simplifies to

$$\left((\mathbf{k}+x\mathbf{q}-z\mathbf{p})^2-(1-z)^2m_\mu^2-zm_Z^2\right)^3 \quad .$$

After shifting $\mathbf{k}\to\mathbf{k}-x\mathbf{q}+z\mathbf{p}$ this becomes

$$\left(\mathbf{k}^2-A\right)^3$$

where $A=(1-z)m_\mu^2+zm_Z^2>0$.
Looking at the numerator before the shift one finds

$$\begin{aligned}
&\gamma^\lambda\left(c_1+c_2\gamma^5\right)\left(\not{k}+\not{q}+m_\mu\right)\gamma^\mu\left(\not{k}+m_\mu\right)\gamma_\lambda\left(c_1+c_2\gamma^5\right)\\
&=c_1^2\gamma^\lambda\left(\not{k}+\not{q}+m_\mu\right)\gamma^\mu\left(\not{k}+m_\mu\right)\gamma_\lambda\\
&+c_2^2\gamma^\lambda\left(\not{k}+\not{q}-m_\mu\right)\gamma^\mu\left(\not{k}-m_\mu\right)\gamma_\lambda\\
&+c_1c_2\gamma^\lambda\left[\left(\not{k}+\not{q}+m_\mu\right)\gamma^\mu\left(\not{k}+m_\mu\right)+\left(\not{k}+\not{q}-m_\mu\right)\gamma^\mu\left(\not{k}-m_\mu\right)\right]\gamma_\lambda\gamma^5 \quad .
\end{aligned}$$

At this point it is easiest to drop the γ^5 term due to parity reasons outlined in [3]. The fact that it does indeed vanish can be easily verified using the projection formula (2.16). One is then left with

$$\begin{aligned}
&(c_1^2+c_2^2)\gamma^\lambda\left(\not{k}\gamma^\mu\not{k}+\not{q}\gamma^\mu\not{k}+m_\mu^2\gamma^\mu\right)\gamma_\lambda\\
&+(c_1^2-c_2^2)\gamma^\lambda\left(m_\mu\not{k}\gamma^\mu+m_\mu\not{q}\gamma^\mu+m_\mu\gamma^\mu\not{k}\right)\gamma_\lambda
\end{aligned}$$

Using the Chisholm identity (4.4) yields

$$-2(c_1^2+c_2^2)\left(\not{k}\gamma^\mu\not{k}+\not{k}\gamma^\mu\not{q}+m_\mu^2\gamma^\mu\right)+4(c_1^2-c_2^2)\left(m_\mu k^\mu+m_\mu q^\mu+m_\mu k^\mu\right) \quad .$$

As seen in section 2.1 terms proportional only to γ^μ can be dropped as they do not contribute to a_μ. After sorting the strings of Gamma matrices and only keeping the relevant terms the numerator becomes

$$-2(c_1^2+c_2^2)\left(2\not{k}k^\mu-\not{k}\not{q}\gamma^\mu+2\not{k}q^\mu\right)+4m_\mu(c_1^2-c_2^2)\left(2k^\mu+q^\mu\right) \quad .$$

Now the shift $\mathbf{k}\to\mathbf{k}-x\mathbf{q}+z\mathbf{p}$ can be executed and terms linear in $\mathbf{k}$ can be dropped, because the integral over them vanishes for symmetry reasons.

One is left with

$$
\begin{aligned}
&-2(c_1^2+c_2^2)\left(2\not{k}k^\mu - 2x\not{q}(zp^\mu - xq^\mu)+2z\not{p}(zp^\mu - xq^\mu) - z\not{p}\not{q}\gamma^\mu + 2(z\not{p} - x\not{q})q^\mu\right) \\
&+4m_\mu(c_1^2-c_2^2)\left(2zp^\mu - 2x^\mu + q^\mu\right) \quad .
\end{aligned}
$$

Using the relations (4.5) this can be simplified to

$$
\begin{aligned}
&-4(c_1^2+c_2^2)\left(\not{k}k^\mu + z^2 m_\mu p^\mu + z m_\mu p^\mu - xz m_\mu q^\mu + z m_\mu q^\mu\right) \\
&+4m_\mu(c_1^2-c_2^2)\left(2zp^\mu - 2x^\mu + q^\mu\right) \quad .
\end{aligned}
$$

Equation 4.2 shows that the integral over the term $\not{k}k^\mu$ is proportional to $\gamma_\nu \eta^{\nu\mu}$ and can thus be dropped. The numerator then is

$$
\begin{aligned}
4m_\mu \Big[& c_1^2\left(z(1-z)p^\mu + (xz - 2x - z + 1)q^\mu\right) \\
& + c_2^2\left((-z^2 - 3z)p^\mu + (2x - 1 + xz - z)q^\mu\right)\Big]
\end{aligned}
$$

which is now independent of $\mathbf{k}$. The momentum integral can then be solved using equation 4.1:

$$
\int \frac{d^4\mathbf{k}}{(2\pi)^4} \frac{1}{\left(\mathbf{k}^2 - A\right)^3} = \frac{-i}{32\pi^2 A}
$$

Putting everything together we find

$$
\begin{aligned}
&\bar{u}(\mathbf{p}+\mathbf{q})\frac{i m_\mu e^3}{4c_w^2 s_w^2 \pi^2}\left[\int_0^1 dx dz \Theta(1-x-z)\frac{c_1^2\left(z(1-z)p^\mu + (xz-2x-z+1)q^\mu\right)}{m_\mu^2(1-z)^2 + z m_Z^2}\right. \\
&\qquad \left. + \int_0^1 dx dz \Theta(1-x-z)\frac{c_2^2\left((-z^2-3z)p^\mu + (2x-1+xz-z)q^\mu\right)}{m_\mu^2(1-z)^2 + z m_Z^2}\right] u(\mathbf{p}) \\
&= \bar{u}(\mathbf{p}+\mathbf{q})\frac{i m_\mu e^3}{8c_w^2 s_w^2 \pi^2}(2p^\mu + q^\mu)\int_0^1 dz \frac{c_1^2 z(1-z)^2 + c_2^2 z(-3+2z+z^2)}{m_\mu^2(1-z)^2 + z m_Z^2} u(\mathbf{p}) \quad .
\end{aligned}
$$

