Quantum Chromodynamics (QCD)

Thu 12.15 - 13.45 → 12.30 - 14.00 ?
Fri 10.15 - 11.45

4S, E6-118, 6221, yorks@...

www.physik.uni-bielefeld.de/~yorks/qcd11

prerequisites: QFT (maybe G. Akemann's QFT SS11 ?)
Elementary Particle Physics (useful for context)

literature: → see webpage, Semester Apparat

topics: → ./.

credits: 6 = 3 + 3
oral exam after end of semester
attending lecture ⇝ signup sheet

exercises: will be occasionally offered in the lecture

1. Introduction

Nature is extremely strange – but also very beautiful.
We have built a system of understanding (her):

- QM + Special Rel ⇒ QFT
- objects: space-filling fields;
  excitations: particles
- "Standard Model" $\hat{=}$ 3 basic conceptual structures
  - → gauge system: $SU(3)_c \times SU(2)_L \times U(1)_Y$
    ~ 3 parameters g_i
  - → gravity system: Einstein-Hilbert action
    + minimal matter coupling
    → 2 parameters G_N, Λ
  - → Higgs system: no deep principle
    ~ many parameters
    provisional concept ?
- (extremely) accurately tested/confirmed by many experiments

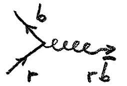
1.1. QCD Appetizer

"zoom" into part of gauge system: QCD (the $SU(3)_c$ above)
theory of strong interactions
$\hat{=}$ models you know from particle physics (quarks, color; partons)
+ mathematical structure you know from QFT
(non-Abelian gauge theory, [Yang, Mills 1954])

in analogy to QED: specify QCD via Feynman rules

Vertices: 

Some qualitative remarks:

- gluon couples to <u>color</u> charge
  color of quark typically changes at qqg - vertex
  eg. 

  , gluon carries the difference

  ((the fact that quarks carry 3 (eg. red/green/blue) color charges has been determined experimentally; more later))

- gluons therefore interact also among themselves
  (in contrast to the electrically neutral photon)

- QCD has very few parameters
  "gauge invariance" requires $\sim g_s$, $\sim g_s^2$

- define $\alpha_s \equiv \frac{g_s^2}{4\pi}$ (cf. $\alpha_{EM} = \frac{e^2}{4\pi} \approx \frac{1}{137}$)

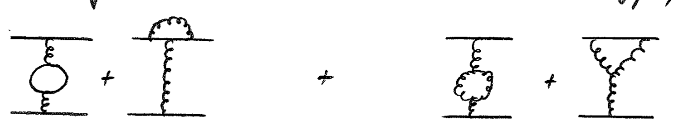
  now, $\alpha_s \gtrsim \frac{1}{10}$ is "large"
  $\Rightarrow$ perturbation theory is not "as perfect" as in QED
  $\rightarrow$ QCD is "more interesting", has a very rich structure, features surprising effects (more later; material of this semester)
  $\rightarrow$ theoretical calculations typically have errors $\gtrsim 1\%$
  $\rightsquigarrow$ one important method to "solve" QCD is (numerical) <u>Lattice - QCD</u>

Some highlights:

- central feature: asymptotic freedom
  QCD shows different faces at long and short distances

long distance		short distance
low energy		high energy ($Q \sim 1/r$)
confinement	$\xleftrightarrow{\text{interplay}}$	asymptotic freedom
non-perturbative hadronic structure		hard scattering
		cross sections
eg. Lattice QCD		perturbative methods

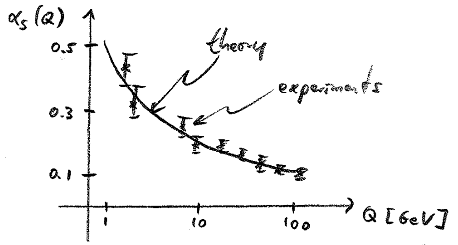
- rough qualitative picture of asymptotic freedom:
  (more later $\rightarrow$ "renormalization")
  value of α_s depends on distance (i.e. energy)



screening of the charge like in QED | anti-screening non-abelian

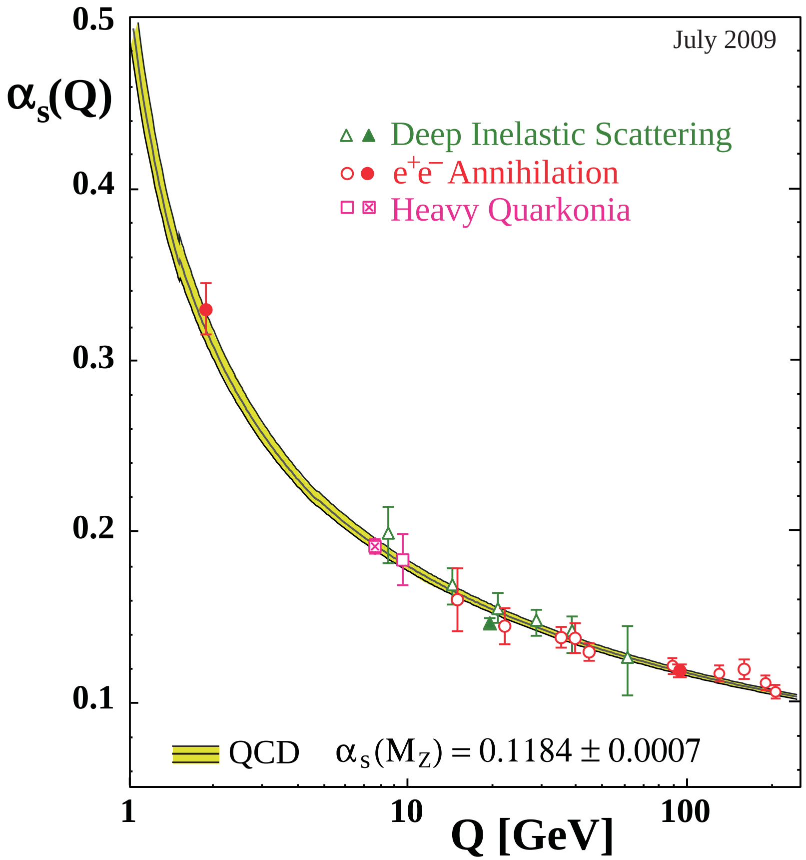
$\alpha_s(r) \nearrow$ if $r \searrow$ | $\alpha_s(r) \searrow$ if $r \searrow$

who wins? $\alpha_s(Q^2) \approx \frac{4\pi}{(-\frac{2}{3}N_f + 11)\ln(Q^2/\Lambda^2)}$

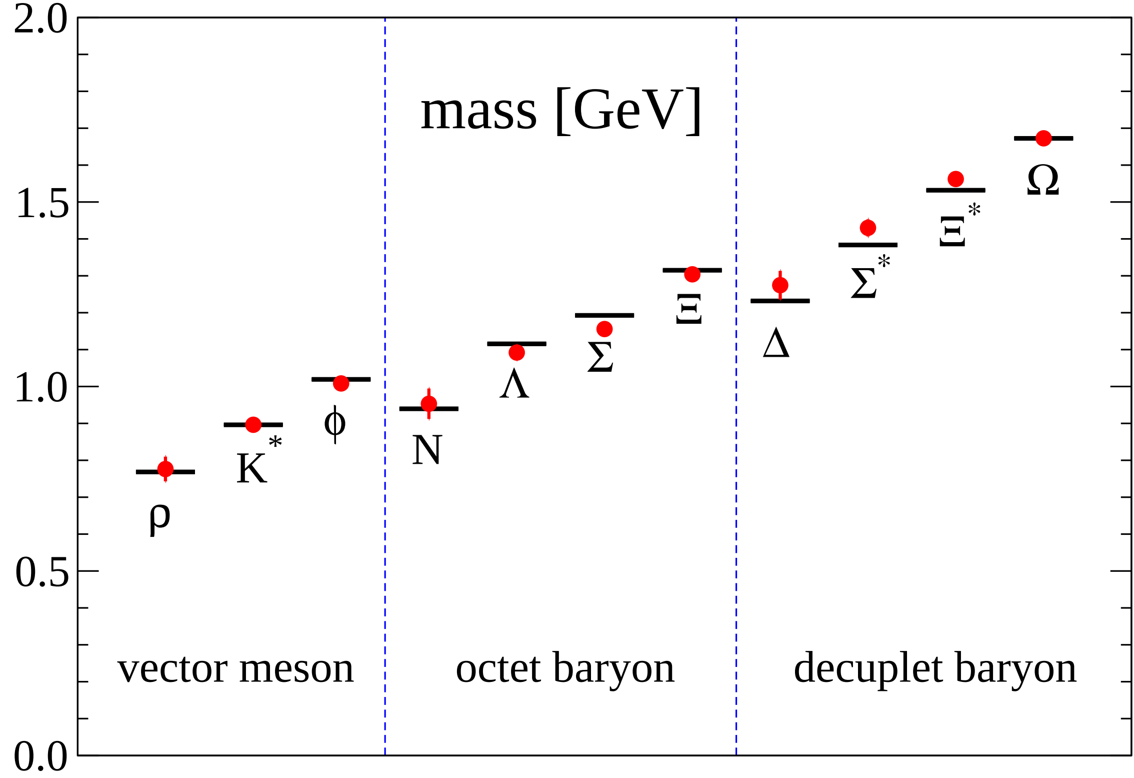


Nobel 2004: Gross / Politzer / Wilczek

$\leftarrow$ plot online [PDG; LEP EWWG]



July 2009
α_s(Q)
Deep Inelastic Scattering
e+e− Annihilation
Heavy Quarkonia
QCD
α_s(M_Z) = 0.1184 ± 0.0007
Q [GeV]
0.5
0.4
0.3
0.2
0.1
1
10
100



mass [GeV]
vector meson
octet baryon
decuplet baryon
ρ
K*
φ
N
Λ
Σ
Ξ
Δ
Σ*
Ξ*
Ω
2.0
1.5
1.0
0.5
0.0

1.2 reality checks

- hadron spectrum

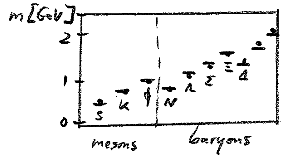
↳ (bound states of quarks; eg $K = s\bar{d}$, $p = uud$, $\Lambda = uds$, ...)

in "our" world, at long distances, observe not quarks + gluons, but hadrons (mesons $q\bar{q}$, baryons qqq)

→ "solve" QCD eqs by computer : Lattice QCD ← powerful: presently $O(10)$ TFlops

⇒ what one gets are just the observed particles + masses (no gluons ; no fractional charges)

upshot : QCD postdicts the low-lying hadron masses



← plot online [Aoki et al, PACS-CS 2008]

((discretize eg. $V_4(3fm)^4 \to 48^3 \times 64$ points ; Euclidean time $t \to -it_E$; physics may differ))

- experimental checks of QCD

collider physics

eg. LEP (@ CERN, 1989-2000) : $e^+e^- \to X$ ← stuff hitting the detectors

→ asymptotic freedom enables us to compute interactions of quarks + gluons at short distances ; detectors are a long distance away, see hadrons (not free partons)

⇒ for comparison of theory ⇔ experiment, need also

infrared safety : classes of quantities which are independent of long-distance physics, hence pQCD calculable

factorization : even wider class of processes, can be factorized into universal long-distance piece and process-dependent (but pQCD calculable) short-distance pc.

(more later!)

get (QM!) (1) $X = e^+e^-$ or $\tau^+\tau^-$ or ... ⇒ detailed QED check

(2) $X > 10$ particles : $\pi, S, p, \bar{p}, \ldots$ ⇒ QCD "jets"

case (1) : no color charge → mainly QED interactions

simple final states: coupling $\alpha_{em} \approx \frac{1}{137}$ small

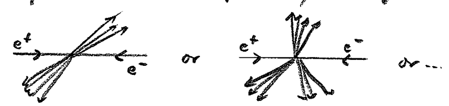
→ most of the time (99%) nothing happens

→ $e^+e^-\gamma$ ~ 1% ⇒ check QED details

→ $e^+e^-\gamma\gamma$ ~ 0.01% ⇒ ...

case (2) : $X \in \{$"greek & latin soup" constructed from quarks + gluons

observed pattern :



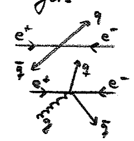
flow of energy + momentum in "jets"

$\alpha_s \gtrsim \frac{1}{10}$

→ 2 jets 90%

→ 3 jets 9%

→ 4 jets 0.9%



- perturbative QCD and hard physics

we have seen (pg 4) ((and will calculate later)) that $\alpha_s(Q^2) =$ [sketch: α_s falling with Q]

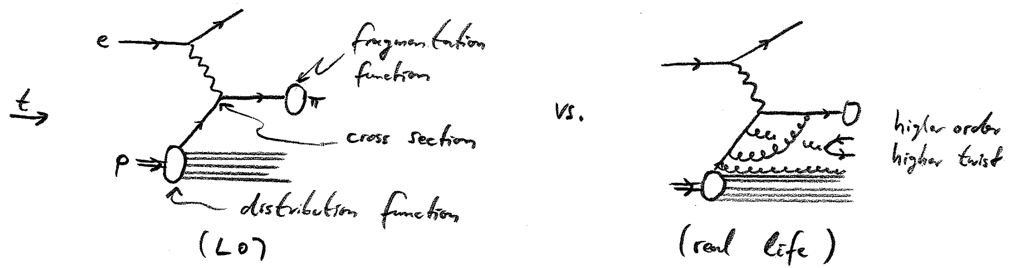
⇒ for large enough 4-momentum squared Q^2 coupling should be small enough for perturbation theory to converge

((more precisely : one gets asymptotic expansions which converge only when "higher twist" and genuine non-perturbative contributions such as "instantons" are also accounted for))

→ perturbative QCD is the basis for interpreting most experiments.

⇒ so pQCD is the most important topic to learn here.

eg. deep inelastic scattering :



- QCD and search for "New Physics"

specific example: anomalous magnetic moment of muon a_μ

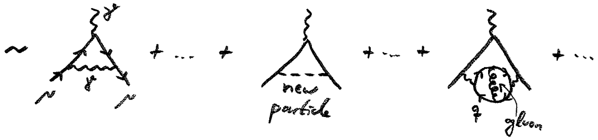
→ determined experimentally and theoretically (within the SM) with such high precision that it became a very sensitive test for many ideas for "physics beyond the SM"

$a_\mu(\text{exp}) = 11659208\,(\pm 6)\cdot 10^{-10}$
$a_\mu(\text{theor}) = 11659186\,(\pm 8)\cdot 10^{-10}$

deviation: 2-3 σ
not "significant" yet

← dominated by uncertainty of QCD contributions

→ strategy for "New Physics" search:

$a_\mu \sim$ 

typically get rather stringent limits on e.g. the minimal allowed mass of hypothetical new particles;
obviously, any deviation between $a_\mu(\text{exp})$ and $a_\mu(\text{theor})$ could be a signal for new physics.

