3. Fundamentals

→ our $\mathcal{L}_{QCD}$ contains operators of dimension ≤ 4

⇒ theory is renormalizable : all divergences can be removed by a finite number of counterterms

here: illustrate some divergences in QCD

important physical consequence: asymptotic freedom !

- mini-review: renormalization [see e.g. Peskin / Schroeder §10]

loops → $\int d^4k$ → ultraviolet (UV) divergences (from large k ⇔ small x)

common and natural in QFT.

→ counting of UV div's (in 1PI diags): superficial degree of divergence

→ idea: "hide" div's in $\mathcal{L}$ by rescaling fields + couplings

$$\underbrace{\phi_B}_{\text{bare}} \to Z_\phi \underbrace{\phi_R}_{\text{renormalized}} \quad \text{etc} \quad , \quad Z = 1 + \mathcal{O}(g_R^2)$$

→ need intermediate regularization of divergent loop integrals

many possibilities (discretize space-time; cutoff large momenta; Pauli-Villars; ...)

most elegant for us: dimensional regularization $\int \frac{d^4k}{(2\pi)^4} \to \int \frac{d^dk}{(2\pi)^d}$,

$\left(\begin{array}{l} d = 4-2\varepsilon \\ \varepsilon \to 0 \end{array} \right)$

$d \in \mathbb{C}$; analytic cont. $d \to 4$; div's will be poles $\sim \frac{1}{d-4}$

introduce artificial mass-scale μ

$$\text{e.g.} \quad (m^2)^{-\varepsilon} = \mu^{-2\varepsilon} \left(\frac{\mu^2}{m^2}\right)^{\varepsilon} \overset{x^\varepsilon = e^{\varepsilon \ln x} \approx \text{Taylor} \ldots}{=} \mu^{-2\varepsilon} \left(1 + \varepsilon \ln \frac{\mu^2}{m^2} + \mathcal{O}(\varepsilon^2)\right)$$

→ the renormalization group (RG) equations

describe the μ-dependence of parameters / Greens functions

- mini-review: dimensional regularization

$$\int d^dk\, f(k+q) = \int d^dk\, f(k)$$

$$\int d^dk\, f(\lambda k) = |\lambda|^{-d} \int d^dk\, f(k) \qquad \Rightarrow \qquad \int d^dk = 0 = \int \frac{d^dk}{(k^2)^{\alpha \neq d/2}} \quad (!)$$

$$\int d^dk\, e^{-\alpha k^2} = \left[\int_{-\infty}^{\infty} dk_1\, e^{-\alpha k_1^2} \right]^d = \left(\frac{\pi}{\alpha}\right)^{d/2}$$

$$\int d^dk\, f(|k|) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dk\, k^{d-1} f(k) \qquad \text{d-dim. spherical coordinates}$$

$$\int d^dk\, k^\mu f(|k|) = 0 \qquad (\text{ odd in } k \,)$$

$$\int d^dk\, k^\mu k^\nu f(|k|) \overset{!}{=} g^{\mu\nu} I = \int d^dk\, \frac{g^{\mu\nu} k^2}{d} f(|k|) \qquad \left(\text{since } \int d^dk\, k^2 f = g^\mu{}_\mu I = d\, I\right)$$

• important integrals

$$\Gamma(n) = (n-1)! = \int_0^\infty dt\, t^{n-1} e^{-t}$$

$$\int_0^\infty dx \, \frac{x^{2a-1}}{(x^2+1)^b} = \frac{\Gamma(a)\,\Gamma(b-a)}{2\,\Gamma(b)} \qquad (\text{if } \mathrm{Re}[a]>0,\ \mathrm{Re}[b-a]>0)$$

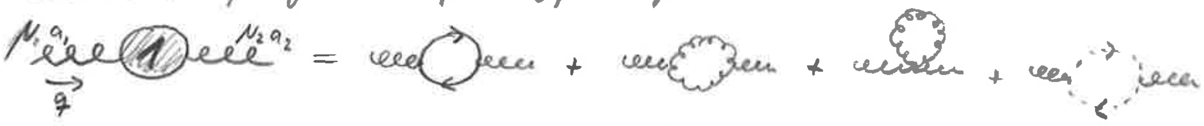
$(A_i = \hat{A}_i + i\varepsilon)$

$$\frac{1}{A_1^{a_1} A_2^{a_2} \cdots A_n^{a_n}} = \frac{\Gamma(a_1+\dots+a_n)}{\Gamma(a_1)\cdot\ldots\cdot\Gamma(a_n)} \int_0^1 dx_1 \dots dx_n \, \frac{x_1^{a_1-1}\cdot\ldots\cdot x_n^{a_n-1}\, \delta(1-\sum_{i=1}^n x_i)}{[x_1 A_1 + \dots + x_n A_n]^{a_1+\dots+a_n}}$$

Feynman parametrization

3.1 one-loop divergences in QCD

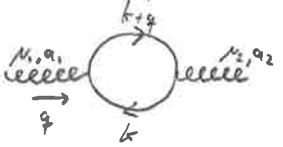
→ goal: evaluate 1-loop gluon self-energy diagrams



use dimensional regularization, $d^4k \to d^dk$, $d = 4-2\varepsilon$

use Feynman gauge $\xi=1$ for gluon propagator (for simplicity)

• 1st diagram: consider one quark flavor, do $\sum_f$ in the end



$$= -\mathrm{Tr}\int \frac{d^dk}{(2\pi)^d} \left(ig\gamma^{\mu_1}T^{a_1}\right) \frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon} \left(ig\gamma^{\mu_2}T^{a_2}\right) \frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2+i\varepsilon}$$

trace over γ's and T's

one closed fermion loop

remember: Dirac matrices

def $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$; $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$

$tr(\mathbb{1}) \equiv N$ (we take $N=4$; also $N=2^{d/2}$, $N=d$ seen)

$tr(\gamma^\mu) = 0$; $tr(\gamma^\mu\gamma^\nu\gamma^5) = 0$; ...