From the Gordon identity (2.6) we know that we can replace $(2p^\mu + q^\mu)$ by $-i\sigma^{\mu\nu}q_\nu$ and by comparison with equations 2.8 and 2.14 we find

$$
a_\mu^{(1,\gamma/Z)} = \frac{m_\mu^2 e^2}{4c_w^2 s_w^2 \pi^2}\int_0^1 dz \frac{c_1^2 z(1-z)^2 + c_2^2 z(-3+2z+z^2)}{m_\mu^2(1-z)^2 + z m_Z^2} \quad . \tag{5.3}
$$

5.1 Photon

By setting the parameters $m_Z = 0$, $c_1 = -1$, $c_2 = 0$ and $c_w = s_w = 1$ we can evaluate the diagram in figure 4 for the photon case. The result is

$$a_\mu^{(1,\gamma)} = \frac{e^2}{8\pi^2} = \frac{\alpha}{2\pi} \tag{5.4}$$

where $\alpha = e^2/4\pi$ is the fine-structure constant. This confirms the Schwinger result (eq. 3.1).

5.2 Z boson

The parameter choice $c_w = \cos(\Theta_W)$, $s_w = \sin(\Theta_W)$, $c_1 = \frac{1}{4} - \sin^2(\Theta_W)$, $c_2 = \frac{1}{4}$ together with the approximation $\frac{m_\mu^2}{m_Z^2} \approx 0$ yields the result for the one-loop Z boson exchange (fig. 4):

$$\begin{aligned} a_\mu^{(1,Z)} &= \frac{g^2 m_\mu^2}{192\pi^2 m_W^2}((1 - 4\sin^2(\Theta_w))^2 - 5) \\ &= \frac{\sqrt{2} G_F m_\mu^2}{16\pi^2} \frac{(1 - 4\sin^2(\Theta_w))^2 - 5}{3} \end{aligned} \tag{5.5}$$

where $m_W = cos(\Theta_W) m_Z$, $g = e/sin(\Theta_W)$ and $G_F = \frac{\sqrt{2}}{8}\frac{g^2}{m_W^2}$ is the Fermi coupling constant.

6 JFeyn

As seen in section 5, the evaluation of even a single one-loop diagram is quite tedious. Matters become worse for higher numbers of loops, since calculations become more difficult and more diagrams have to be taken into account. It is then natural to seek ways to automate the calculations. To this end I have written *JFeyn*, a program capable of algebraically manipulating four-vectors, Gamma matrices, traces and a range of common momentum integrals.

6.1 Structure

JFeyn is programmed in Java and was developed with an object-oriented approach in mind. At its heart is the top-level class *Term*. Every mathematical term is represented by an object of an appropriate subclass to *Term*. Classes range from basic algebraic constructs such as *Sum*, *Product* or *Power* to specialised classes like *GammaMatrix*, *GammaFunction* or *Integral*. Most classes can have subterms called children, such as the factors in a product or the limits and integrand of an integral. The concept is illustrated in figure 5.

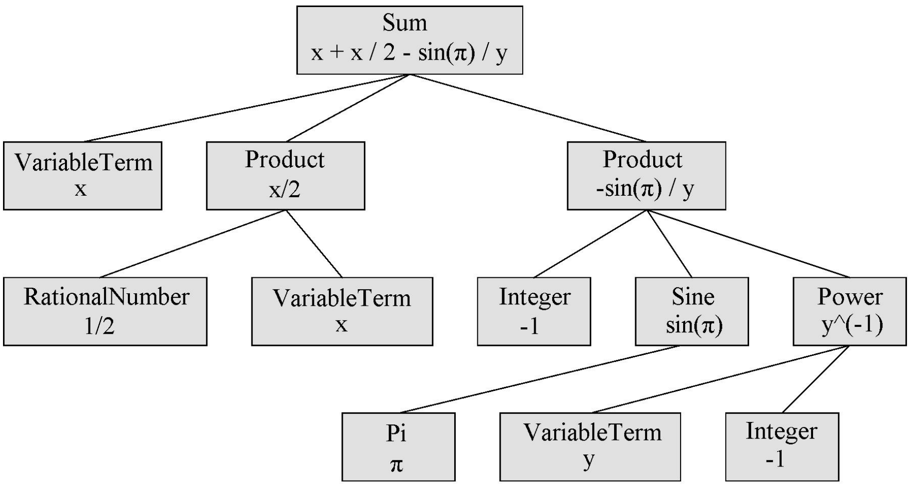


Figure 5: Internal representation of the term $x + \frac{x}{2} - \frac{sin(\pi)}{y}$

This approach enables *JFeyn* to analytically determine the order of calculations rather than rely on a predetermined course of action. Instead of following an external control flow, *JFeyn* tries to simplify a given term as much as possible. This requires every mathematical object to be capable of simplifying itself. For this purpose the class *Term* implements the method `simplify()`:

```
public Term simplify(){
        //simplify all children of this term
        simplifyChildren();

        //create a copy that can be safely modified
        Term x = this.cloneTerm();

        //simplifications provided by the class of this term
        Simplification sim = x.simplifySelf();
        if(sim.easier){ //if a simplification was found
            x = sim.term;
        }

        //simplifications provided by the term's children
        for(int i = 0; i < x.children.size(); i++){
            sim = x.children.get(i).simplifyParent(x, i);
            if(sim.easier){
                x = sim.term;
                break;
            }
        }

        return x;
    }
```

When called upon to simplify, a term thus first ensures that all its children are already simplified. This bottom-up approach is used for its performance benefits over a top-down strategy.
The next step is to check whether any simplifications specific to the term's class can be performed via the method `simplifySelf()`. Examples of such simplifications are the combining of powers with the same base within a product or the returning of known values of a trigonometric function.
Another set of simplifications is triggered not by the class of a term itself, but by its children. This improves performance by reducing the amount of checks the program needs to do. When one considers the identity

$$\gamma^5\gamma^5 = \mathbb{1}_4 \quad ,$$

it is obvious that a product is a much more common term than γ^5. Having every product check for this identity would be very ineffiecient. Instead, the class *GammaFive* triggers the check in its method `simplifyParent()`, if the enclosing term is a product.

By default *JFeyn* uses the method `completeSimplify()` on an input term, which calls the method `simplify()` until input and output of the latter are the same.