⇒ does the lack of precision in our QCD calculations keep us from clearly "seeing" signals of exciting new physics?

1.3 Color charge in QCD

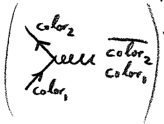
- in addition to its electric charge $\left(\left(\begin{matrix} u\ c\ t \leftarrow +\tfrac{2}{3} \\ d\ s\ b \leftarrow -\tfrac{1}{3}\end{matrix}\ ;\ \bar{q}\ \text{neg.}\right)\right)$

each quark carries a color charge.

3 possible values, e.g. r = red, g = green, b = blue

((experimentally determined, more later; often we generalize $3 \to N_c$))

- if a quark emits a gluon, its color may or may not change



→ 9 ways of coupling a gluon between initial + final quark

e.g. $g_1 = r\bar{g}$, $g_2 = r\bar{b}$, $g_3 = g\bar{r}$, $g_4 = g\bar{b}$, $g_5 = b\bar{r}$,
$g_6 = b\bar{g}$, $g_7 = \frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$, $g_8 = \frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$, } SU(3) color "octet"
$g_0 = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$ } SU(3) "singlet"

- experimental evidence: from scattering expt's we know that matter (mesons = $q\bar{q}$, baryons = qqq) is composed of quarks, yet those hadrons must be neutral to the strong force.

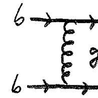
⇒ stable particles (hadrons) are "colorless";
more precisely: they are in "color singlet state"

⇒ color singlet gluon state g_0 is not needed / observed.

- strength of coupling between 2 quarks ~ color factors:

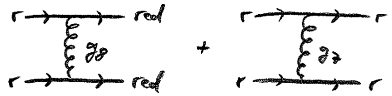
((QED:  $\sim e_1 e_2 \alpha_{em}$ where e.g. $e_{u,c,t} = +\frac{2}{3}$, $e_{d,s,b} = -\frac{1}{3}$ etc.))

QCD:  $\sim \frac{c_1}{\sqrt{2}}\frac{c_2}{\sqrt{2}}\alpha_s$ where $\frac{1}{\sqrt{2}}$ are historical; c_i from g_i above

example: 

$g_8 = \frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b}) \quad \sim \frac{1}{2}\left(-\frac{2}{\sqrt{6}}\right)\left(-\frac{2}{\sqrt{6}}\right) = \frac{1}{3} \qquad (\times \alpha_s)$

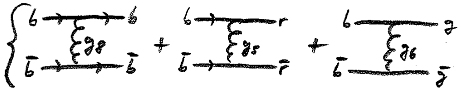
same result due to color symmetry

vs 

$\sim \frac{1}{2}\frac{1}{\sqrt{6}}\frac{1}{\sqrt{6}} + \frac{1}{2}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}$

example: single gluon exchange between q and $\bar{q}$ in color singlet state

$(q\bar{q})_{\text{singlet}} = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$ ⇒ consider e.g. $b\bar{b}$, mult. ×3

$3\Big\{$  $\Big\} \sim 3\,\frac{1}{2}\,\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}\left\{-\frac{2}{\sqrt{6}}\frac{2}{\sqrt{6}} - 1\cdot 1 - 1\cdot 1\right\} = -\frac{4}{3}$

(($\bar{q}$ opposite charge to q ⇒ − sign for $\bar{q}$ vertex))

⇒ color force can be both repulsive and attractive.

1.4 Elements of group theory

- the color charge introduced above can be treated much more rigorously.
  - → symmetry at work.
  - → before (re-)learning the connection symmetry ⇔ charge from QED (cf. § 2.1), let us review some basic facts about the theory of continuous symmetry groups
  - → (much) more detail eg. in [H. Georgi: Lie Algebras in Particle Physics] or at http://www.physik.uni-bielefeld.de/~laine/symmetrien/cover.html

- our provisional color assignment to gluons (cf. § 1.3) $g_1 \dots g_8$ can be rewritten in a different basis (just different linear combinations) of 3×3-Matrices (label eg. rows by colors, columns by anticolors $\begin{array}{c|ccc} & \bar r & \bar g & \bar b \\ \hline r & \cdot & \cdot & \cdot \\ g & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot \end{array}$)

$$T^1 = \frac{1}{2}\begin{pmatrix} 0&1&0\\1&0&1\\0&0&0 \end{pmatrix} \quad T^2 = \frac{1}{2}\begin{pmatrix} 0&-i&0\\i&0&0\\0&0&0 \end{pmatrix} \quad T^3 = \frac{1}{2}\begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{pmatrix} \quad T^4 = \frac{1}{2}\begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix}$$

$$T^5 = \frac{1}{2}\begin{pmatrix} 0&0&-i\\0&0&0\\i&0&0 \end{pmatrix} \quad T^6 = \frac{1}{2}\begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix} \quad T^7 = \frac{1}{2}\begin{pmatrix} 0&0&0\\0&0&-i\\0&i&0 \end{pmatrix} \quad T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1&0&0\\0&1&0\\0&0&-2 \end{pmatrix}$$

("generalization of Pauli matrices")

- → actually, $T^a = \frac{1}{2}\lambda^a$, where λ^a are "Gell-Mann matrices", $a = 1 \dots 3^2-1 = 1 \dots 8$
- → they form a possible representation of the infinitesimal generators of the "special unitary group" $SU(3)$, the fundamental representation
- → some important properties (check ?!):

$$[T^a, T^b] = i f^{abc} T^c \qquad \text{Lie algebra}$$

↑ antisymmetric structure constants

$$\{T^a, T^b\} = \frac{1}{3}\delta^{ab}\mathbb{1}_{3\times3} + d^{abc} T^c$$

↑ symmetric structure constants

$$\Rightarrow T^a T^b = \frac{1}{2}\left(\frac{1}{3}\delta^{ab}\mathbb{1}_{3\times3} + (d^{abc} + i f^{abc}) T^c\right)$$

$$T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{kl}\right) \qquad \text{Fierz identity}$$

- → often we will need traces

$$Tr(T^a) = 0$$
$$Tr(T^a T^b) = \frac{1}{2}\delta^{ab} \qquad \text{normalization}$$

etc.

- → could calculate f^{abc} by multiplying Lie algebra with T^d, then taking trace:

$$f^{abc} = \frac{2}{i}\left(Tr(T^a T^b T^c) - Tr(T^b T^a T^c)\right)$$

result (check ?!): $f^{123} = 1$, $f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}$, $f^{458} = f^{678} = \frac{\sqrt{3}}{2}$; rest by antisymmetry

- from a more general viewpoint, we have just seen one example of a broader mathematical concept: representations of Lie Groups
  - math: group contains abstract entities that obey certain algebraic rules
  - QM: interested in groups of unitary operators acting on vector space of states
  - here: interested in continuously generated groups
    - contain elements arbitrarily close to identity.
    - can reach general group element by repeated action of infinitesimal ones

$$g(\alpha) = 1 + i\alpha^a T^a + \mathcal{O}(\alpha^2)$$

↑ group parameters; ↑ Hermitean op's; "generators" of symm. group

a group with this structure is called a "Lie group"

- the set T^a spans space of infinitesimal group transformations
  - ⇒ Commutator is a linear combination of generators

$$[T^a, T^b] = i f^{abc} T^c$$

vector space spanned by generators + commutator ≡ Lie Algebra

$$\rightarrow [T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0 \qquad \text{Jacobi identity}$$

$$\Rightarrow f^{abe} f^{ecd} + f^{ace} f^{edb} + f^{ade} f^{ebc} = 0$$

- for us, symmetry ≙ unitary transformation of a set of fields
  - → interested in Lie groups with finite # of generators: "compact"
- classification of Lie Algebras
  - → if one T^a commutes with all others: Abelian subgroup, $\psi \to e^{i\alpha}\psi$, $U(1)$ (group of phase rotations)
  - → if set of T^a's cannot be divided into two mutually commuting sets: "simple"
  - → general Lie algebra ≡ direct sum of non-Abelian simple components + additional Abelian generators
  - → $SU(N)$ $((UU^\dagger = 1, \det U = 1))$, $SO(N)$ $((RR^T = 1, \det R = 1))$, $Sp(N)$; $G_2, F_4, E_{6,7,8}$ is complete set of compact simple Lie groups! (Killing, Cartan ca. 1894)

1.5 Notation and conventions

- natural units $\quad \hbar = c = k_B = 1$

  $\Rightarrow [\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1} = \text{GeV}^{-1}$

- vectors + tensors

  indices $\mu = 0,1,2,3$ or t, x, y, z

  metric tensor $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1,-1,-1,-1)$

  four-vectors $x^\mu = (x^0, \vec{x})$; $\partial_\mu = \partial_{x^\mu} = (\partial_0, \vec{\nabla})$

  totally antisymmetric tensor $\varepsilon^{0123} \equiv 1 \quad ((\Rightarrow \varepsilon_{0123} = -1,\ \varepsilon^{1230} = -1$ etc$))$

- matrices

  Pauli $\quad \sigma^i \sigma^j = \delta^{ij} \mathbb{1}_{2\times 2} + i \varepsilon^{ijk} \sigma^k$

  $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

  Dirac $\quad \{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}$

  standard basis: $\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$

  $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$

- Einstein summation convention

  e.g. $\quad p_\mu x^\mu \equiv \sum_{\mu=0}^{3} p_\mu x^\mu = p_0 x^0 + (-\vec{p})\cdot\vec{x} = p^0 x^0 - \vec{p}\cdot\vec{x}$

  e.g. $\quad A^a T^a \equiv \sum_a A^a T^a$

2. Basics

2.1 Reminder: QED and gauge invariance

- gauge symmetry is a fundamental principle that determines the form of the Lagrangian
- consider $\psi(x)$ (complex-valued Dirac-field)

  we now demand the theory to be invariant under local phase transformations: $\boxed{\psi(x) \to e^{i\alpha(x)} \psi(x)}$

  Q: which Lagrangian terms can we construct that are invariant?

  A1: terms that are also invariant under global trafos

  e.g. $\bar{\psi}(x)\psi(x) \quad$ ((recall: Dirac-adjoint $\bar{\psi} \equiv \psi^\dagger \gamma^0$))

  A2: for terms with derivatives, we need some preparation:

  ((recall: derivative in e.g. n^μ-direction def'd as differential quotient))

  $\dfrac{\psi(x+\varepsilon n) - \psi(x)}{\varepsilon} \xrightarrow{\varepsilon\to 0} n^\mu \partial_\mu \psi(x)$

  feel completely different phase transformation! ↯

  $\Rightarrow$ for meaningful comparison, introduce a compensating (scalar) phase factor, transforming as $U(y,x) \to e^{i\alpha(y)} U(y,x) e^{-i\alpha(x)}$

  ($U(x,x) = 1$; $\psi(y)$, $U(y,x)\psi(x)$ transform the same)

  $\Rightarrow$ def. covariant derivative $\quad \dfrac{\psi(x+\varepsilon n) - U(x+\varepsilon n, x)\psi(x)}{\varepsilon} \xrightarrow{\varepsilon \to 0} n^\mu D_\mu \psi(x)$

  for infinitesimal separation of y, x, expand:

  $U(x+\varepsilon n, x) \approx 1 \underbrace{-\,ie}_{\text{definition}}\ \varepsilon n^\mu A_\mu(x) + O(\varepsilon^2)$

  new vector field! "connection"

  $\Rightarrow$ cov. deriv.: $\quad D_\mu \psi(x) = \partial_\mu \psi(x) + ie A_\mu(x) \psi(x)$

  where $\quad A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x) \quad$ (consistent w/ U-transformation)

  now: $\boxed{D_\mu \psi(x) \xrightarrow{\text{(check!)}} \cdots = e^{i\alpha(x)} D_\mu \psi(x)}$

  transforms the same way as the field $\psi(x)$ ✓✓

  $\Rightarrow \bar{\psi}(x) D_\mu \psi(x)$ also invariant.

- <u>summary 1:</u> local phase rotation symmetry
  $\Rightarrow$ def. of covariant derivative
  <u>and</u> existence of vector field A_μ (connection)
  <u>and</u> transformation properties of A_μ

$\rightarrow$ all terms that are globally ($\psi \to e^{i\alpha}\psi$, α const) invariant are also locally invariant if we replace all $\partial_\mu \to D_\mu$.

- how about (locally invariant) kinetic terms for A_μ?

(a) construction using $U(y,x)$

U is pure phase: $U(y,x) = e^{iu(y,x)}$, $u(y,x) \in \mathbb{R}$

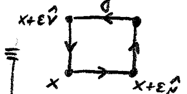
($u(x,x)=0$ from $U(x,x)=1$)

assume $U(y,x) = [U(x,y)]^\dagger$ $\quad (\Rightarrow \; u(y,x) = -u(x,y))$

$\Rightarrow u(y,x) = \sum_{n=0}^{\infty} (y-x)^{2n+1} f_n(y+x)$ is odd under $y \leftrightarrow x$

$\rightsquigarrow$ can write $U(x+\varepsilon n, x) = e^{-ie\,\varepsilon n^\mu A_\mu(x+\frac{\varepsilon n}{2})} + O(\varepsilon^3)$

now, use this for comparing phase products around a small square



$U(x) \equiv$ (unit vector in μ-direction, e.g. $\hat{1}$)

$$
\begin{aligned}
&= U(x, x+\varepsilon\hat\nu)\, U(x+\varepsilon\hat\nu, x+\varepsilon\hat\mu+\varepsilon\hat\nu)\, U(x+\varepsilon\hat\mu+\varepsilon\hat\nu, x+\varepsilon\hat\mu)\, U(x+\varepsilon\hat\mu, x) \\
&= e^{-ie\varepsilon\{-A_\nu(x+\frac{\varepsilon\hat\nu}{2}) - A_\mu(x+\frac{\varepsilon\hat\mu}{2}+\varepsilon\hat\nu) + A_\nu(x+\varepsilon\hat\mu+\frac{\varepsilon\hat\nu}{2}) + A_\mu(x+\frac{\varepsilon\hat\mu}{2})\} + O(\varepsilon^3)} \\
&\approx 1 - ie\varepsilon\frac{\varepsilon}{2}\{-\cancel{\partial_\nu A_\nu(x)} - \cancel{\partial_\mu A_\mu(x)} - 2\partial_\nu A_\mu + 2\partial_\mu A_\nu(x) + \cancel{\partial_\nu A_\nu(x)} + \cancel{\partial_\mu A_\mu(x)}\} + O(\varepsilon^3) \\
&= 1 - ie\underbrace{\varepsilon^2}_{\text{(area of square)}}\underbrace{\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right)}_{\equiv F_{\mu\nu}(x)} + O(\varepsilon^3)
\end{aligned}
$$

$\equiv F_{\mu\nu}(x) \;\hat{=}\;$ electromagnetic field strength tensor

but $U(x)$ is locally invariant by construction!