$tr(\gamma^\mu\gamma^\nu) = N g^{\mu\nu}$; $tr(\gamma^{1234}) = N(g^{12}g^{34} - g^{13}g^{24} + g^{14}g^{23})$; ...

$$= -g^2 (tr\mathbb{1}) \overbrace{(tr\, T^{a_1}T^{a_2})}^{\frac{1}{2}\delta^{12},\ \text{see p. 9}} \int \frac{d^dk}{(2\pi)^d} \frac{N^{\mu_1\mu_2}(k,k+q)}{(k^2-m^2+i\varepsilon)((k+q)^2-m^2+i\varepsilon)}$$

$$N^{13}(k,p) \overset{tr(\gamma's)}{=} m^2 g^{13} + m\cdot 0 + (g^{12}g^{34} - g^{13}g^{24} + g^{14}g^{23})\,k_2 p_4$$

$$= (m^2 - k\cdot p)\,g^{13} + k^1 p^3 + p^1 k^3$$

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for denominator, use Feynman parameters, see pg. 30

$$\frac{1}{(k^2-m^2+i\varepsilon)((k+q)^2-m^2+i\varepsilon)} = \int_0^1 dx \frac{1}{[(1-x)(k^2-m^2+i\varepsilon)+x(k^2+2kq+q^2-m^2+i\varepsilon)]^2}$$

$$= \int_0^1 dx \frac{1}{[(k+xq)^2+\underbrace{x(1-x)q^2-m^2}_{\equiv -\Delta}+i\varepsilon]^2}$$

now, shift $k \to k - xq$ under integral $\int d^dk$

$$= -g^2 \cdot 4 \cdot \tfrac{1}{2}\delta^{a_1a_2} \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{N^{\mu_1\mu_2}(k-xq,\, k+(1-x)q)}{[k^2-\Delta+i\varepsilon]^2}$$

numerator $= \left(m^2-(k^2-x(1-x)q^2+\cancel{O(k)}^{\,0})\right)g^{\mu_1\mu_2} + 2\boxed{k^{\mu_1}k^{\mu_2}} - 2x(1-x)q^{\mu_1}q^{\mu_2} + \cancel{O(k)}^{\,0}$

linear (in k) terms integrate to 0; $\quad \boxed{k^{\mu_1}k^{\mu_2}} \to \dfrac{g^{\mu_1\mu_2}k^2}{d}$, see pg. 29 bottom

$$= \left(m^2+\left(\tfrac{2}{d}-1\right)k^2+x(1-x)q^2\right)g^{12} - 2x(1-x)q^1q^2$$

to evaluate the $\int d^dk$ integral, perform a Wick rotation $k^0 = ik_E^0$

$$k^2-m^2+i\varepsilon = k_0^2-\vec{k}^2-m^2+i\varepsilon = (k_0+\sqrt{\vec{k}^2+m^2}-i\varepsilon)(k_0-\Gamma+i\varepsilon)$$

$0 = \oint$, $\quad \to \; = -\downarrow \; = +\uparrow \; : \; k^0 = ik_E^0$

$$= -g^2\cdot 4\cdot\tfrac{1}{2}\delta^{a_1a_2}\int_0^1 dx\; i\left\{\left[\left(m^2+x(1-x)q^2\right)g^{12}-2x(1-x)q^1q^2\right]I_2^0(\Delta)+\left(1-\tfrac{2}{d}\right)g^{12}I_2^1(\Delta)\right\}$$

where $\displaystyle I_n^a(\Delta) \equiv \int \frac{d^dk_E}{(2\pi)^d}\frac{(k_E^2)^a}{[k_E^2+\Delta]^n}$ (basic 1-loop "tadpole" integral)

$$= \frac{1}{(2\pi)^d}\frac{2\pi^{d/2}}{\Gamma(d/2)}\underbrace{\int_0^\infty dk_E \frac{k_E^{d-1+2a}}{(k_E^2+1)^n}}_{=\frac{\Gamma(a+d/2)\Gamma(n-a-d/2)}{2\Gamma(n)}}|\Delta|^{d/2+a-n}$$ see pg. 30

$$= \frac{\Gamma(a+\frac{d}{2})\Gamma(n-a-\frac{d}{2})}{(4\pi)^{d/2}\,\Gamma(\frac{d}{2})\Gamma(n)\,|\Delta|^{n-a-d/2}}$$

note that $\displaystyle I_2^1(\Delta) = I_2^0(\Delta)\cdot|\Delta|\cdot\frac{d}{2}\frac{1}{1-\frac{d}{2}} = I_2^0(\Delta)\frac{|\Delta|}{\frac{2}{d}-1}$ $\quad$ $((x\Gamma(x)=\Gamma(x+1)))$

$$= -g^2\cdot 4\cdot\tfrac{1}{2}\delta^{12}\int_0^1 dx\; i\,I_2^0(\Delta)\left\{\underbrace{\left(m^2+x(1-x)q^2-|\Delta|\right)}_{\Delta>0:\; = 2x(1-x)q^2}g^{12}-2x(1-x)q^1q^2\right\}$$

$$= -4ig^2\delta^{a_1a_2}\left(g^{\mu_1\mu_2}q^2-q^{\mu_1}q^{\mu_2}\right)\int_0^1 dx\; I_2^0(m^2-x(1-x)q^2)\,x(1-x)$$

$$= \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\int_0^1 dx\frac{x(1-x)}{[m^2-x(1-x)q^2]^{2-d/2}}$$