6.2 Examples

6.2.1 Chisholm Identity

An important simplification to strings of Gamma matrices is given by the Chisholm identity (4.4). In *JFeyn* it is triggered when a product detects two matching indices. When this happens, the `contractWith()` method of the first term with the relevant index is called with the second term and all terms in between as arguments. The method then checks whether any terms in between can commute or anticommute to the outside. If after this step any terms are left between the two matching indices, the method `contractWithNonCommutingInBetween()` is called:

```
public Simplification contractWithNonCommutingInBetween(Term x,
         TermList termsInBetween) {
      boolean chisholm = true;
      if(!(x instanceof GammaMatrix)){
         chisholm = false;
      }
      for(int i = 0; chisholm && i < termsInBetween.size(); i++){
         if(!(termsInBetween.get(i) instanceof GammaMatrix)){
            chisholm = false;
         }
      }
      if(chisholm){
         int number = termsInBetween.size();
         if(number % 2 == 0){ //even number in between
            TermList backwards = termsInBetween.cloneTerms();
            TermList cycled = termsInBetween.cloneTerms();
            backwards.reverseOrder();
            //cycling the list forwards by 1 position
```

```
            cycled.cycle(true);
            backwards.cycle(false); //cycling backwards
            Product p1 = new Product(cycled);
            Product p2 = new Product(backwards);
            TermList list = new TermList();
            list.add(p1.multiplyWith(new IntegerTerm(2)));
            list.add(p2.multiplyWith(new IntegerTerm(2)));
            Product p3 = new Product(termsInBetween);
            Term d4 = Settings.getFourDimension().addTo(
               new IntegerTerm(-4)); //d-4
            list.add(p3.multiplyWith(d4));

            Term ret = new Sum(list);
            return new Simplification(ret);
         }
         else{
         ...(odd number in between)}
      }
      return new Simplification(); //no simplification found
   }
```

The following results are taken directly from the programs LaTeX-output:

$$\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\nu\gamma^\mu = -2\gamma^\nu\gamma^\beta\gamma^\alpha + 4\gamma^\alpha\gamma^\beta\gamma^\nu - d\gamma^\alpha\gamma^\beta\gamma^\nu$$
$$\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\nu\gamma^\lambda\gamma^\mu = -4\gamma^\alpha\gamma^\beta\gamma^\nu\gamma^\lambda + 2\gamma^\lambda\gamma^\alpha\gamma^\beta\gamma^\nu + 2\gamma^\nu\gamma^\beta\gamma^\alpha\gamma^\lambda + d\gamma^\alpha\gamma^\beta\gamma^\nu\gamma^\lambda$$

6.2.2 Traces of Gamma matrices

Since the program makes use of the projection formula (2.16), it needs to be able to evaluate traces of Gamma matrices. It is easy to show (see [3]) that traces of an odd number of Gamma matrices vanish. Using the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_4$ and the cylicity of the trace one finds for an even number n of Gamma matrices:

$$Tr\left[\gamma^{\mu_1}\cdots\gamma^{\mu_n}\right] = \sum_{k=2}^{n}(-1)^k\,\eta^{\mu_1\mu_k}Tr\left[\gamma^{\mu_2}\cdots\gamma^{\mu_{k-1}}\gamma^{\mu_{k+1}}\cdots\gamma^{\mu_n}\right] \tag{6.1}$$

where the right-hand side traces now contain $n-2$ Gamma matrices each. *JFeyn* uses this identity in a recursive algorithm that is able to evaluate traces of any even number of Gamma matrices:

```
private Term traceGammaMatrices(IndexList indices){
   if(indices.size() == 0){
      //trace over remaining identity matrix
      return new IntegerTerm(4);
   }
   TermList sumList = new TermList();
   for(int i = 1; i < indices.size(); i++){
      //contraction of two Gamma matrices
      Term eta = new MinkowskiMetric(indices.get(0),
         indices.get(i));
      if(i % 2 == 0){
         //add factor of -1 for every second contraction
         eta = eta.multiplyWith(new IntegerTerm(-1));
      }
      //create list of indices for next level of recursion
      IndexList shortenedList = indices.cloneIndices();
      shortenedList.remove(i);
      shortenedList.remove(0);
      //add term to sum
      sumList.add(eta.multiplyWith(
         traceGammaMatrices(shortenedList)));
   }
   return new Sum(sumList);
}
```

Sample results taken from the LaTeX-output are given below:

$$
\begin{aligned}
Tr\left[\gamma^{\alpha}\gamma^{\beta}\gamma^{\nu}\gamma^{\mu}\right] &= -4\eta^{\alpha\nu}\eta^{\beta\mu} + 4\eta^{\alpha\beta}\eta^{\mu\nu} + 4\eta^{\alpha\mu}\eta^{\beta\nu} \\
Tr\left[\gamma^{\alpha}\gamma^{\beta}\gamma^{\nu}\gamma^{\mu}\gamma^{\sigma}\right] &= 0 \\
Tr\left[\gamma^{\alpha}\gamma^{\beta}\gamma^{\nu}\gamma^{\mu}\gamma^{\sigma}\gamma^{\lambda}\right] &= -4\eta^{\alpha\beta}\eta^{\lambda\mu}\eta^{\nu\sigma} - 4\eta^{\alpha\lambda}\eta^{\beta\mu}\eta^{\nu\sigma} - 4\eta^{\alpha\mu}\eta^{\beta\sigma}\eta^{\lambda\nu} \\
&\quad - 4\eta^{\alpha\nu}\eta^{\beta\lambda}\eta^{\mu\sigma} - 4\eta^{\alpha\nu}\eta^{\beta\mu}\eta^{\lambda\sigma} - 4\eta^{\alpha\sigma}\eta^{\beta\lambda}\eta^{\mu\nu} \\
&\quad - 4\eta^{\alpha\sigma}\eta^{\beta\nu}\eta^{\lambda\mu} + 4\eta^{\alpha\beta}\eta^{\lambda\nu}\eta^{\mu\sigma} + 4\eta^{\alpha\beta}\eta^{\lambda\sigma}\eta^{\mu\nu} \\
&\quad + 4\eta^{\alpha\lambda}\eta^{\beta\nu}\eta^{\mu\sigma} + 4\eta^{\alpha\lambda}\eta^{\beta\sigma}\eta^{\mu\nu} + 4\eta^{\alpha\mu}\eta^{\beta\lambda}\eta^{\nu\sigma} \\
&\quad + 4\eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\lambda\sigma} + 4\eta^{\alpha\nu}\eta^{\beta\sigma}\eta^{\lambda\mu} + 4\eta^{\alpha\sigma}\eta^{\beta\mu}\eta^{\lambda\nu}
\end{aligned}
$$