$\Rightarrow F_{\mu\nu}(x)$ is a locally invariant function of A_μ!

(b) construction using D_μ

Since (see above) $\psi \to e^{i\alpha(x)}\psi$, $D_\mu\psi \to e^{i\alpha(x)} D_\mu\psi$

it also follows that $D_\mu D_\nu \psi \to e^{i\alpha(x)} D_\mu D_\nu \psi$

or $[D_\mu, D_\nu]\psi \to e^{i\alpha(x)}[D_\mu, D_\nu]\psi \qquad (*)$

now, note that

$$
\begin{aligned}
[D_\mu, D_\nu]\psi &= \underbrace{[\partial_\mu,\partial_\nu]}_{0}\psi + ie\left([\partial_\mu, A_\nu] + [A_\mu, \partial_\nu]\right)\psi - e^2\underbrace{[A_\mu, A_\nu]}_{0}\psi \\
&= ie\left(\partial_\mu A_\nu\psi + \cancel{\partial_\mu A_\nu\psi} - \cancel{A_\nu\partial_\mu\psi} + \cancel{A_\mu\partial_\nu\psi} - \partial_\nu A_\mu\psi - \cancel{\partial_\nu A_\mu\psi}\right) \\
&= ie\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\cdot\psi \qquad \text{has } \underline{\text{no}} \text{ derivative acting outside ()!}
\end{aligned}
$$

$\Rightarrow [D_\mu, D_\nu] = ie F_{\mu\nu}$

$\Rightarrow$ in $(*)$, $F_{\mu\nu}$ is just a multiplicative factor, must be <u>invariant</u>.

- can now write the <u>most general locally invariant Lagrangian</u> (for the electron field ψ and its associated vector field A_μ)

$$\mathcal{L}_{QED} = \bar\psi(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - c\,\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

remarks: ▶ used operators of dimension ≤ 4 here

in general, there are many additional gauge-invariant op's, e.g.:

$\mathcal{L}_5 \sim \bar\psi[\gamma^\mu, \gamma^\nu]F_{\mu\nu}\psi$

$\mathcal{L}_6 \sim (\bar\psi\psi)^2, \; (\bar\psi\gamma^5\psi)^2, \ldots$

$\rightarrow$ all these are <u>non-renormalizable</u> interactions (see later)

$\rightarrow$ irrelevant for physics, in Wilsonian sense

(see e.g. § 12.1 in Peskin/Schroeder)

▶ the coefficient $c \equiv 0$ if we postulate invariance under (discrete) P, T symmetries

$\rightarrow$ then <u>only 2 free parameters</u> in $\mathcal{L}$: m, e (hidden in D_μ)

- <u>Summary 2:</u> local phase rotation symmetry of electron field ψ
  $\Rightarrow$ existence + transformation properties of em. vector potential A_μ
  $\Rightarrow$ most general (4d, renormalizable, T or P invariant) Lagrangian is <u>unique</u>: Maxwell-Dirac-Lagrangian!

2.2 Generalization : Yang-Mills Lagrangian

geometric construction can be generalized :
invariance under local phase rotations
→ invariance under any (continuous) symmetry group

here, use 3d rotation group (O(3) or SU(2)) for brevity
→ in the end, simple generalization to arbitrary local symmetry.

- consider $\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$ (doublet of Dirac fields)

demand invariance under local 3d rotations : $\boxed{\psi(x) \to \boxed{e^{i\frac{\sigma^i}{2}\alpha^i(x)}}\,\psi(x)}$ $\equiv V(x)$

((where σ^i = Pauli matrices $= \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$; $\sum_{i=1}^{3}$ suppressed))

($e^{\text{Matrix}} = ?$ $= \sum_{n=0}^{\infty} \frac{(\text{Matrix})^n}{n!}$)

Q : construct invariant Lagrangian ?

→ need again a covariant derivative !
→ now, compensating phase factor has to be a matrix,
with transformation $U(y,x) \to V(y)\, U(y,x)\, V^\dagger(x)$
→ again, $U(x,x) = 1$ and $U^\dagger U = U U^\dagger = \mathbb{1}$ unitary

⇒ can expand in terms of (hermitean : $\sigma^\dagger = \sigma$) SU(2)-generators :

$$U(x+\varepsilon n, x) \approx \underbrace{\mathbb{1} + ig}_{\text{convention}}\, \varepsilon n^\mu \underbrace{A_\mu^i \frac{\sigma^i}{2}}_{\text{new (matrix-valued) vector field}} + O(\varepsilon^2)$$

(sign different from QED, since we take $e<0$, $g>0$)

⇒ covariant derivative : (($n^\mu D_\mu \psi = \lim_{\varepsilon \to 0} \frac{\psi(x+\varepsilon n) - U(x+\varepsilon n, x)\psi(x)}{\varepsilon}$))

$$D_\mu = \partial_\mu - ig A_\mu^i \frac{\sigma^i}{2}$$

where $A_\mu^i(x)\frac{\sigma^i}{2} \longrightarrow V(x)\left(A_\mu^i(x)\frac{\sigma^i}{2} + \frac{i}{g}\partial_\mu \right) V^\dagger(x)$ (consistency w/ U-trafo)

now, infinitesimally, $\psi \to \left(1 + i\alpha^i \frac{\sigma^i}{2} + \dots \right)\psi$

$$D_\mu \psi \to \underset{\text{(check!)}}{\dots} = \left(1 + i\alpha^i \frac{\sigma^i}{2} \right) D_\mu \psi$$

again, transforms the same way as field $\psi(x)$ ✓
((also valid for finite transformations (check!)))

⇒ again, $\bar\psi(x) \left(i\gamma^\mu D_\mu - m \right) \psi(x)$ is locally invariant.

- gauge-invariant terms containing A_μ^i only ?

→ here, use construction via D_μ :
from above, we have $[D_\mu, D_\nu]\psi(x) \to V(x)[D_\mu, D_\nu]\psi(x)$ (**)

now, note that

$$[D_\mu, D_\nu]\psi = \overset{0}{[\partial_\mu, \partial_\nu]}\psi - ig\left([\partial_\mu, A_\nu^i \tfrac{\sigma^i}{2}] + [A_\mu^i \tfrac{\sigma^i}{2}, \partial_\nu] \right)\psi - g^2 \underbrace{[A_\mu^i \tfrac{\sigma^i}{2}, A_\nu^j \tfrac{\sigma^j}{2}]}\psi$$

does not vanish, as in QED → $= A_\mu^i A_\nu^j [\frac{\sigma^i}{2}, \frac{\sigma^j}{2}] = A_\mu^i A_\nu^j\, i\varepsilon^{ijk}\frac{\sigma^k}{2}$

($\sigma^i\sigma^j = \delta^{ij}\mathbb{1} + i\varepsilon^{ijk}\sigma^k$)

$$= -ig \underbrace{\left(\partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g\varepsilon^{ijk} A_\mu^j A_\nu^k \right)}_{\equiv F_{\mu\nu}^i(x)} \frac{\sigma^i}{2}\psi$$

$\equiv F_{\mu\nu}^i(x)$ = (non-Abelian) field strength tensor

→ as before, $[D_\mu, D_\nu]$ is not a derivative, but a constant (matrix)!

⇒ from (**), the field strength is not invariant now,
but transforms as $F_{\mu\nu}^i \frac{\sigma^i}{2} \to V(x)\, F_{\mu\nu}^i \frac{\sigma^i}{2}\, V^\dagger(x)$

⇒ can construct the locally invariant terms from
traces (using cyclicity and $V^\dagger V = \mathbb{1}$)
e.g. $\mathrm{Tr}\left(F_{\mu\nu}^i \frac{\sigma^i}{2} F^{\mu\nu\, j} \frac{\sigma^j}{2} \right) = \frac{1}{2} F_{\mu\nu}^i F^{\mu\nu\, i} \equiv \frac{1}{2}(F_{\mu\nu}^i)^2$

- adding up : $\boxed{\mathcal{L} = \bar\psi\left(i\gamma^\mu D_\mu - m \right)\psi - \frac{1}{4}\left(F_{\mu\nu}^i \right)^2}$ Yang-Mills Lagrangian

- two parameters : m, g
- variations → equations of motion : Dirac eqn + eqn for vector field

- generalize to other continuous symmetry groups :

V → n×n unitary matrices ; $\psi(x)$ is n-plet ; $\psi(x) \to V(x)\psi(x)$

expand $V(x) \approx \mathbb{1} + i\, T^a \alpha^a(x) + O(\alpha^2)$
↑ $(T^a)^\dagger = T^a$ set of generators of symmetry group

all as above, with $\frac{\sigma^i}{2} \to T^a$

for def. of $F_{\mu\nu}^a$ use $[T^a, T^b] = i f^{abc}\, T^c$
↑ completely antisy. structure const.

- summary:

  invariance of n-plet ψ under local "gauge" transformations $\psi(x) \to V(x)\psi(x)$

  with $V(x) \equiv n\times n$ unitary matrices $= e^{iT^a\alpha^a(x)}$

  where $(T^a)^\dagger = T^a$ are hermitean generators

  with structure constants f^{abc} given by $[T^a, T^b] = if^{abc}T^c$

  $\Rightarrow$ covariant derivative $D_\mu = \partial_\mu - igA_\mu^a T^a$

  contains one vector field for each independent generator of local symmetry

  $A_\mu^a T^a \to V(x)\left(A_\mu^a T^a + \frac{i}{g}\partial_\mu\right)V^\dagger(x)$ guarantees $D_\mu\psi(x) \to V(x)D_\mu\psi(x)$

  $\Rightarrow$ field strength tensor $[D_\mu, D_\nu] = -ig F_{\mu\nu}^a T^a$

  ((or $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$))

  transforms as $F_{\mu\nu}^a T^a \to V(x)F_{\mu\nu}^a T^a V^\dagger(x)$

  $\rightsquigarrow$ for later reference: infinitesimal transformations

  $$\psi \to \psi + iT^a\alpha^a(x)\psi + O(\alpha^2)$$

  $$A_\mu^a \to A_\mu^a + \left(f^{abc}A_\mu^b + \frac{1}{g}\delta^{ac}\partial_\mu\right)\alpha^c(x) + O(\alpha^2)$$

  $$F_{\mu\nu}^a \to F_{\mu\nu}^a + f^{abc}F_{\mu\nu}^b\alpha^c(x) + O(\alpha^2)$$

  $\Rightarrow$ most general gauge-invariant renormalizable Lagrangian

  (conserving P, T): $\mathcal{L} = \bar\psi\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{1}{4}\left(F_{\mu\nu}^a\right)^2$

- jargon: Abelian symmetry group of QED

  vs non-Abelian symmetry group of the more general theories above.

  $\to$ non-Abelian gauge theory $\equiv$ QFT associated with a non-commuting local symmetry

2.3 QCD and its symmetries

- Quantum Chromodynamics (QCD) is a Yang-Mills theory with gauge group SU(3).

  - matter fields (the ψ above) are quarks;

    they are in the fundamental representation of SU(3), have spin $\frac{1}{2}$

    there are six types ("flavors") of quarks: u, d, c, s, t, b

    index of gauge group is called color index

    $\Rightarrow$ write as $\psi^{\alpha,A}$; color index $\alpha = 1,2,3$; flavor index $A = u,d,c,s,t,b$

  - the $3^2 - 1 = 8$ vector fields (or gauge bosons) A_μ^a, $a = 1,\dots,8$ are called gluons

- $$\mathcal{L}_{QCD} = \bar\psi^{\alpha,A}\left(i\gamma^\mu\left(\partial_\mu\delta_{\alpha\beta} - igA_\mu^a T^a_{\alpha\beta}\right) - m_A\delta_{\alpha\beta}\right)\psi^{\beta,A} - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu}$$

  ((sum over color indices α,β; sum over flavor index A))

  m_A: each quark flavor can have a different mass

  $T^a_{\alpha\beta}$: generators of SU(3) in fundamental rep.

- sometimes, it is useful to consider the generalizations

  $SU(3) \to SU(N_c)$ $\Rightarrow$ colors: $\alpha,\beta = 1,\dots,N_c$; gluons: $a = 1,\dots,N_c^2-1$

  6 quark flavors $\to N_f$ quark flavors $\Rightarrow A = 1,\dots,N_f$

- QCD possesses not only the exact local $SU(N_c)$ color symmetry, but has also important approximate global symmetries:

  $\to$ consider (x-independent) rotations in flavor space

  ((note: global phase-redefinition for each flavor ($A = u,d,\dots$) separately $\Rightarrow \mathcal{L}_{QCD}$ invariant $\Rightarrow$ quark number conserved))

  $\to$ rotations between different flavors make sense if some masses are (approximately) degenerate

  ((note: in nature, $m_u \sim 3$ MeV, $m_d \sim 5$ MeV

  $\Rightarrow m_d - m_u \ll m_{\text{Hadron}} \sim 150$ MeV $\Rightarrow \mathcal{L}$ has increased symmetry))

$\rightarrow$ assume e.g. $\underline{m_u \approx m_d}$ $\Rightarrow$ $M \equiv \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \approx m_u \mathbb{1}_{2\times2}$

write $q = \begin{pmatrix} q^u \\ q^d \end{pmatrix}$, then $\mathcal{L}_{QCD} \ni \bar{q}(i\not{D} - M)q$

is invariant under $q \to \underbrace{e^{i\sum_{j=0}^{3}\alpha_j \sigma^j}}_{\in U(2) \,=\, U(1)\otimes SU(2)} q$ $\quad$ (($\sigma^{1,2,3}$ = Pauli, $\sigma^0 \equiv \mathbb{1}_{2\times2}$))

quark number symmetry (see above) $\quad$ "isospin symmetry" exact only if $m_u = m_d$

((note: these symmetries are, via Noether's theorem, associated with vector currents $j^i_\mu = \bar{q}\gamma_\mu \sigma^i q$, hence often $SU(2)_V$))

((note: if e.g. $m_u \approx m_d \approx m_s$ is a useful approximation, then symmetry is enhanced, $SU(3)_V$; etc))

$\rightarrow$ for massless flavors, the symmetry becomes even larger: $M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

use left- and right-handed projectors

$$\gamma_{L/R} \equiv \frac{1 \mp \gamma^5}{2} \qquad ((\Rightarrow \gamma_L^2 = \gamma_L ,\ \gamma_R^2 = \gamma_R ,\ \gamma_L \gamma_R = 0))$$

decompose $q^u = (\gamma_L + \gamma_R) q^u \equiv q^u_L + q^u_R$ $\quad$ etc

now $q_L \equiv \begin{pmatrix} q^u_L \\ q^d_L \end{pmatrix}$, and $\mathcal{L}_{QCD} \ni \bar{q}_L i\not{D} q_L + \bar{q}_R i\not{D} q_R$

((note: $M \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ would have coupled L, R : $\mathcal{L} \ni -q_L^\dagger M q_R$ + h.c.))