Unfortunately, the above recursion identity (6.1) does not hold for traces that involve γ^5. Using the four-dimensional identity

$$\gamma^5 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma \quad [3] \tag{6.2}$$

where $\epsilon^{\mu\nu\rho\sigma}$ ($\epsilon^{0123} = +1$) is the totally antisymmetric tensor, it is straightforward to show that

$$Tr\left[\gamma^{\mu_1}\cdots\gamma^{\mu_n}\gamma^5\right] = 0 \qquad \text{for } n < 4,\ n \text{ odd.} \tag{6.3}$$

For even $n \geq 4$ one could use eq. 6.2 to reduce the problem to evaluating a trace over Gamma matrices without γ^5. However, since a trace over an even number n Gamma matrices yields $(n-1)!!$ terms (of which many would vanish afterwards when contracted with $\epsilon^{\mu\nu\rho\sigma}$), adding an extra 4 Gamma matrices is very inefficient. Instead, *JFeyn* makes use of the identity

$$\gamma^\mu\gamma^\nu\gamma^\lambda = \eta^{\mu\nu}\gamma^\lambda + \eta^{\nu\lambda}\gamma^\mu - \eta^{\mu\lambda}\gamma^\nu + i\epsilon^{\mu\nu\lambda\sigma}\gamma_\sigma\gamma^5. \tag{6.4}$$

It can be easily verified by checking all possible choices for the indices. Every application of this identity reduces the amount of Gamma matrices in the trace by 2, until one is left with either traces that involve γ^5 and less than 4 Gamma matrices (and therefore vanish) or traces that do not involve γ^5. The first two nontrivial results calculated by *JFeyn* are

$$\begin{aligned}
Tr\left[\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\mu\gamma^5\right] &= -4i\epsilon^{\alpha\beta\lambda\mu} \\
Tr\left[\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^5\right] &= -4i\eta^{\alpha\beta}i\epsilon^{\lambda\mu\nu\sigma} - 4i\eta^{\beta\lambda}\epsilon^{\alpha\mu\nu\sigma} - 4i\eta^{\mu\nu}\epsilon^{\alpha\beta\lambda\sigma} \\
&\quad - 4i\eta^{\nu\sigma}\epsilon^{\alpha\beta\lambda\mu} + 4i\eta^{\alpha\lambda}\epsilon^{\beta\mu\nu\sigma} + 4i\eta^{\mu\sigma}\epsilon^{\alpha\beta\lambda\nu}.
\end{aligned}$$

These results are valid in four dimensions. There are various implementations of a dimensional regularisation for terms involving γ^5 (see [8]), but they are beyond the scope of this thesis.

6.2.3 Tensor integrals

As seen in section 4.2, it is possible to convert integrals of the form

$$\int d^d\mathbf{k}\, k^{\mu_1} k^{\mu_2} \cdots k^{\mu_n} f(\mathbf{k}^2)$$

to ones that do not feature any free indices (see eq. 4.3). If the integration variable of an integral is a four-vector and the integrand is a product, *JFeyn* analyses every factor sequentially. Indices of factors k^{μ_i} are stored in a list, while all other factors are checked to only contain terms of $\mathbf{k}^2$. If the requirements are met, a symmetric sum of Minkowski metrics is generated from the list of indices (0 for odd numbers) and the integral is converted to the form seen in eq. 4.3. Results are then of the form

$$\int d^d\mathbf{k}\, \frac{k^\mu k^\nu}{\left(\mathbf{k}^2 - m^2\right)^3} = \frac{\eta^{\mu\nu}}{d} \int d^d\mathbf{k}\, \frac{\mathbf{k}^2}{\left(\mathbf{k}^2 - m^2\right)^3}$$

$$\int d^d\mathbf{k}\, \frac{k^\alpha k^\beta k^\mu k^\nu}{\left(\mathbf{k}^2 - m^2\right)^3} = \frac{\eta^{\alpha\beta}\eta^{\mu\nu} + \eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu}}{d\,(2+d)} \int d^d\mathbf{k}\, \frac{\left(\mathbf{k}^2\right)^2}{\left(\mathbf{k}^2 - m^2\right)^3} \quad .$$

6.2.4 Momentum integrals

The most complex task in the evaluation of a Feynman diagram usually is the integration over the internal momenta. When *JFeyn* encounters such an integral, it first checks whether a Feynman parametrisation (see eq. 4.6) is needed. Afterwards, the integrand is of the form

$$\frac{f(\mathbf{k})}{\left(a\mathbf{k}^2 + 2k^\mu b_\mu + c\right)^n}$$

where $\mathbf{k}$ is the integration variable. The square in the denominator is then completed and $\mathbf{k}$ is transformed to $\frac{\mathbf{k}-\mathbf{b}}{\sqrt{a}}$ to yield a denominator of the form

$$\left(\mathbf{k}^2 - A\right)^n \quad .$$

The integral can then be reduced to the standard momentum integrals seen in eq. 4.1 by dropping odd powers of $\mathbf{k}$ in the numerator for symmetry reasons and applying eq. 4.2 where needed. Remaining integrals are then solved by the class *Integral* using its built-in library. The relevant library term in this case is

$$\left((?\mathbf{x})^2\right)^{?m\{-?\mathbf{x},\#\text{integer}\}}\left((?\mathbf{x})^2+?a\{<0,-?\mathbf{x}\}\right)^{?n\{-?\mathbf{x},\#\text{integer}\}}$$