$\Rightarrow$ independent transformations $q_L \to U_L q_L$, $q_R \to U_R q_R$ permitted!

$\rightsquigarrow$ $U(2)_L \otimes U(2)_R$ symmetry

$= U(1)_L \otimes U(1)_R \otimes SU(2)_L \otimes SU(2)_R$, called chiral symmetry ((since acting separately on L, R))

((note: the symmetry $SU(N_f)_L \otimes SU(N_f)_R$ is sometimes rewritten as the product $SU(N_f)_V \otimes SU(N_f)_A$ "axial",

using $Q \equiv \begin{pmatrix} q_L \\ q_R \end{pmatrix}$, $\mathcal{L}_{QCD} \ni \bar{Q}\, i\not{D}\, Q$,

invariant under $Q \to e^{i\alpha^a T^a} Q$ and $Q \to e^{i\beta^a T^a \gamma^5} Q$))

generators of $SU(N_f)$ flavor symmetry in fundamental rep. $\quad$ to check this invariance, see (left)

$\{\gamma^5, \gamma^\mu\} = 0$, $(\gamma^5)^\dagger = \gamma^5$

$\Rightarrow \gamma^\mu e^{i\beta\gamma^5} = e^{-i\beta\gamma^5}\gamma^\mu$ via $e^a = \sum \frac{a^n}{n!}$

$\Rightarrow \gamma^0\gamma^\mu e^{i\beta\gamma^5} = e^{i\beta\gamma^5}\gamma^0\gamma^\mu$

$\Rightarrow \bar{q}\not{D}q = q^\dagger\gamma^0\gamma^\mu\partial_\mu q \to q^\dagger e^{-i\beta\gamma^5}\gamma^0\gamma^\mu\partial_\mu e^{i\beta\gamma^5} q = \bar{q}\not{D}q$

- similar to the above approximate global symmetries for light quarks (neglecting effects of order m_q), can also consider heavy quark symmetries (neglecting effects of order $1/m_Q$).

  $\rightarrow$ systematics, "heavy quark effective theories", see e.g. [M. Neubert, Phys. Rept. 245 (1994) 259]

- other important exact symmetries of QCD are the discrete global symmetries : C, P, T

  ((these agree with the observed properties of the strong interactions; for tests and limits, see Particle Data Group, pdg.lbl.gov))

  $\rightarrow$ analysis of $\mathcal{L}_{QCD}$ under C, P, T is complicated (at quantum level) due to the possible dim-4 operator we had discovered (see pg. 14)

$$\mathcal{L}_\theta = \frac{\theta g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \quad , \text{ where } \quad \tilde{F}^{a\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F^a_{\rho\sigma}$$

conventional normalization ; on pg. 14: "c"

$\rightarrow$ $\mathcal{L}_\theta$ would violate both P and T , in contradiction to observations

$\Rightarrow$ set $\theta = 0$, or at least $\theta \ll 1$?!

$\mathcal{L}$ (could be regenerated by known CP effects in weak int.)

$\rightarrow$ actually, $F^a_{\mu\nu}\tilde{F}^{a\mu\nu} = \partial_\alpha \left\{ 2\varepsilon^{\alpha\beta\gamma\delta} A^a_\beta \left(\partial_\gamma A^a_\delta - \frac{2}{3} g f^{abc} A^b_\gamma A^c_\delta \right) \right\}$

is a total derivative

contributes only surface term to action $S = i\int d^4x\, \mathcal{L}$

therefore plays no role in perturbative QCD

$\rightarrow$ however, $\mathcal{L}_\theta$ can have real physical effects due to non-perturbative effects (QCD vacuum can have non-trivial topology $\Rightarrow$ surface terms contribute; the $\{\ldots\}$ is not gauge-invariant) [see e.g. Erice lectures by S. Coleman (1977), F. Wilczek (1983)]

$\rightarrow$ problem: observations tell $\theta < 10^{-9}$ (neutron dipole moment) "naturally", θ should be large (coming from strong interactions)

$\Rightarrow$ "strong CP problem"

$\Rightarrow$ several proposed solutions; e.g. Peccei-Quinn-symmetry

$\Rightarrow$ new particles: Axions

2.4 Quantization, path integral (remarks only)

- so far, have seen non-Abelian gauge symmetry at work $\Rightarrow \mathcal{L}_{QCD}$.

  now, work out consequences for particle physics interactions

  $\rightarrow$ need rules for computing Feynman diagrams

  $\rightarrow$ apply rules to compute amplitudes, cross sections

- local gauge symmetry $\Rightarrow$ some Lagrangian dof's are unphysical

  ($\hat{=}$ can be adjusted arbitrarily by gauge trafos)

cf. QED: in functional integral $\int \mathcal{D}A\, e^{iS[A]}$ the photon part

was $S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)$

$= \frac{1}{2} \int d^4x\, A_\mu(x) \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu\right) A_\nu(x)$

$= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(k) \left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right) \tilde{A}_\nu(-k)$

(FT $A(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{A}(k)$, $\int d^4x\, e^{-ikx} = (2\pi)^4 \delta^{(4)}(k)$)

$\Rightarrow$ for $\tilde{A}_\mu(k) = k_\mu a(k)$, $S = 0 \Rightarrow \int \mathcal{D}A\, e^{i0}$ diverges ! ↯

$a(k)$: arbitrary scalar fct

(($\Leftrightarrow$ (...) has no inverse: cannot solve $(-k^2 g_{\mu\nu} + k_\mu k_\nu) \tilde{D}_F^{\nu s}(k) = i g_\mu{}^s$ for Feynman propag. $\tilde{D}_F$))

recall Abelian gauge invariance $A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x)$

$\Rightarrow$ field configurations that are gauge-equivalent to $A_\mu(x) = 0$ did ↯

$\rightarrow$ the way out was Faddeev-Popov gauge fixing

[Phys. Lett. 25B (1967) 29]

result: $S \rightarrow S + \int d^4x \left(-\frac{1}{2\xi} (\partial_\mu A^\mu)^2\right)$

$\Leftrightarrow$ can solve $\left(-k^2 g_{\mu\nu} + (1 - \frac{1}{\xi}) k_\mu k_\nu\right) \tilde{D}_F^{\nu s}(k) = i g_\mu{}^s$:

$$\tilde{D}_F^{\mu\nu}(k) = \frac{-i}{k^2 + i\varepsilon} \left(g^{\mu\nu} - (1-\xi) \frac{k^\mu k^\nu}{k^2}\right) \quad \text{photon propagator}$$

$\rightarrow$ propagator depends on arbitrary parameter ξ ? ↯

physics does not: QED vertex is such that ξ drops out of S-matrix elements

(due to the Ward-Takahashi identities)

$\rightsquigarrow$ similar structure in QCD; ξ-cancellations more complicated.

- we will make use of functional methods

  $\rightarrow$ most useful for interacting QFT's:

  path integral method, relying on functional integration

  $\rightarrow$ for (many) more details: [QFT lecture]

  [Peskin/Schröder, §9]

- reminder of a functional derivative:

def. $\delta_{J(x)} J(y) = \delta^{(4)}(x-y)$ or $\delta_{J(x)} \int d^4y\, J(y) \phi(y) = \phi(x)$

$\Rightarrow$ can take functional derivatives as usual,

e.g. $\delta_{J(x)}\, e^{i\int d^4y\, J(y)\phi(y)} = i\phi(x)\, e^{i\int d^4y\, J(y)\phi(y)}$

e.g. $\delta_{J(x)} \int d^4y\, (\partial_\mu J(y)) A^\mu(y) = -\partial_\mu A^\mu(x)$ ((after partial integration))

- reminder of the generating functional of correlation functions

$$Z[J] = \int \mathcal{D}\phi\, e^{i\int d^4x\, [\mathcal{L} + J(x)\phi(x)]}$$

$J(x)\phi(x)$: source term

such that $\langle 0 | T \phi(x_1) \phi(x_2) | 0 \rangle = \frac{\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)\, e^{i\int d^4x\, \mathcal{L}}}{\int \mathcal{D}\phi\, e^{i\int d^4x\, \mathcal{L}}}$ (T: time ordering)

$= \frac{1}{Z[0]} \left(-i\delta_{J(x_1)}\right)\left(-i\delta_{J(x_2)}\right) Z[J] \Big|_{J=0}$ very elegant!

- to see the elegance of the $Z[J]$ formulation,

consider a free scalar theory, $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$

$\Rightarrow \int d^4x\, [\mathcal{L}_0 + J\phi] \overset{PI}{=} \int d^4x \left[\frac{1}{2}\phi(-\partial^2 - m^2 + i\varepsilon)\phi + J\phi\right]$

$i\varepsilon$: convergence factor for functional integral, $\varepsilon > 0$

complete the square: $\phi \rightarrow \phi + i\int d^4y\, D_F(x-y) J(y)$

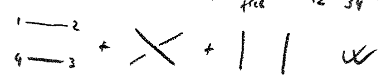
where $(-\partial^2 - m^2 + i\varepsilon) D_F(x-y) = -i\delta^{(4)}(x-y)$, D_F is Greens fct

$= \int d^4x \left[\mathcal{L}_0 + \frac{i}{2} J(x) \int d^4y\, D_F(x-y) J(y)\right]$

$\Rightarrow Z_{free}[J] = Z_{free}[0]\, e^{-\frac{1}{2}\int d^4x\, d^4y\, J(x) D_F(x-y) J(y)}$

$\Rightarrow$ two-point function $\langle 0 | T \phi(x_1)\phi(x_2) | 0 \rangle_{free} \overset{\text{(check?!)}}{=} \cdots = D_F(x_1 - x_2)$

$\Rightarrow$ four-point function $\langle 0 | T \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) | 0 \rangle_{free} = D_{12} D_{34} + D_{13} D_{24} + D_{14} D_{23}$

((where $D_{ij} \equiv D_F(x_i - x_j)$; ))

$\Rightarrow$ etc

- now, consider an interacting theory , $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ (e.g. $-\frac{\lambda}{4!}\phi^4$)

look at generating functional again

$$Z[J] = \int \mathcal{D}\phi\, e^{i\int d^4x [\mathcal{L}_0 + \mathcal{L}_I + J\phi]}$$
$$= \int \mathcal{D}\phi\, \underbrace{e^{i\int d^4x \mathcal{L}_I(\phi \to -i\delta_J)}}_{\phi\text{-independent!}} \underbrace{e^{i\int d^4x [\mathcal{L}_0 + J\phi]}}_{\text{as in free theory} \Rightarrow \text{shift as above}}$$
$$= e^{i\int d^4x \mathcal{L}_I(\phi\to -i\delta_J)} \cdot e^{-\frac{1}{2}\int d^4x \int d^4y\, J(x) D_F(x-y) J(y)} \cdot \underbrace{\int \mathcal{D}\phi\, e^{i\int d^4x \mathcal{L}_0}}_{Z_{free}[0]}$$

such that the correlation functions follow from

$$\langle 0 | T\, \mathcal{O}(\phi) | 0 \rangle = \frac{1}{Z[0]} \mathcal{O}(\phi\to -i\delta_J)\, Z[J]\Big|_{J=0}$$

$$= \frac{\mathcal{O}(\phi\to -i\delta_J)\, e^{i\int d^4x \mathcal{L}_I(\phi\to -i\delta_J)}\, e^{-\frac{1}{2}\int d^4x\int d^4y\, J(x) D_F(x-y) J(y)}\Big|_{J=0} \cdot \cancel{Z_{free}[0]}}{e^{i\int d^4x \mathcal{L}_I(\phi\to -i\delta_J)}\, e^{-\frac{1}{2}\int d^4x \int d^4y\, J(x) D_F(x-y) J(y)}\Big|_{J=0} \cdot \cancel{Z_{free}[0]}}$$

((note: in denominator, sum of "vacuum diagrams"))

→ perturbative expansion (Feynman diagrams) follow from expanding $e^{i\int \mathcal{L}_I}$ in terms of (small) coupling constants (here: λ)

→ all combinatorics for evaluating correlation fcts is just in exponentials !

⇒ two-point function $\langle 0 | T\, \phi(x_1)\phi(x_2) | 0\rangle =$

$$= \frac{\delta_{iJ(x_1)}\, \delta_{iJ(x_2)} \left\{ 1 + \int d^4z \left(-\frac{i\lambda}{4!}\right) \delta^4_{iJ(z)} + \mathcal{O}(\lambda^2) \right\} e^{-\frac{1}{2}\int d^4x \int d^4y\, J(x) D_F(x-y) J(y)}\Big|_{J=0}}{\{ \ldots \}\, e^{-\frac{1}{2}\ldots}\Big|_{J=0}}$$

((check!)) $$= \frac{D_{12} + \left(-\frac{i\lambda}{4!}\right)\int d^4z \left(3 D_{12} D_{zz} D_{zz} + 12\, D_{1z} D_{2z} D_{zz}\right) + \mathcal{O}(\lambda^2)}{1 + \left(-\frac{i\lambda}{4!}\right)\int d^4z\; 3\, D_{zz} D_{zz} + \mathcal{O}(\lambda^2)}$$

$(D_{ij} = D_F(x_i - x_j))$

$$= \frac{\ldots + \ldots + \ldots}{1 + \ldots + \ldots} = \ldots + \ldots + \mathcal{O}(\lambda^2)$$

→ this cancellation is actually generic : $\frac{(\text{connected pieces}) \cdot e^{(\text{disconnected pieces})}}{e^{(\text{disconnected pieces})}}$ works for all higher correlation fcts as well.

2.5 QCD Feynman rules

→ recall from above remarks that for a perturbative treatment, we need to read off propagators (cf pg 21) and vertices (cf pg 23) from the Lagrangian $\mathcal{L}$.

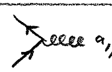
→ $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$
 $\mathcal{L}_0$: bilinear in fields ⇒ propagators
 $\mathcal{L}_I$: interactions ⇒ vertices

→ recall : $\mathcal{L}_{QCD} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar\psi(i\not{D} - m)\psi$ (pg 18)

$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$ (pg 17)

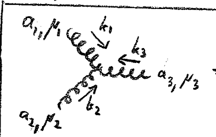
$D_\mu = \partial_\mu - ig A^a_\mu T^a$

$= \mathcal{L}_0 + \bar\psi A^a_\mu g\gamma^\mu T^a \psi - g f^{abc} g^{\nu\sigma} (\partial^\mu A^a_\nu) A^b_\mu A^c_\sigma - \frac{1}{4} g^2 f^{eab} f^{ecd} g^{\mu\sigma} g^{\nu\tau} A^a_\mu A^b_\nu A^c_\sigma A^d_\tau$

- quark-gluon vertex : $\boxed{i \cdot g\gamma^\mu T^a \;\hat{=}\;}$  a, μ
 from $e^{i\int \mathcal{L}_I}$

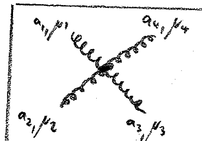
$\left(A(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{A}(k) \right)$

- 3-gluon-vertex : need to fix conventions
 in Fourier space , $\partial_\mu \to -ik_\mu$ ⇒ $i \cdot (-g f^{abc} g^{\nu\sigma})(-ik^\mu)$
 symmetrize this wrt A : 3! possible permutations



$$\hat{=}\; g f^{123} \left\{ (k_1 - k_2)_3\, g_{12} + (k_2 - k_3)_1\, g_{23} + (k_3 - k_1)_2\, g_{31} \right\}$$
(f^{123}: color indices $a_1, \ldots$; g_{12}: Lorentz indices $\mu_1, \ldots$)

- 4-gluon-vertex : $i \cdot \left(-\frac{1}{4} g^2 f^{e12} f^{e34} g^{13} g^{24}\right)$, 4! possible permutations (sets of 4 are equal)



$$\hat{=}\; -ig^2 \left\{ f^{12e} f^{e34} \left(g_{13} g_{24} - g_{14} g_{23}\right) + (1324) + (1423) \right\}$$

→ for the propagators, need to look at

$$S_0 = \int d^4x\, \mathcal{L}_0 = \int d^4x \left\{ \tfrac{1}{2} A_\mu^a \delta^{ab} (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu^b + \sum_{\text{flavor}} \bar{\psi}_f (i\partial\!\!\!/ - m_f) \psi_f \right\}$$

- mini-review: anticommuting (Grassmann) numbers

$\{\theta, \eta\} = 0 \Rightarrow \theta^2 = 0$, Taylor $f(\theta) = a + b\theta$ terminates!

integrals: $\int d\theta = 0$, $\int d\theta\, \theta \equiv 1$

complex Grassmann #s: $\theta = \theta_1 + i\theta_2$, $\theta^* = \theta_1 - i\theta_2$, $(\theta\eta)^* = \eta^*\theta^* = -\theta^*\eta^*$

complex Gauss int: $\int d\theta^* d\theta\, e^{-\theta^* b \theta} = \int d\theta^* d\theta (1 - \theta^* b \theta) = \int d\theta^* d\theta (1 + \theta\theta^* b) = b$

another one: $\int d\theta^* d\theta\, \theta\theta^* e^{-\theta^* b\theta} = \int d\theta^* d\theta\, \theta\theta^* = 1$

higher dim Gauss int: $\left(\prod_i \int d\theta_i^* d\theta_i\right) e^{-\theta_i^* B_{ij} \theta_j} = \left(\ \right) e^{-\sum_i \theta_i^* b_i \theta_i} = \prod_i b_i = \det B$

hermitean; diagonalize by unitary trafo.

derivatives: $\partial_\theta \theta = 1$; e.g. $\partial_\theta \eta\theta = -\partial_\theta \theta\eta = -\eta$ etc.

- quark propagator: consider one quark flavor, $\mathcal{L}_0 \ni \bar{\psi}(i\partial\!\!\!/ - m)\psi$

Grassmann-valued source fields

$$Z^q_{free}[\bar\eta, \eta] = \int \mathcal{D}\bar\psi \mathcal{D}\psi\, e^{i\int d^4x [\bar\psi (i\partial\!\!\!/ - m + i\varepsilon)\psi + \bar\eta\psi + \bar\psi\eta]}$$

shift ψ to complete square (see pg. 22)

$$= Z^q_{free}[0,0]\; e^{-\int d^4x \int d^4y\, \bar\eta(x)\, S_F(x-y)\, \eta(y)}$$

where $S_F(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{i}{k\!\!\!/ - m + i\varepsilon}$

((is again Greens fct: $(i\partial\!\!\!/ - m + i\varepsilon) S_F(x-y) = i\delta^{(4)}(x-y)$))

now e.g. $\langle 0 | T \psi(x_1) \bar\psi(x_2) | 0 \rangle = \dfrac{\int \mathcal{D}\bar\psi \mathcal{D}\psi\, \psi(x_1) \bar\psi(x_2)\, e^{i\int d^4x \mathcal{L}}}{\int \mathcal{D}\bar\psi \mathcal{D}\psi\, e^{i\int d^4x \mathcal{L}}}$

$$= \frac{1}{Z^q_{free}[0,0]} \left(-i\delta_{\bar\eta(x_1)}\right)\left(+i\delta_{\eta(x_2)}\right) Z^q_{free}[\bar\eta, \eta] \Big|_{\bar\eta = \eta = 0}$$

$= S_F(x_1 - x_2)$ Feynman propagator ✓

$$\boxed{\xleftarrow[k]{\hspace{2em}} \;\hat{=}\; \frac{i}{k\!\!\!/ - m + i\varepsilon} = \frac{i(k\!\!\!/ + m)}{k^2 - m^2 + i\varepsilon}}$$

(($(k\!\!\!/ - m)(k\!\!\!/ + m) = k\!\!\!/ k\!\!\!/ - m^2$, $k\!\!\!/ k\!\!\!/ = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} k_\mu k_\nu = k^2$))

→ for gluon propagator, will have same problem as in QED:
$(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu)$ has no inverse (see pg 21)
need ~~Faddeev~~ Faddeev-Popov gauge fixing

- mini-review: defining the functional integral of a gauge theory

$Z \equiv \int \mathcal{D}\phi\, G(\phi)$; ϕ some gauge-fields (A_μ^a) (($G(A) = e^{i\int d^4x (-\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu})}$))

gauge invariance: $G(\phi) = G(\phi_\alpha)$, $\int \mathcal{D}\phi = \int \mathcal{D}\phi_\alpha$ (($(A_\mu)_\alpha = V(A_\mu + \frac{i}{g}\partial_\mu)V^\dagger$, $V = e^{iT^a\alpha^a}$)) see §2.3

$= \int \mathcal{D}\phi\, G(\phi) \underbrace{\Delta(\phi, b) \int \mathcal{D}\alpha\, \delta(f(\phi_\alpha) - b)}_{\equiv 1 \text{ (defines } \Delta)}$ ← arbitrary field b; "gauge condition" f (($f^a(A) = \partial^\mu A_\mu^a$)) covariant gauge

note: $\Delta(\phi, b) = \Delta(\phi_\alpha, b)$ owing to $\int \mathcal{D}\alpha$ in its definition

$= \int \mathcal{D}\alpha \int \mathcal{D}\phi_\alpha\, G(\phi_\alpha)\, \Delta(\phi_\alpha, b)\, \delta(f(\phi_\alpha) - b)$ used gauge invariance of $\int \mathcal{D}\phi, G, \Delta$ (at each α)

$= \left(\int \mathcal{D}\alpha\right) \int \mathcal{D}\phi\, G(\phi)\, \Delta(\phi, b)\, \delta(f(\phi) - b)$ renamed int variable $\phi_\alpha \to \phi$

"volume" of gauge orbit factors out; cancels in expectation values!

now average over b, with weight $B(b)$ (($B(b) = e^{-\frac{i}{2\xi}\int d^4x\, b^a b^a}$))

$= \dfrac{\left(\int \mathcal{D}\alpha\right)}{\left(\int \mathcal{D}b\, B(b)\right)} \int \mathcal{D}\phi\, G(\phi) \int \mathcal{D}b\, B(b)\, \Delta(\phi, b)\, \delta(f(\phi) - b)$

$= \dfrac{\left(\int \mathcal{D}\alpha\right)}{\left(\int \mathcal{D}b\, B(b)\right)} \int \mathcal{D}\phi\, G(\phi)\, B(f(\phi))\, \Delta(\phi, f(\phi))$ used δ-fct

now compute the "Faddeev-Popov determinant" Δ from its definition:

$\Delta(\phi, f(\phi)) = \left[\int \mathcal{D}\alpha\, \delta(f(\phi_\alpha) - f(\phi))\right]^{-1}$ cross δ-peak with infinit. gauge trafo

$= \left[\int \mathcal{D}\alpha\, \delta(\alpha \mathcal{F}(\phi) + \mathcal{O}(\alpha^2))\right]^{-1}$ $f(\phi_\alpha) \approx f(\phi) + \alpha \mathcal{F}(\phi) + \mathcal{O}(\alpha^2)$

$= \left[\frac{1}{\det \mathcal{F}(\phi)} \int \mathcal{D}\alpha\, \delta(\alpha)\right]^{-1} = \det \mathcal{F}(\phi)$

$= \dfrac{\left(\int \mathcal{D}\alpha\right)}{\left(\int \mathcal{D}b\, B(b)\right)} \int \mathcal{D}\phi\, G(\phi)\, B(f(\phi)) \det \mathcal{F}(\phi)$, where $\mathcal{F}(\phi) = (\partial_\phi f(\phi))(\partial_\alpha \phi_\alpha)_{\alpha=0}$

(($\mathcal{F}(A) = \partial^\mu (f^{abc} A_\mu^b + \frac{1}{g}\delta^{ac}\partial_\mu)$, pg 17))

note: in QED, $(A_\mu)_\alpha = A_\mu - \frac{1}{e}\partial_\mu \alpha$ (pg 12),
so $\mathcal{F}$ does not depend on A, hence $\det \mathcal{F}$ cancels in correlators.
⇒ in QCD, $\det \mathcal{F}(A)$ remains inside the functional integral.

- <u>gluon propagator</u>

collection from above, $G(\phi)\, B(f(\phi)) = e^{i\int d^4x\, \mathcal{L}_{QCD}}\, e^{i\int d^4x \left(-\frac{1}{2\xi}\right)(\partial^\mu A_\mu^a)^2}$

such that $S_0 \ni \int d^4x\ \frac{1}{2} A_\mu^a\, \delta^{ab} \left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu + \frac{1}{\xi}\partial^\mu\partial^\nu\right) A_\nu^b$

$= \int \frac{d^4k}{(2\pi)^4}\ \frac{1}{2} \tilde{A}_\mu^a(k)\, \delta^{ab} \underbrace{\left(-k^2 g^{\mu\nu} + \left(1-\frac{1}{\xi}\right) k^\mu k^\nu\right)}\, A_\nu^b(-k)$

((check?!))

$$\boxed{a,\mu \sim\sim\sim b,\nu \ \hat{=}\ \frac{-i}{k^2+i\varepsilon} \left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) \delta^{ab}}$$

$\left(\left(\text{since } (\sim\sim)^{ab}_{\mu\nu} \cdot (\sim\sim)^{bc,\nu\varsigma} \overset{\checkmark}{=} i\,\delta^{ac} g_\mu{}^\varsigma\right)\right)$

ξ: <u>gauge parameter</u>; often, use $\xi = 1$ <u>Feynman gauge</u>

as in QED, physics is ξ-independent

- <u>Faddeev-Popov ghost fields</u>

have to take care of $\det \mathcal{F}(\phi)$ factor (on bottom of pg 26)

using Grassmann numbers again (see Gauss integral on pg 25), rewrite

$\det \mathcal{F}(A) = \det\left(\frac{1}{g}\partial^\mu \left[g f^{abc} A_\mu^b + \delta^{ac}\partial_\mu\right]\right)$

$= \int \mathcal{D}c\, \mathcal{D}\bar{c}\ e^{i\int d^4x\ \bar{c}^a \left(-\partial^\mu\left[\partial_\mu \delta^{ac} + g f^{abc} A_\mu^b\right]\right) c^c}$

where the "<u>FP ghosts</u>" are anticommuting fields

(but there are no γ-matrices $\Rightarrow$ spin 0 !)

$\Rightarrow$ ghost propagator	$a \cdots\leftarrow\cdots b \ \hat{=}\ \frac{i}{k^2+i\varepsilon}\, \delta^{ab}$
$\Rightarrow$ ghost-gluon-vertex	$a,\ k,\ c$ $\sim\sim\sim b,\mu \ \hat{=}\ -g f^{abc} k_\mu$

((for physical interpretation of ghosts, see e.g. Peskin/Schroeder § 16.3))

$\longrightarrow$ now, know all propagators and vertices of QCD,

so we can again (as in ϕ^4 theory, see § 2.4) do perturbative expansions via generating functional.

- short summary:

from pg. 18, 26, 27 we have now

$$\boxed{\mathcal{L}_{QCD} = \sum_f \bar{\psi}_f (i\not{D} - m_f)\psi_f - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar{c}^a\left(-\partial^\mu D_\mu^{ac}\right) c^c}$$

where $D_\mu^{ac} = \partial_\mu \delta^{ac} + g f^{abc} A_\mu^b$ is the "covariant derivative in the adjoint representation"

$\longrightarrow$ this expression is still invariant, but <u>not</u> under a local gauge transformation as in § 2.2; the relevant transformation now includes the ghost fields $\bar{c}, c$ in an essential way and is called "<u>BRST</u>" transformation

[Becchi/Rouet/Stora, Ann. Phys. 98 (1976) 287;
Tyutin, Theor. Math. Phys. 27 (1976) 316]

(a symmetry with continuous but anticommuting parameters)

((more: QFT lecture; e.g. Peskin/Schroeder §))

$\longrightarrow$ the formal treatment of pg. 26 had implicitly <u>assumed</u> that the gauge condition $f(\phi)$ $((f^a(A_\mu^a)))$ selects (via the Delta-fct) one unique representative ϕ $((A_\mu^a))$ for each "gauge orbit" ϕ_Ω.

However, <u>Gribov</u> [Gribov, Nucl. Phys. B 139 (1978) 1] has demonstrated that for <u>non-Abelian</u> theories, this cannot always be guaranteed.

In practice, this fact has little relevance.