where $?\mathbf{x}$ is a wildcard for the integration variable and $?a$, $?m$ and $?n$ are further wildcard terms. The curly brackets denote constraints on the wildcard terms. In this particular case they all need to be independent of the integration variable, $?m$ and $?n$ need to be integers and we also require $?a < 0$.
The method `checkLibrary2()` then checks whether a given integrand matches the library term and returns the solution given in eq. 4.1 if it does:

```
private Simplification checkLibrary2(){
   if(metric instanceof MinkowskiMetric && isOverWholeRoom()){
      Term integrand = children.get(0);
      Fit fit = library2.fits(integrand, new Fit());
      if(fit.fits){ //integrand is a match
         //get wildcard term values
         Term m = fit.getTerm("m");
         Term n = fit.getTerm("n").multiplyWith(
            new IntegerTerm(-1));
         Term a = fit.getTerm("a").multiplyWith(
            new IntegerTerm(-1));

         Term d = Settings.getFourDimension();

         //check level of divergence
         int n1 = ((IntegerTerm)n).getNumber();
         int m1 = ((IntegerTerm)m).getNumber();
         if(d.equalsTerm(new IntegerTerm(4)))
         if(4+2*m1-2*n1 > 0){
            return new Simplification();
         }

         [... Construction of solution "ret"
            using m, n, a and d]
         return new Simplification(ret);
      }
   }
   return new Simplification();
}
```

The process described above yields the following results:

$$\int \frac{d^d\mathbf{k}}{\left(\mathbf{k}^2 - m^2\right)^3} = -\frac{i}{2}\Gamma\left(3 - \frac{d}{2}\right)\pi^{\frac{d}{2}} m^{(-6+d)}$$

$$\int \frac{d^d\mathbf{k}\,\mathbf{k}^2}{\left(\mathbf{k}^2 - m^2 - 2\mathbf{p}\cdot\mathbf{k}\right)^3} = -\frac{i}{2}\mathbf{p}^2\Gamma\left(3 - \frac{d}{2}\right)\pi^{\frac{d}{2}}\left(m^2 + \mathbf{p}^2\right)^{\left(-3+\frac{d}{2}\right)} + \frac{i}{2}\frac{\Gamma\left(1+\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\pi^{\frac{d}{2}}\left(m^2 + \mathbf{p}^2\right)^{\left(-2+\frac{d}{2}\right)}$$

$$\int \frac{d^d\mathbf{k}}{\left(\mathbf{k}^2 - M^2\right)\left(\mathbf{k}^2 - m^2\right)^2} = -i\Gamma\left(3 - \frac{d}{2}\right)\pi^{\frac{d}{2}} \times \times \int_0^1 dx_2\, x_2 \left(M^2 - x_2 M^2 + x_2 m^2\right)^{\left(-3+\frac{d}{2}\right)}$$

where m and M have been declared to have real values.

6.3 Evaluation of Feynman Diagrams

To determine the contribution of a single Feynman diagram, *JFeyn* requires the vertex function $\Gamma_\mu(\mathbf{P},\mathbf{q})$ (see sect. 2.2) as input. It then calculates

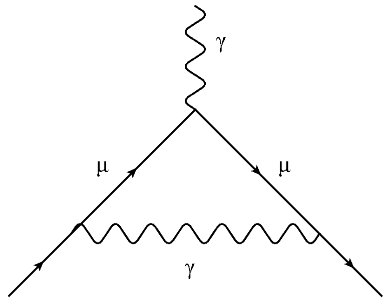
$$V_\mu(\mathbf{p}) = \Gamma_\mu(\mathbf{P},0) \quad \text{and} \quad T_{\nu\mu}(\mathbf{p}) = \frac{\partial}{\partial q^\nu}\Gamma_\mu(\mathbf{P},\mathbf{q})|_{\mathbf{q}=0}$$

and applies the projection formula seen in eq. 2.16. Since certain identities depend on the sign of a given term, the program also requires the additional information that the mass terms are real (and positive). It is also useful to provide the on shell condition $\mathbf{p}^2 = m_\mu^2$ from the start.

The whole expression for a_μ is first simplified by calling `completeSimplify()` with integral evaluation disabled. All traces are calculated in this step. Since many of them vanish, the program is prevented from evaluating the integrals they multiply for efficiency reasons.
`completeSimplify()` is then called again with integral evaluation enabled. All momentum integrals are calculated in d dimensions as seen in section 6.2.4 and d is set to be $4-2\epsilon$ afterwards to provide a dimensional regularisation. If masses other than m_μ appear in the diagram, it might at this point be necessary to use the approximation $\frac{m_1}{m_2} \approx 0$ to be able to analytically solve all integrals over the Feynman parameters (see e.g. calculation in sect. 5).

This is done by Taylor expanding integrands in $\frac{m_1^2}{m_2^2}$ and only keeping the first two nonvanishing terms. Higher orders have been included in test runs and have been shown not to affect the results for any of the one-loop diagrams. After the expansion a last call of `completeSimplify()` is performed to solve the remaining integrals.
Due to *JFeyn's* limited factorising capabilities the results are usually lengthy terms involving rational functions and Gamma functions of ϵ. The output is then written to a file and imported to Mathematica. There it is simplified and finally the limit $\epsilon \to 0$ is taken.

6.3.1 Photon diagram



The easiest contribution to calculate is the Schwinger term, where no Taylor expansion is needed. After simplification in Mathematica the result is

```
In[1]:= << "JFeyn\\1loop-photon.txt"
```

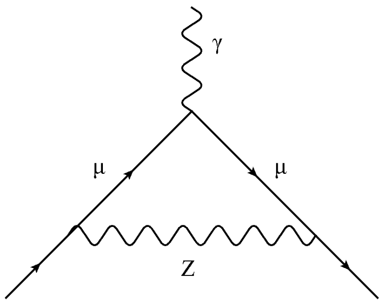
Out[1]= $-\left(2^{-1+2\varepsilon}\, m^{-2\varepsilon}\, \pi^{-1+\varepsilon}\, \alpha\, \left((-1+2\varepsilon)\,\text{Gamma}[3-\varepsilon]\,\text{Gamma}[\varepsilon] + 2\,\text{Gamma}[2-\varepsilon]\left(\left(1-3\varepsilon+2\varepsilon^2\right)\text{Gamma}[\varepsilon] + \left(-1-3\varepsilon+2\varepsilon^2\right)\text{Gamma}[1+\varepsilon]\right)\right)\right) \Big/ \left(\left(3-8\varepsilon+4\varepsilon^2\right)\text{Gamma}[2-\varepsilon]\right)$

```
In[2]:= Limit[a, ε → 0]
```

Out[2]= $\frac{\alpha}{2\pi}$

which is in agreement with the manual calculation (eq. 5.4) and Schwinger's original result (eq. 3.1).