- "set of "Feynman rules" as usual

$\longrightarrow$ see QFT lecture; Particle physics lecture; ...

draw diagrams – fix symmetry factors – insert Feynman rules for propagators + vertices – perform traces and Lorentz algebra – regularize divergent integrals – Wick rotation – evaluate loop integrals – ...

3. Fundamentals

→ our $\mathcal{L}_{QCD}$ contains operators of dimension ≤ 4

⇒ theory is renormalizable: all divergences can be removed by a finite number of counterterms

here: illustrate some divergences in QCD

important physical consequence: asymptotic freedom!

- mini-review: renormalization [see e.g. Peskin/Schroeder §10]

loops → $\int d^4k$ → ultraviolet (UV) divergences (from large k ⇔ small x)

common and natural in QFT.

→ counting of UV div's (in 1PI diags): superficial degree of divergence

→ idea: "hide" div's in $\mathcal{L}$ by rescaling fields + couplings

$\underset{\text{bare}}{\phi_B} \to Z_\phi \underset{\text{renormalized}}{\phi_R}$ etc, $Z = 1 + \mathcal{O}(g_R^2)$

→ need intermediate regularization of divergent loop integrals

many possibilities (discretize space-time; cutoff large momenta; Pauli-Villars; ...)

most elegant for us: dimensional regularization $\int \frac{d^4k}{(2\pi)^4} \to \int \frac{d^dk}{(2\pi)^d}$,

$\left(\begin{matrix} d = 4-2\varepsilon \\ \varepsilon \to 0 \end{matrix}\right)$

$d \in \mathbb{C}$; analytic cont. $d \to 4$; div's will be poles $\sim \frac{1}{d-4}$

introduce artificial mass-scale μ

e.g. $(m^2)^{-\varepsilon} = \mu^{-2\varepsilon}\left(\frac{\mu^2}{m^2}\right)^{\varepsilon} \stackrel{\left[x^\varepsilon = e^{\varepsilon \ln x} \approx \text{Taylor...}\right.}{=} \mu^{-2\varepsilon}\left(1 + \varepsilon \ln \frac{\mu^2}{m^2} + \mathcal{O}(\varepsilon^2)\right)$

→ the renormalization group (RG) equations

describe the μ-dependence of parameters / Greens functions

- mini-review: dimensional regularization

$\int d^dk\, f(k+q) = \int d^dk\, f(k)$

$\int d^dk\, f(\lambda k) = |\lambda|^{-d} \int d^dk\, f(k)$ ⇒ $\int d^dk = 0 = \int \frac{d^dk}{(k^2)^{\alpha \neq d/2}}$ (!)

$\int d^dk\, e^{-\alpha k^2} = \left[\int_{-\infty}^{\infty} dk_1\, e^{-\alpha k_1^2}\right]^d = \left(\frac{\pi}{\alpha}\right)^{d/2}$

$\int d^dk\, f(|k|) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dk\, k^{d-1} f(k)$ d-dim. spherical coordinates

$\int d^dk\, k^\mu f(|k|) = 0$ (odd in k)

$\int d^dk\, k^\mu k^\nu f(|k|) \stackrel{!}{=} g^{\mu\nu} I = \int d^dk\, \frac{g^{\mu\nu} k^2}{d} f(|k|)$ (since $\int d^dk\, k^2 f = g^\mu{}_\mu I = dI$)

- important integrals

$\Gamma(n) = (n-1)! = \int_0^\infty dt\, t^{n-1} e^{-t}$

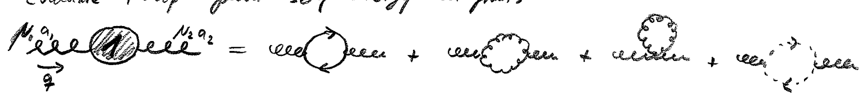
$\int_0^\infty dx\, \frac{x^{2a-1}}{(x^2+1)^b} = \frac{\Gamma(a)\Gamma(b-a)}{2\,\Gamma(b)}$ (if $\text{Re}[a] > 0$, $\text{Re}[b-a] > 0$)

$(A_i = \hat{A}_i + i\varepsilon)$

$\frac{1}{A_1^{a_1} A_2^{a_2} \cdots A_n^{a_n}} = \frac{\Gamma(a_1 + \dots + a_n)}{\Gamma(a_1) \cdot \ldots \cdot \Gamma(a_n)} \int_0^1 dx_1 \ldots dx_n \frac{x_1^{a_1-1} \cdot \ldots \cdot x_n^{a_n-1}\, \delta(1 - \sum_{i=1}^n x_i)}{[x_1 A_1 + \dots + x_n A_n]^{a_1 + \dots + a_n}}$ Feynman parametrization

3.1 one-loop divergences in QCD

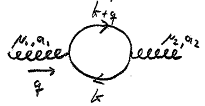
→ goal: evaluate 1-loop gluon self-energy diagrams



use dimensional regularization, $d^4k \to d^dk$, $d = 4 - 2\varepsilon$

use Feynman gauge $\xi = 1$ for gluon propagator (for simplicity)

- 1st diagram: consider one quark flavor, do $\sum_f$ in the end



$= -\,\mathrm{Tr} \int \frac{d^dk}{(2\pi)^d} \left(ig\gamma^{\mu_1} T^{a_1}\right) \frac{i(\not{k}+m)}{k^2 - m^2 + i\varepsilon} \left(ig\gamma^{\mu_2} T^{a_2}\right) \frac{i(\not{k}+\not{q}+m)}{(k+q)^2 - m^2 + i\varepsilon}$

↑ one closed fermion loop; ↑ trace over γ's and T's

remember: Dirac matrices

def $\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \mathbb{1}$; $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$

$\mathrm{tr}(\mathbb{1}) \equiv N$ (we take $N = 4$; also $N = 2^{d/2}$, $N = d$ seen)

$\mathrm{tr}(\gamma^\mu) = 0$; $\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho) = 0$; ...

$\mathrm{tr}(\gamma^\mu \gamma^\nu) = N g^{\mu\nu}$; $\mathrm{tr}(\gamma^{1234}) = N\left(g^{12} g^{34} - g^{13} g^{24} + g^{14} g^{23}\right)$; ...

$= -g^2 (\mathrm{tr}\,\mathbb{1}) \underbrace{(\mathrm{tr}\, T^{a_1} T^{a_2})}_{\frac{1}{2}\delta^{12}\text{, see pg. 9}} \int \frac{d^dk}{(2\pi)^d} \frac{N^{\mu_1\mu_2}(k, k+q)}{(k^2 - m^2 + i\varepsilon)((k+q)^2 - m^2 + i\varepsilon)}$

$N^{13}(k,p) \stackrel{\mathrm{tr}(\gamma\text{'s})}{=} m^2 g^{13} + m \cdot 0 + \left(g^{12} g^{34} - g^{13} g^{24} + g^{14} g^{23}\right) k_2 p_4$

$= (m^2 - k \cdot p) g^{13} + k^1 p^3 + p^1 k^3$

for denominator, use Feynman parameters, see p. 30

$$\frac{1}{(k^2-m^2+i\varepsilon)((k+q)^2-m^2+i\varepsilon)} = \int_0^1 dx \frac{1}{\left[(1-x)(k^2-m^2+i\varepsilon)+x(k^2+2kq+q^2-m^2+i\varepsilon)\right]^2} = \int_0^1 dx \frac{1}{[(k+xq)^2+\underbrace{x(1-x)q^2-m^2}_{\equiv -\Delta}+i\varepsilon]^2}$$

now, shift $k \to k-xq$ under integral $\int d^dk$

$$= -g^2 \cdot 4 \cdot \tfrac{1}{2}\delta^{a_1a_2} \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{N^{\mu_1\mu_2}(k-xq, k+(1-x)q)}{[k^2-\Delta+i\varepsilon]^2}$$

$$\text{numerator} = \left(m^2-(k^2-x(1-x)q^2+\cancel{\mathcal{O}(k)})\right)g^{\mu_1\mu_2} + 2\boxed{k^{\mu_1}k^{\mu_2}} - 2x(1-x)q^{\mu_1}q^{\mu_2} + \cancel{\mathcal{O}(k)}$$

linear (in k) terms integrate to 0; $\quad k^{\mu_1}k^{\mu_2} \to g^{\mu_1\mu_2}\frac{k^2}{d}$, see p. 29 bottom

$$= \left(m^2+(\tfrac{2}{d}-1)k^2+x(1-x)q^2\right)g^{12} - 2x(1-x)q^1q^2$$

to evaluate the $\int d^dk$ integral, perform a Wick rotation $k^0 = ik_E^0$

$$k^2-m^2+i\varepsilon = k_0^2-\vec{k}^2-m^2+i\varepsilon = (k_0+\sqrt{\vec{k}^2+m^2}-i\varepsilon)(k_0-\Gamma+i\varepsilon)$$

$-\Gamma+i\varepsilon$, $\Gamma-i\varepsilon$, $\quad 0 = \oint$, $\to = -\downarrow = +\uparrow$: $k^0 = ik_E^0$

$$= -g^2 \cdot 4 \cdot \tfrac{1}{2}\delta^{a_1a_2}\int_0^1 dx\; i\left\{\left[\left(m^2+x(1-x)q^2\right)g^{12}-2x(1-x)q^1q^2\right]I_2^0(\Delta) + (1-\tfrac{2}{d})g^{12}I_2^1(\Delta)\right\}$$

where $I_n^a(\Delta) \equiv \int \frac{d^dk_E}{(2\pi)^d}\frac{(k_E^2)^a}{[k_E^2+\Delta]^n}$ (basic 1-loop "tadpole" integral)

$$= \frac{1}{(2\pi)^d}\frac{2\pi^{d/2}}{\Gamma(d/2)}\underbrace{\int_0^\infty dk_E \frac{k_E^{d-1+2a}}{(k_E^2+1)^n}}_{=\frac{\Gamma(a+\frac{d}{2})\Gamma(n-a-\frac{d}{2})}{2\Gamma(n)}} |\Delta|^{\frac{d}{2}+a-n}$$ see p. 30

$$= \frac{\Gamma(a+\frac{d}{2})\Gamma(n-a-\frac{d}{2})}{(4\pi)^{d/2}\,\Gamma(\frac{d}{2})\Gamma(n)\,|\Delta|^{n-a-\frac{d}{2}}}$$

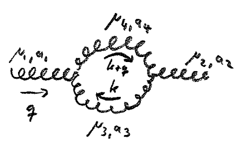
note that $I_2^1(\Delta) = I_2^0(\Delta)\cdot|\Delta|\cdot\frac{d}{2}\frac{1}{1-\frac{d}{2}} = I_2^0(\Delta)\frac{|\Delta|}{\frac{2}{d}-1}$ $\quad (x\Gamma(x)=\Gamma(x+1))$

$$= -g^2\cdot 4\cdot\tfrac{1}{2}\delta^{12}\int_0^1 dx\; i\, I_2^0(\Delta)\left\{\underbrace{\left(m^2+x(1-x)q^2-|\Delta|\right)}_{\Delta>0:\ = 2x(1-x)q^2}g^{12} - 2x(1-x)q^1q^2\right\}$$

$$= -4ig^2\delta^{a_1a_2}\left(g^{\mu_1\mu_2}q^2-q^{\mu_1}q^{\mu_2}\right)\underbrace{\int_0^1 dx\, I_2^0(m^2-x(1-x)q^2)\,x(1-x)}$$

$$= \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\int_0^1 dx\frac{x(1-x)}{[m^2-x(1-x)q^2]^{2-d/2}}$$

- 2^{nd} diagram : calculation parallels the one above!



symmetry factor, Feynman gauge, color indices a_i

$$= \frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\left(\frac{-i}{k^2}\right)\left(\frac{-i}{(k+q)^2}\right)g^2 f^{134}f^{243} \times$$

Lorentz indices μ_i

$$\times\left[(q-k)_4 g^{13} + (2k+q)_1 g^{34} + (-k-2q)_3 g^{41}\right]\times$$
$$\times\left[(-k-2q)_3 g^2_4 + (2k+q)^2 g_{34} + (-k+q)_4 g_3^{\ 2}\right]$$

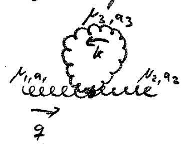
Feynman parameters; shift $k\to k-xq$; $\Delta \equiv -x(1-x)q^2$; in numerator, linear $k\to 0$, $k^\mu k^\nu \to g^{\mu\nu}\frac{k^2}{d}$

$$= \frac{1}{2}g^2 f^{134}f^{234}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-\Delta)^2}\times$$
$$\times\left\{g^{12}\left[k\cdot k\, 6\frac{d-1}{d} + q\cdot q\left((2-x)^2+(1+x)^2\right)\right] - q^1q^2\left[(2-d)(1-2x)^2+2(1+x)(2-x)\right]\right\}$$

Wick rotate, use basic 1-loop tadpole integrals, $I_2^1 = I_2^0\frac{d}{d-2}\Delta$

$$= ig^2 f^{a_1a_3a_4}f^{a_2a_3a_4}\int_0^1 dx\, I_2^0(-x(1-x)q^2)\;\tfrac{1}{2}\left\{g^{\mu_1\mu_2}q^2\left[6\frac{d-1}{2-d}x(1-x)+5-2x+2x^2\right] - q^{\mu_1}q^{\mu_2}\left[(2-d)(1-2x)^2+2(1+x)(2-x)\right]\right\}$$

- 3^{rd} diagram:



template for $\underset{1\ \ 2}{\overset{4\ \ 3}{\times}}$: $\underset{12}{f^{12\circ}}\underset{34}{f^{\circ34}}(g_{13}g_{24}-g_{14}g_{23}) + 1324 + 1423$ $\quad$ 1323 $\quad$ 1323 $\leftarrow$ here

$$= \frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\left(\frac{-i}{k^2}\right)(-ig^2)\left\{f^{12\circ}\cancel{f^{\circ 33}}^{\,0}(..) + 2f^{13\circ}f^{\circ 23}\underbrace{(g^{12}g^3_{\ 3}-g^{13}g_3^{\ 2})}_{=g^{12}(d-1)}\right\}$$

symm. factor

$$= -g^2 f^{134}f^{234}g^{12}(d-1)\underbrace{\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}}_{\text{would be zero}}\times\underbrace{\frac{k^2+2kq+q^2}{(k+q)^2}}_{\text{try to make it look like }2^{nd}\text{ diag}}$$

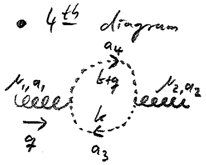
Fey par, shift $k\to k-xq$, $\Delta\equiv -x(1-x)q^2$, numerator $k=0$

$$= -g^2 f^{134}f^{234}g^{12}(d-1)\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{k^2+(1-x)^2q^2}{(k^2-\Delta)^2}$$

Wick rotate, use basic 1-loop tadpole ints

$$= ig^2 f^{a_1a_3a_4}f^{a_2a_3a_4}\int_0^1 dx\, I_2^0(-x(1-x)q^2)\; g^{\mu_1\mu_2}q^2(1-d)\left[\frac{d}{2-d}x(1-x)+(1-x)^2\right]$$

• 4^{th} diagram



closed loop of anticomm. fields; color; Lorentz

$$= -\int \frac{d^d\ell}{(2\pi)^d} \frac{i}{\ell^2} \frac{i}{(\ell+q)^2} (-g)^2 f^{413} (\ell+q)^1 f^{324} \ell^2$$

$$= -g^2 f^{134} f^{234} \int \frac{d^d\ell}{(2\pi)^d} \frac{(\ell+q)^1 \ell^2}{\ell^2 (\ell+q)^2}$$

Fey par, shift $\ell \to \ell - xq$, $\Delta \equiv -x(1-x)q^2$, numerator $\ell = 0$, $\ell^\mu \ell^\nu \to g^{\mu\nu}\frac{\ell^2}{d}$

$$= -g^2 f^{134} f^{234} \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta)^2} \left\{ \frac{g^{12}\ell^2}{d} - x(1-x) q^1 q^2 \right\}$$

Wick rotate, use 1-loop tadpole integrals

$$= ig^2 f^{a_1 a_3 a_4} f^{a_2 a_3 a_4} \int_0^1 dx\, I_2^0(-x(1-x)q^2)\; x(1-x) \left\{ -g^{\mu_1\mu_2} q^2 \frac{1}{2-d} + q^{\mu_1} q^{\mu_2} \right\}$$

• sum diagrams 2+3+4

$$2+3+4 = ig^2 f^{134} f^{234} \int_0^1 dx\, I_2^0(-x(1-x)q^2)\; \times$$

$$\times \Big\{ g^{\mu_1\mu_2} q^2 \Big[\underbrace{3\tfrac{d-1}{2-d} x(1-x) + \tfrac{1}{2}(5-2x+2x^2)}_{} + \underbrace{(1-d)\tfrac{d}{2-d} x(1-x) + (1-d)(1-x)^2}_{} - \underbrace{\tfrac{1}{2-d} x(1-x)}_{} \Big]$$

$$- q^{\mu_1} q^{\mu_2} \Big[\underbrace{\tfrac{2-d}{2}(1-2x)^2 + (1+x)(2-x)}_{\text{from}} \qquad\qquad \underbrace{- x(1-x)}_{} \Big] \Big\}$$

note that first line is invariant under $x \to 1-x$

$\{\dots\}$ is polynomial $\sim 1 + x + x^2$

re-express this in terms of $a \equiv 1-2x$ which is odd under $x \to 1-x$

((i.e. $x^2 = \frac{1}{4}(a^2-2a+1)$, $x = \frac{1}{2}(1-a)$))

$$= ig^2 f^{134} f^{234} \int_0^1 dx\, I_2^0(-x(1-x)q^2) \left\{ g^{12} q^2 \Big[(1-\tfrac{d}{2})a^2 + \underbrace{\tfrac{1-d}{2}a}_{\text{integrates to 0 (odd under } x\to 1-x)} + 2 \Big] - q^1 q^2 \Big[(1-\tfrac{d}{2})a^2 + 2 \Big] \right\}$$

$$= ig^2 \underbrace{f^{a_1 a_3 a_4} f^{a_2 a_3 a_4}}_{= N_c \delta^{a_1 a_2} \text{ for } SU(N_c)} \left(g^{\mu_1\mu_2} q^2 - q^{\mu_1} q^{\mu_2} \right) \int_0^1 dx\, I_2^0(-x(1-x)q^2) \left[(1-\tfrac{d}{2})(1-2x)^2 + 2 \right]$$

((wow: same Lorentz-structure as !))

derivation: from Lie algebra (pg. 9) $[T^a, T^b] = if^{abc} T^c$

$$\Rightarrow f^{abc} = -2i\, \mathrm{tr}([T^a, T^b] T^c)$$

$$= -2i \{ (T^a)_{ij} (T^b)_{jk} (T^c)_{ki} - (T^b)_{ij} (T^a)_{jk} (T^c)_{ki} \}$$

then use Fierz identity (pg. 9) $(T^a)_{ij} (T^a)_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)$,

normalization (pg. 9) $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$, and $\delta_{ii} = N_c$

• summing up all four diagrams (see pg. 31, 33)

$$= ig^2 \delta^{a_1 a_2} \left(g^{\mu_1\mu_2} q^2 - q^{\mu_1} q^{\mu_2} \right) \int_0^1 dx \; *$$

$$* \left\{ -4x(1-x) \sum_f I_2^0\left(m_f^2 - x(1-x)q^2\right) + N_c \left[(1-\tfrac{d}{2})(1-2x)^2 + 2 \right] I_2^0(-x(1-x)q^2) \right\}$$

now, use (pg. 31) $I_2^0(\Delta) = \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-d/2}} = \frac{1}{2-\frac{d}{2}} \frac{\Gamma(3-\frac{d}{2})}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-d/2}}$

$\overset{d=4-2\varepsilon}{\approx} \frac{1}{\varepsilon} \frac{1}{(4\pi)^2} + \mathcal{O}(\varepsilon^0)$

$$\Rightarrow \{\dots\} \approx \frac{1}{\varepsilon} \frac{1}{(4\pi)^2} \left\{ -4x(1-x) N_f + N_c\left[-(1-2x)^2 + 2 \right] \right\} + \mathcal{O}(\varepsilon^0)$$

$$\Rightarrow \int_0^1 dx \{\dots\} \approx \frac{1}{\varepsilon} \frac{1}{(4\pi)^2} \left\{ -\frac{2}{3} N_f + \frac{5}{3} N_c \right\} + \mathcal{O}(\varepsilon^0)$$

$$\approx i \frac{g^2}{16\pi^2} \delta^{a_1 a_2} \left(g^{\mu_1\mu_2} q^2 - q^{\mu_1} q^{\mu_2} \right) \frac{1}{\varepsilon} \left\{ \boxed{\tfrac{5}{3}} N_c - \frac{2}{3} N_f + \mathcal{O}(\varepsilon) \right\}$$

$\hookrightarrow \left(\frac{13}{6} - \frac{\xi}{2}\right)$ for general value of gauge parameter ξ

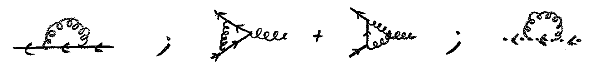
⤳ cannot (yet) take limit $\varepsilon \to 0$

need to remove the $\frac{1}{\varepsilon}$ divergence by a counterterm (see pg 29; § 3.3 below)

⤳ want to first compute other 1-loop divergences

3.2 more 1-loop divergences in QCD

→ goal: evaluate 1-loop diagrams (again, in dim. reg. $d=4-2\varepsilon$ and Feynman gauge $\xi = 1$)



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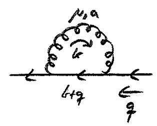


$$= \int \frac{d^d\ell}{(2\pi)^d} (ig\gamma^\mu T^a) \frac{i(\not{\ell} + \not{q} + m)}{(\ell+q)^2 - m^2 + i\varepsilon} (ig\gamma_\mu T^a) \left(\frac{-i}{\ell^2}\right)$$

$$\left[\ \gamma^\mu\gamma_\mu = \gamma^\mu\gamma^\nu g_{\mu\nu} = \tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\} g_{\mu\nu} = \tfrac{1}{2} 2 g^{\mu\nu} \mathbb{1} g_{\mu\nu} = d\,\mathbb{1}\right.$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = \{\gamma^\mu, \gamma^\nu\}\gamma_\mu - \gamma^\nu\gamma^\mu\gamma_\mu = 2\gamma^\nu - \gamma^\nu d = (2-d)\gamma^\nu$$

$$\left. T^a_{ij} T^a_{jk} = \tfrac{1}{2}\left(\delta_{ik}\delta_{jj} - \tfrac{1}{N_c}\delta_{ij}\delta_{jk}\right) = \frac{N_c^2 - 1}{2N_c}\delta_{ik}\right.$$

$$= -g^2 \frac{N_c^2 - 1}{2N_c} \mathbb{1}_{N\times N} \int \frac{d^d\ell}{(2\pi)^d} \frac{(2-d)(\not{\ell} + \not{q}) + md\,\mathbb{1}_{Dirac}}{(\ell^2)\left((\ell+q)^2 - m^2\right)}$$

Fey par, $\frac{1}{()()} = \int_0^1 dx \frac{1}{[(1-x)k^2 + x(k^2+2kq+q^2-m^2)]^2} = \int_0^1 dx \frac{1}{[(k+xq)^2 + \underbrace{x(1-x)q^2 - xm^2}_{\equiv -\Delta}]^2}$

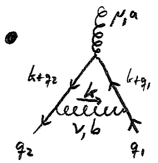
shift $k \to k - xq$

$$= -g^2 \frac{N^2-1}{2N} \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{(2-d)(\underbrace{\not k}_{\to 0 \text{ (odd)}} + (1-x)\not q) + md}{(k^2-\Delta)^2}$$

Wick rotate, use basic tadpole from pg 31

$$= -ig^2 \frac{N^2-1}{2N} \int_0^1 dx \underbrace{I_2^0(xm^2 - x(1-x)q^2)}_{\approx \frac{1}{\varepsilon}\frac{1}{(4\pi)^2}\text{, see pg. 34}} \{(2-d)(1-x)\not q + md\}$$

$$\approx i \frac{g^2}{16\pi^2} \frac{N_c^2-1}{2N_c} \frac{1}{\varepsilon} \{\underbrace{\boxed{1}}_{\to \xi} \not q - \underbrace{\boxed{4}}_{\to (3+\xi)} m + O(\varepsilon)\}$$ for general value of ξ

- 

$$= \int \frac{d^dk}{(2\pi)^d} (ig)^3 T^b T^a T^b \frac{\gamma^\nu i(\not k + \not q_2 + m)\gamma^\mu i(\not k + \not q_1 + m)\gamma_\nu}{((k+q_2)^2 - m^2)\quad((k+q_1)^2-m^2)} \frac{(-i)}{k^2}$$

$$\gamma^{\mu\nu\rho}{}_\mu = (2g^{\mu\nu} - \gamma^\nu\gamma^\mu)\gamma^\rho{}_\mu = \underbrace{2\gamma^{\rho\nu}}_{=2g^{\rho\nu}-\gamma^\nu\gamma^\rho} - \gamma^\nu \underbrace{\gamma^{\mu\rho}{}_\mu}_{=(2-d)\gamma^\rho} = 4g^{\nu\rho} - (4-d)\gamma^{\nu\rho}$$

$$\gamma^{\mu\nu\rho\sigma}{}_\mu = (2g^{\mu\nu} - \gamma^\nu\gamma^\mu)\gamma^{\rho\sigma}{}_\mu = \underbrace{2\gamma^{\rho\sigma}\gamma^\nu}_{=\cancel{2g^{\rho\sigma}}-\gamma^{\sigma\rho}} - \gamma^\nu(\cancel{4g^{\rho\sigma}} - (4-d)\gamma^{\rho\sigma}) = -2\gamma^{\sigma\rho\nu} + (4-d)\gamma^{\nu\rho\sigma}$$

$$T^b_{ij} T^a_{jk} T^b_{kl} = T^a_{jk} \frac{1}{2}(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}) = \frac{1}{2}\delta_{il}\overset{0}{T^a_{jj}} - \frac{1}{2N}T^a_{il} = -\frac{1}{2N}T^a$$

$$= g^3(-\tfrac{1}{2N}T^a) \int \frac{d^dk}{(2\pi)^d} \frac{N^\mu}{((k+q_2)^2-m^2)((k+q_1)^2-m^2)k^2}$$

$$N^\mu = m^2(2-d)\gamma^\mu + m(k+q_2)_\nu(4g^{\nu\mu} - (4-d)\gamma^{\nu\mu}) + m(k+q_1)_\nu(4g^{\mu\nu} - (4-d)\gamma^{\mu\nu})$$
$$+ (k+q_2)_\nu(k+q_1)_\sigma[-2\gamma^{\sigma\mu\nu} + (4-d)\gamma^{\nu\mu\sigma}]$$

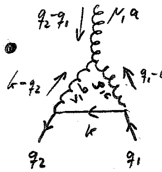
superficial degree of divergence of this integral: $\frac{k^4}{k^6} \to \log$!