6.3.2 Z boson diagram



As seen in section 5, the one-loop Z boson exchange has a similar vertex function to the photon exchange, but needs the approximation $m_Z \gg m_\mu$. The result then is

In[1]:= `<< "JFeyn\\1loop-Z.txt"`

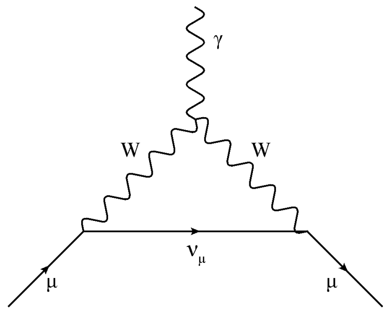
Out[1]=
$$\Big((-1)^{-2\,\varepsilon}\, 2^{-\frac{1}{2}+2\,\varepsilon}\, m^2\, \pi^{-2+\varepsilon}\, \text{Cos}[\theta_W]^{2\,\varepsilon}\, \text{Gamma}[1+\varepsilon]\, \big(1 - (-2+11\,\varepsilon-6\,\varepsilon^2+\varepsilon^3)\, \text{Sin}[\theta_W]^2 + 2\,(-2+11\,\varepsilon-6\,\varepsilon^2+\varepsilon^3)\, \text{Sin}[\theta_W]^4\big)\, G_F\, m_W^{-2\,\varepsilon}\Big) \Big/ \big((-3+\varepsilon)\,(-2+\varepsilon)\,(-1+\varepsilon)\big)$$

In[2]:= `Limit[a, ε → 0]`

Out[2]=
$$\frac{m^2\,(-1-2\,\text{Cos}[2\,\theta_W]+\text{Cos}[4\,\theta_W])\,G_F}{12\,\sqrt{2}\,\pi^2}$$

which can be shown to be identical to the result in eq. 5.5 by applying $\cos(2x) = 1 - 2\sin^2(x)$.

6.3.3 W boson diagram



JFeyn's result for the W boson diagram is given by

In[1]:= `<< "JFeyn\\1loop-W.txt"`

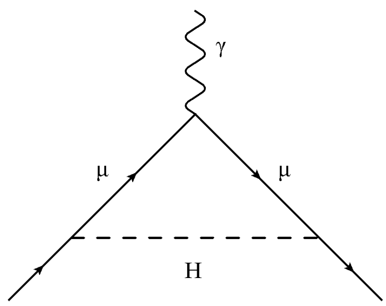
Out[1]=
$$\Big(2^{-\frac{5}{2}+2\,\varepsilon}\; m^2\; \pi^{-2+\varepsilon}\;\big((-3+\varepsilon)\;\varepsilon\;\mathrm{Gamma}[3-\varepsilon]\;\mathrm{Gamma}[\varepsilon]\;-\;(14-9\,\varepsilon+\varepsilon^2)\;\mathrm{Gamma}[2-\varepsilon]\;\mathrm{Gamma}[1+\varepsilon]\big)\;G_F\;m_W^{-2\,\varepsilon}\Big)\Big/\Big((-3+\varepsilon)\;(-2+\varepsilon)^2\;\mathrm{Gamma}[2-\varepsilon]\Big)$$

In[2]:= `Limit[a, ε → 0]`

Out[2]=
$$\frac{5\;m^2\;G_F}{12\;\sqrt{2}\;\pi^2}$$

This is consistent with the result given in [1] (see also eq. 3.5).

6.3.4 Higgs boson diagram



For the Higgs boson exchange a Taylor series approach in $\frac{m_\mu}{m_H}$ on the integrand level leads to divergent integrals. The result before performing the remaining integral is:

```
In[1]:= << "JFeyn\\1loop-H.txt"
```

Out[1]= $\texttt{Integral}\left[2^{-\frac{5}{2}+2\varepsilon}\, m^4\, \pi^{-2+\varepsilon}\, (-2+x)\, x^2 \left(mh^2\,(-1+x) - m^2 x^2\right)^{-1-\varepsilon} \texttt{Gamma}[1+\varepsilon]\, G_F,\ \{x,0,1\}\right]$

```
In[2]:= ε = 0;
        a
```

Out[3]= $\texttt{Integral}\left[\frac{m^4\,(-2+x)\,x^2\,G_F}{4\sqrt{2}\,\pi^2\left(mh^2\,(-1+x) - m^2 x^2\right)},\ \{x,0,1\}\right]$

The result differs from the one given in [1] (see eq. 3.6) by a factor of 2. This might be due to different conventions for the constants, since the same result was obtained in manual calculation (see appendix B).
For $m_\mu \gg m_H$ the remaining integral trivially yields

$$a_\mu^{(1,W)} \approx \frac{\sqrt{2} G_F m_\mu^2}{8\pi^2} \frac{3}{2} \quad .$$

The limit $m_\mu \ll m_h$ is more difficult to perform and results in

$$a_\mu^{(1,W)} \approx \frac{\sqrt{2} G_F m_\mu^2}{8\pi^2} \frac{m_\mu^2}{m_H^2} \ln \frac{m_H^2}{m_\mu^2}$$

as can be seen in Mathematica:

In[1]:= $\texttt{b = FullSimplify}\left[\texttt{Integrate}\left[\frac{(-2+x)\,x^2}{(-1+x) - t\,x^2},\ \{x,0,1\}\right] \Big/ \texttt{Log}[1/t],\ \texttt{Assumptions} \to \{t < 0.1,\ t > 0\}\right]$

Out[1]= $\frac{1}{2\,t^3\sqrt{-1+4t}\ \texttt{Log}[t]}\Big(2\,(-1+t)\,(-1+4t)\,\texttt{ArcCot}\left[\sqrt{-1+4t}\right] + 2\,(-1+t)\,(-1+4t)\,\texttt{ArcTan}\left[\frac{-1+2t}{\sqrt{-1+4t}}\right] + \sqrt{-1+4t}\ \left((2-3t)\,t + \texttt{Log}[t] - 3t\,\texttt{Log}[t]\right)\Big)$

```
In[2]:= Limit[b, t → 0]

Out[2]= 1
```

where $t = \frac{m_\mu^2}{m_H^2}$.

6.4 Performance

Since one-loop calculations are relatively short compared to the contributions of higher orders, *JFeyn* was created with a focus on versatility rather than optimum performance. All calculations were done on an Intel® Core^{TM}2 Duo Processor P8600 (3M Cache, 2.40 GHz, 1066 MHz FSB) and finished within a reasonable amount of time.

Table 2: Calculation times for traces of Gamma matrices in ms

# of γs	simplify()	completeSimplify()
6	0.08	46
$6 + \gamma^5$	0.07	15
8	0.78	280
$8 + \gamma^5$	0.34	110
10	12	14,000
$10 + \gamma^5$	2.5	1,100
12	190	$1,8 \times 10^6$
$12 + \gamma^5$	31	52,000
14	2,800	
$14 + \gamma^5$	300	

Table 2 lists the calculation times of the methods `simplify()` and `complete-Simplify()` for traces of Gamma matrices. While `simplify()` only returns the initial simplification of the trace, `completeSimplify()` also includes sorting of terms and the attempt to contract indices. From the vast differences in durations we can see that the most time-consuming task seems to be the sorting of terms, which is to be expected for the large numbers of terms resulting from such traces. Traces involving γ^5 are solved quicker, because each iteration of eq. 6.4 yields 4 terms regardless of the number n of Gamma matrices, while the recursion formula 6.1 yields $n - 1$ terms. The difference is thus particularly noticeable for large n.

Table 3: Performance for one-loop diagrams

Diagram	Calculation time [s]	Peak memory usage [Mb]
Photon	13.9	16.3
Z boson	68.4	62.0
W boson	95.9	96.3
Higgs boson	7.1	16.1

Calculation times for the one-loop QED and electroweak contributions to a_μ are given in table 3. They show a strong dependency on the number of terms and especially Gamma matrices in the relevant Feynman rules. This is due mostly to *JFeyn's* limited factorising capabilites and the lack of a database of already performed calculations. These limitations force *JFeyn* to evaluate many similar or identical integrals individually. Solutions to these problems would likely vastly improve performance, but were deemed unnecessary for the evaluation of one-loop diagrams, since none of the calculation times or peak memory values exceed tolerable values.

6.5 Higher loop contributions and possible improvements

To allow for evaluation of higher order contributions to a_μ, certain aspects of *JFeyn* would need to be enhanced. As already seen in section 6.4, several performance improvements could be implemented to decrease the dependency on the number of terms, which could otherwise become a limiting factor for 2-loop diagrams and above.

While according to [8] the lack of a coherent scheme for the dimensional regularisation of γ^5 (see section 6.2.2) does not have any influence on the results for most calculations, it is best to ensure there is no possibility of errors occuring due to this problem.

Since every loop in a diagram adds an additional integral over a four-momentum, higher order contributions feature more and more complex integrals. This would have to be accounted for in *JFeyn's* class *Integral* by implementing the capability to separate integrals over different momenta where possible and extending the library of built-in integrals. It might also be useful to implement simplifications based on integration by parts to decrease the complexity of encountered integrals.

Above the level of one-loop diagrams results start to exhibit divergences that need to be treated using renormalisation [1]. This requires procedures such as integral cutoffs and counterterms, which are not currently implemented in *JFeyn*.

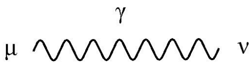
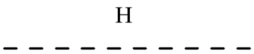
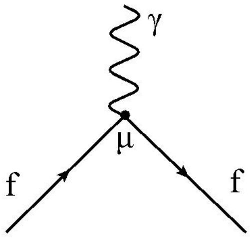
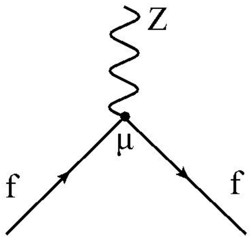
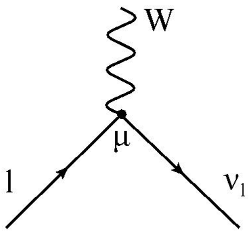
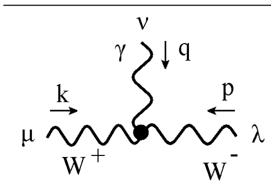
7 Conclusion and outlook

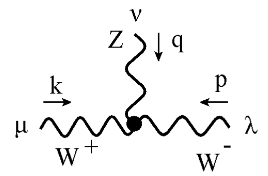
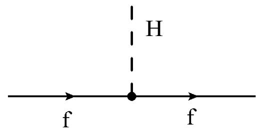
In sections 1-3 of this thesis I gave an overview on the topic of the anomalous magnetic moment of the muon. The current discrepancy of 3.2 standard deviations between the experimental and theoretical values allows room for new theories beyond the Standard Model, several of which are described in [1]. For the near future, experimental precision is expected to improve by approximately one order of magnitude [1]. This would lead to more stringent constraints on new theories, but also requires the theoretical prediction to improve drastically to fully understand the magnitude of the discrepancy between the two values. Automation will play a key role in this process, as the number and complexity of Feynman diagrams that need to be taken into account vastly increase for higher precisions.

JFeyn, the computer algebra system presented in section 6, is able to algebraically perform all calculations necessary to evaluate the one-loop electroweak contributions to a_μ. This includes Taylor series, four-vectors, Gamma matrices, traces thereof and a range of common momentum and one-dimensional integrals. The results for all QED and electroweak one-loop diagrams were presented in section 6.3 and agree with published results [1] (apart from a factor of 2 for the Higgs boson diagram). This proves the viability of an object-oriented computer algebra system for Feynman diagram calculations. A logical next step would be to move on to higher loop contributions. Several requirements and possible improvements towards this were identified in section 6.5.

A Feynman rules

The Feynman rules in this section are taken from [7] and given in the unitary gauge for the 'mostly minus' convention $\eta_{\mu\nu} = \text{Diag}\,[+ - - -]$. The Feynman diagrams have been created with *JaxoDraw* [6].

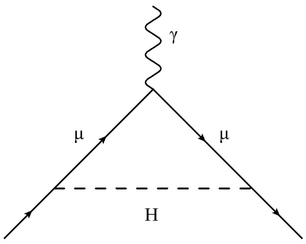
	$\frac{i(\not{p}+m)}{\mathbf{p}^2-m^2+i\epsilon}$	Fermion propagator
	$\frac{-i\eta_{\mu\nu}}{\mathbf{p}^2+i\epsilon}$	Photon propagator
	$\frac{-i\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{m^2}\right)}{\mathbf{p}^2-m^2+i\epsilon}$	Massive spin-1 propagator
	$\frac{i}{\mathbf{p}^2-m_H^2+i\epsilon}$	Higgs propagator
	$ieQ_f\gamma^\mu$	$\gamma f f$-vertex
	$ie_Z\gamma^\mu(g_V + g_A\gamma^5)$	Zff-vertex
	$ie_W\gamma^\mu(1-\gamma^5)$	$Wl\nu_l$-vertex
	$ie\left[\eta^{\mu\lambda}(p-k)^\nu + \eta^{\mu\nu}(k-q)^\lambda + \eta^{\nu\lambda}(q-p)^\mu\right]$	$WW\gamma$-vertex

	$ig\cos\Theta_W\left[\eta^{\mu\lambda}(p-k)^\nu+\eta^{\mu\nu}(k-q)^\lambda+\eta^{\nu\lambda}(q-p)^\mu\right]$	WWZ-vertex
	$-i\frac{m_f}{v}$	Hff-vertex

where $e_Z=\frac{e}{\sin\Theta_W\cos\Theta_W}$, $e_W=\frac{e}{2\sqrt{2}\sin\Theta_W}$, $g_A=\frac{1}{2}T_3,\ g_V=\frac{1}{2}T_3-Q\sin^2\Theta_W$, $g=\frac{e}{\sin\Theta_W}$, $v=\frac{2m_W\sin\Theta_W}{e}$ is the Higgs expectation value and Θ_W is the weak mixing angle. The Fermi coupling constant is given by

$$G_F=\frac{\sqrt{2}g^2}{8m_W^2}=\frac{1}{\sqrt{2}v^2}\quad .$$

B Higgs diagram calculation



The calculation will follow closely the one presented in section 5 and make use of the identities and relations of section 4. Terms proportional only to γ^μ will be dropped without notice. From the Feynman rules (and leaving out spinors) we get:

$$
\begin{aligned}
&\frac{m^2 e}{v^2}\int \frac{d^4\mathbf{k}}{(2\pi)^4}\frac{\not{k}+\not{q}+m}{(\mathbf{k}+\mathbf{q})^2-m^2}\gamma^\mu\frac{\not{k}+m}{\mathbf{k}^2-m^2}\frac{1}{(\mathbf{p}-\mathbf{k})^2-m_H^2}\\
&=\frac{m^2 e}{8\pi^4 v^2}\int_0^1 dxdydz\delta\left(1-x-y-z\right)\times\\
&\quad\times\int d^4\mathbf{k}\frac{\not{k}\gamma^\mu\not{k}+2mk^\mu+\not{q}\gamma^\mu\not{k}+m\not{q}\gamma^\mu}{\left(\mathbf{k}^2-2\mathbf{k}\cdot\left(z\mathbf{p}-x\mathbf{q}\right)+\left(2z-1\right)m^2-zm_H^2\right)^3}\\
&=\frac{m^2 e}{8\pi^4 v^2}\int_0^1 dxdydz\delta\left(1-x-y-z\right)\times\\
&\quad\times\int d^4\mathbf{k}\frac{\left(z\not{p}-x\not{q}\right)\gamma^\mu\left(z\not{p}-x\not{q}\right)+2zmp^\mu-2xmq^\mu+\not{q}\gamma^\mu\left(z\not{p}-x\not{q}\right)+m\not{q}\gamma^\mu}{\left(\mathbf{k}^2-\left(1-z\right)^2m^2-zm_H^2\right)^3}\\
&=-i\frac{m^2 e}{16\pi^2 v^2}\int_0^1 dxdz\Theta\left(1-x-z\right)\times\\
&\quad\times\frac{\left(z\not{p}-x\not{q}\right)\gamma^\mu\left(z\not{p}-x\not{q}\right)+2zmp^\mu-2xmq^\mu+\not{q}\gamma^\mu\left(z\not{p}-x\not{q}\right)+m\not{q}\gamma^\mu}{\left(1-z\right)^2m^2+zm_H^2}\\
&=-i\frac{m^3 e}{8\pi^2 v^2}\int_0^1 dxdz\Theta\left(1-x-z\right)\frac{\left(z^2-1\right)p^\mu+\left(-zx-x\right)q^\mu}{\left(1-z\right)^2m^2+zm_H^2}\\
&=\frac{im^3 e}{16\pi^2 v^2}\left(2p^\mu+q^\mu\right)\int_0^1 dz\frac{\left(2-z\right)z^2}{z^2m^2+\left(1-z\right)m_H^2}\\
&=\frac{m^3 e}{16\pi^2 v^2}\sigma^{\mu\nu}q_\nu\int_0^1 dz\frac{\left(2-z\right)z^2}{z^2m^2+\left(1-z\right)m_H^2}\\
\Rightarrow a_\mu &=\frac{m^4}{8\pi^2 v^2}\int_0^1 dz\frac{\left(2-z\right)z^2}{z^2m^2+\left(1-z\right)m_H^2}=\frac{\sqrt{2}m^4G_F}{8\pi^2}\int_0^1 dz\frac{\left(2-z\right)z^2}{z^2m^2+\left(1-z\right)m_H^2}
\end{aligned}
$$

References

[1] F. Jegerlehner and A. Nyffeler, Phys. Rept. **477** (2009) 1 [arXiv:0902.3360 [hep-ph]].

[2] K. Melnikov and A. Vainshtein, *Berlin, Germany: Springer (2006) 176 p*

[3] M. E. Peskin and D. V. Schroeder, *Reading, USA: Addison-Wesley (1995) 842 p*

[4] N. E. Mavromatos, Lecture notes to *Standard Model Physics and Beyond*, King's College London, (2010)

[5] J. S. Schwinger, Phys. Rev. **73** (1948) 416.

[6] D. Binosi and L. Theussl, Comput. Phys. Commun. **161** (2004) 76 [arXiv:hep-ph/0309015].

[7] C. P. Burgess and G. D. Moore, *Cambridge, UK: Cambridge Univ. Pr. (2007) 542 p*

[8] A. J. Buras, arXiv:hep-ph/9806471.