$\Rightarrow$ look at $k \gg \{q_1, q_2, m\}$ only, to extract this leading log div

$$\sim g^3(-\tfrac{1}{2N}T^a) \int \frac{d^dk}{(2\pi)^d} \frac{[-2\gamma^{\sigma\mu\nu} + (4-d)\gamma^{\nu\mu\sigma}]k_\nu k_\sigma}{(k^2)^3} + \text{const.}$$

numerator $\to [\dots] \frac{g_{\nu\sigma}k^2}{d} = \frac{k^2}{d}[-2\gamma^{\sigma\mu}{}_\sigma + (4-d)\gamma^{\nu\mu}{}_\nu] \overset{\text{(pg. 34)}}{=} \frac{k^2}{d}(2-d)^2\gamma^\mu$

$$= g^3(-\tfrac{1}{2N}T^a)\frac{(2-d)^2}{d}\gamma^\mu \underbrace{\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^2}}_{= iI_2^0(0) \text{ (after Wick rot.)}}$$

- 

$$= \int \frac{d^dk}{(2\pi)^d} (ig)^2 T^b T^c \frac{\gamma_\nu\, i(\not k + m)\,\gamma_\rho}{(k^2-m^2)} \frac{(-i)}{(k-q_2)^2}\frac{(-i)}{(k-q_1)^2}\, g f^{abc} *$$
$$* [(2q_2 - q_1 - k)^\rho g^{\mu\nu} + (2k - q_1 - q_2)^\mu g^{\nu\rho} + (2q_1 - q_2 - k)^\nu g^{\rho\mu}]$$

$$T^b T^c f^{abc} = \frac{1}{2}[T^b, T^c] f^{abc} = \frac{i}{2}\underbrace{f^{bcd} T^d f^{abc}}_{= N_c \delta^{ad} \text{ (see pg. 33)}} = \frac{i}{2}N_c T^a$$

again, extract only leading log div, $k \gg \{q_1, q_2, m\}$

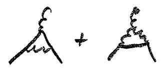
$$\sim -g^3 \frac{N_c}{2} T^a \int \frac{d^dk}{(2\pi)^d} \frac{\gamma_\nu \not k \gamma_\rho [-k^\rho g^{\mu\nu} + 2k^\mu g^{\nu\rho} - k^\nu g^{\rho\mu}]}{(k^2)^3} + \text{const.}$$

replace $k^\mu k^\nu \to \frac{g^{\mu\nu}k^2}{d}$

numerator $\to \gamma_\nu\gamma_\sigma\gamma_\rho[-g^{\sigma\rho}g^{\mu\nu} + 2g^{\sigma\mu}g^{\nu\rho} - g^{\sigma\nu}g^{\rho\mu}]$
$= -\gamma^{\mu\rho}{}_\rho + 2\gamma^{\rho\mu}{}_\rho - \gamma_\nu{}^{\nu\mu} \overset{\text{(pg. 34)}}{=} -d\gamma^\mu + 2(2-d)\gamma^\mu - d\gamma^\mu = 4(1-d)\gamma^\mu$

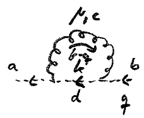
$$= -g^3\, 2N_c T^a \frac{1-d}{d}\gamma^\mu \underbrace{\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^2}}_{= iI_2^0(0) \text{ (after Wick rot.)}}$$

- sum of last two diagrams



$$= -ig^3 T^a \gamma^\mu I_2^0(0) \{\frac{1}{2N}\frac{(2-d)^2}{d} + 2N\frac{1-d}{d}\} + \text{const.}$$

$$\overset{d=4-2\varepsilon}{\approx} -igT^a\gamma^\mu \frac{g^2}{16\pi^2}\frac{1}{\varepsilon}\{\underbrace{\boxed{1}}_{\to \xi}\frac{1}{2N} - \frac{3}{4}\underbrace{\boxed{2}}_{\to (\xi+1)}N + O(\varepsilon)\}$$ for general value of ξ

- 

$$= \int \frac{d^dk}{(2\pi)^d} (-gf^{acd} q^\mu)(-gf^{deb} k_\mu) \frac{i}{k^2}\frac{-i}{(k-q)^2}$$

$$= -g^2 \underbrace{f^{acd} f^{cdb}}_{= N_c \delta^{ab} \text{ (pg. 33)}} \int \frac{d^dk}{(2\pi)^d} \frac{q \cdot k}{k^2(k-q)^2}$$

denominator invariant under $k \to q - k$

write numerator $k = \frac{1}{2}k + \frac{1}{2}k \to \frac{1}{2}k + \frac{1}{2}(q-k) = \frac{1}{2}q$

$$= -g^2 N_c \delta^{ab} \frac{q^2}{2} \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k-q)^2}$$

extract leading log div, $k \gg q$; Wick rot; use tadpole

$$\sim -ig^2 N_c \delta^{ab} \frac{q^2}{2} I_2^0(0) + \text{const.}$$

$$\overset{d=4-2\varepsilon}{\approx} -i\frac{g^2}{16\pi^2}\delta^{ab} q^2 \frac{1}{\varepsilon}\{\underbrace{\boxed{2}}_{\to (3-\xi)}\frac{N_c}{4} + O(\varepsilon)\}$$ for general value of ξ

3.3 one-loop counterterms in QCD

$\rightsquigarrow$ now, renormalize the theory.

use freedom of redefining fields, parameters/couplings

schematically: $\phi_B = \sqrt{Z_\phi}\,\phi_R \quad , \quad \phi \in \{\psi, A, c\}$ $\qquad B \hat{=}$ "bare"

$\lambda_B = Z_\lambda \lambda_R \quad , \quad \lambda \in \{m, g, \xi\}$ $\qquad R \hat{=}$ "renormalized"

where the multiplicative renormalization factors Z_i depend on the renormalized parameters (and the dimension d), and are taken to be dimension<u>less</u>, $Z_i = 1 + \delta Z_i \;,\; \delta Z_i \sim g^2$ (see below)

- recall (pg. 28)

$$\mathcal{L}_B = \bar\psi_B\,(i\gamma^\mu \underbrace{D_\mu}_{\partial_\mu - ig_B A^a_\mu T^a} - m_B)\,\psi_B - \frac14 \underbrace{F^a_{\mu\nu}}_{\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu} F^{a\mu\nu} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar c^a(-\partial^\mu \underbrace{D^{ac}_\mu}_{\partial_\mu \delta^{ac} + gf^{abc}A^b_\mu})c^c$$

(in this line, all ψ, A, c, m, g, ξ should have an index $\underline{B}$)

$$= \overbrace{\bar\psi\, i\partial\!\!\!/\,\psi}^{Z_\psi} - \overbrace{m\bar\psi\psi}^{Z_m Z_\psi} + \overbrace{g\bar\psi\gamma^\mu A^a_\mu T^a \psi}^{Z_g Z_\psi Z_A^{1/2}}$$

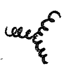
$$- \overbrace{\frac14(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2}^{Z_A} - \overbrace{\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2}^{Z_A Z_\xi^{-1}} - \overbrace{\frac12 g f^{abc} A^b_\mu A^c_\nu (\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu})}^{Z_g Z_A^{3/2}}$$

$$- \overbrace{\frac14 g^2 f^{abc} A^b_\mu A^c_\nu f^{ade} A^{d\mu} A^{e\nu}}^{Z_g^2 Z_A^2} - \overbrace{\bar c^a \partial^\mu \partial_\mu c^a}^{Z_c} - \overbrace{g f^{abc}\bar c^a \partial^\mu A^b_\mu c^c}^{Z_g Z_c Z_A^{1/2}}$$

(here, without Z's : index B ; <u>with</u> Z's : index R for all ψ, A, c, m, g, ξ)

$$= \mathcal{L}_R + \underbrace{\mathcal{L}^{c.t.}}_{} \longleftarrow \text{counterterms}$$

$= (Z_\psi - 1)\bar\psi\left(i\partial\!\!\!/ - \frac{Z_m Z_\psi - 1}{Z_\psi - 1}m\right)\psi \;+ (Z_g Z_\psi Z_A^{1/2} - 1)$ 

$- (Z_A - 1)\left[\frac14(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \frac{Z_A Z_\xi^{-1} - 1}{Z_A - 1}\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2\right] + (Z_g Z_A^{3/2} - 1)$ 

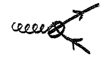
$+ (Z_g^2 Z_A^2 - 1)$  $- (Z_c - 1)\,\bar c\,\partial^\mu\partial_\mu c \;+ (Z_g Z_c Z_A^{1/2} - 1)$ 

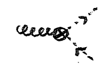
(now, all indices are $\underline{R}$, and omitted)

$\rightarrow$ treat $\mathcal{L}^{c.t.}$ as additional interactions.

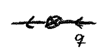
$\rightarrow$ get additional Feynman rules

vertices are easy (have the same form as before),

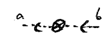
 $= (Z_g Z_\psi Z_A^{1/2} - 1)$  ,  $= (Z_g Z_A^{3/2} - 1)$ 

 $= (Z_g^2 Z_A^2 - 1)$  ,  $= (Z_g Z_c Z_A^{1/2} - 1)$ 

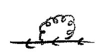
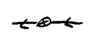
and there are also "two-point-vertices" now:

 $= i\left[(Z_\psi - 1)q\!\!\!/ - (Z_m Z_\psi - 1)m\right]$

 $= -i\delta^{ab}\left[(Z_A - 1)(g^{\mu\nu}q^2 - q^\mu q^\nu) + \underbrace{(Z_A Z_\xi^{-1} - 1)}_{\hat{=}0,\ \text{see pg. 39}}\frac{1}{\xi}q^\mu q^\nu\right]$

 $= i\delta^{ab}(Z_c - 1)\,q^2$

- from our explicit results for 1-loop divergences in §3.1, §3.2, we can now fix the yet-unknown constants Z !

- finite $\stackrel{!}{=}$  +  $+ \mathcal{O}(g^4)$

(pg. 35) $\stackrel{!}{=}\; i\frac{g^2}{16\pi^2}\frac{1}{\varepsilon}\frac{N_c^2-1}{2N_c}\left(q\!\!\!/\,\xi - (\xi+3)m\right) + \mathcal{O}(\varepsilon^0) \;+\; i\left[(Z_\psi - 1)q\!\!\!/ - (Z_m Z_\psi - 1)m\right] + \mathcal{O}(g^4)$

$$\Rightarrow \underline{\underline{Z_\psi \stackrel{!}{=} 1 - \frac{g^2}{16\pi^2}\left(\frac{1}{\varepsilon}\frac{N_c^2-1}{2N_c}\,\xi + \mathcal{O}(\varepsilon^0)\right)}} + \mathcal{O}(g^4)$$

what one puts here is a matter of <u>choice</u>.

often used : "minimal subtraction" (MS) scheme : put 0 here

many other schemes possible,

e.g. modified MS ($\hat{=}\ \overline{\text{MS}}$) , see below

$$\Rightarrow Z_m Z_\psi \stackrel{!}{=} 1 - \frac{g^2}{16\pi^2}\frac{1}{\varepsilon}\frac{N_c^2-1}{2N_c}(3+\xi) + \mathcal{O}(g^4)$$

$$\Leftrightarrow \underline{\underline{Z_m = 1 - \frac{g^2}{16\pi^2}\frac{1}{\varepsilon}\frac{N_c^2-1}{2N_c}\,3 + \mathcal{O}(g^4)}} \qquad \text{in MS scheme}$$

note: ξ-independent

- finite $\stackrel{!}{=}$  $+ \mathcal{O}(g^4)$

(pg. 34) $= \; i \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} \delta^{ab} (g^{\mu\nu} q^2 - q^\mu q^\nu) \left(\left(\frac{13}{6} - \frac{\xi}{2} \right) N_c - \frac{2}{3} N_f \right) + \mathcal{O}(\varepsilon^0)$

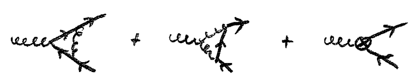
$- i \delta^{ab} \left[(Z_A - 1)(g^{\mu\nu} q^2 - q^\mu q^\nu) + (Z_A Z_\xi^{-1} - 1) \frac{1}{\xi} q^\mu q^\nu \right] \quad + \mathcal{O}(g^4)$

$\Rightarrow \underline{Z_A \stackrel{!}{=} 1 + \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} \left(\left(\frac{13}{6} - \frac{\xi}{2} \right) N_c - \frac{2}{3} N_f \right)} + \mathcal{O}(g^4)$

$\Rightarrow Z_A Z_\xi^{-1} \stackrel{!}{=} 1 + 0 \cdot g^2 + \mathcal{O}(g^4)$

$\Leftrightarrow \underline{Z_\xi = Z_A} + \mathcal{O}(g^4)$

$\rightarrow$ note actually, one can show that $Z_\xi = Z_A$ exactly, to all orders of g^2, due to gauge invariance: the BRST symmetry gives rise to the so-called Ward/Takahashi/Slavnov/Taylor-identities, one of which guarantees that the longitudinal ($q^\mu q^\nu$ piece) part of the gluon propagator does not get radiative corrections, $q^\mu \Pi^{ab}_{\mu\nu}(q) = 0$

- finite $\stackrel{!}{=}$  $+ \mathcal{O}(g^5)$

(pg. 36) $= \; -ig T^a \gamma^\mu \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} \frac{\xi - 3N_c^2 \frac{\xi+1}{2}}{2N_c} + \mathcal{O}(\varepsilon^0) + (Z_g Z_\psi Z_A^{\frac{1}{2}} - 1) \, ig T^a \gamma^\mu + \mathcal{O}(g^5)$

$\Rightarrow Z_g Z_\psi Z_A^{\frac{1}{2}} \stackrel{!}{=} 1 + \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} \frac{\xi - 3N_c^2 \frac{\xi+1}{2}}{2N_c} + \mathcal{O}(g^4)$

$\Leftrightarrow \underline{Z_g = 1 - \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} \left(\frac{11}{6} N_c - \frac{1}{3} N_f \right)} + \mathcal{O}(g^4)$

note: ξ-independent

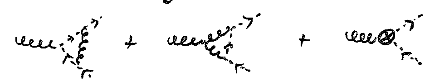
- finite $\stackrel{!}{=}$ 

(pg. 36) $= \; -i \delta^{ab} q^2 \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} (3 - \xi) \frac{N_c}{4} + \mathcal{O}(\varepsilon^0) + i \delta^{ab} q^2 (Z_c - 1)$

$\Rightarrow \underline{Z_c \stackrel{!}{=} 1 - \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} \frac{N_c}{4} (\xi - 3)} + \mathcal{O}(g^4)$

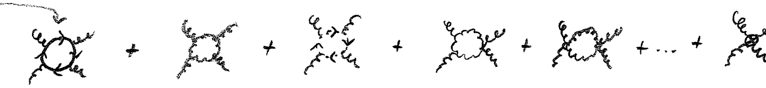
$\rightarrow$ get finite (one-loop) results after fixing Z's as above!

$\rightarrow$ could have computed Z_g also from

finite $\stackrel{?!}{=}$  $+ \mathcal{O}(g^5)$

(check ?!)

finite $\stackrel{?!}{=}$  $+ \mathcal{O}(g^5)$

finite $\stackrel{?!}{=}$  $+ \mathcal{O}(g^6)$

- current status (as of May 2011) of Z's:

$Z \sim 1 + \underbrace{g^2}_{\substack{\text{1-loop} \\ \text{(see above)}}} + g^4 + g^6 + \underbrace{g^8}_{\text{4-loop ✓}} + \underbrace{\mathcal{O}(g^{10})}_{\text{5-loop ??}}$

$\Rightarrow$ Nobel 2004 [Gross/Wilczek, Phys. Rev. Letters 30 (1973) 1343; Politzer, Phys. Rev. Letters 30 (1973) 1346]

4-loop:

Ritbergen/Vermaseren/Larin,	Phys. Lett. B	400 (1997) 379	: Z_g
Chetyrkin,	Phys. Lett. B	404 (1997) 161	: Z_m
Vermaseren/Larin/Ritbergen,	Phys. Lett. B	405 (1997) 327	: Z_m
Chetyrkin/Retey,	Nucl. Phys. B	583 (2000) 3	: Z_ψ
Chetyrkin,	Nucl. Phys. B	710 (2005) 499	: Z_c, Z_A
Czakon,	Nucl. Phys. B	710 (2005) 485	: Z_c, Z_A

3.4 QCD Beta-function, running coupling

$\rightsquigarrow$ recall that we had regularized QCD dimensionally: $d = 4 - 2\varepsilon$.

dimensional analysis: $e^{i \int d^dx \, \mathcal{L}} \Rightarrow [\mathcal{L}] \stackrel{!}{=} d$ (mass-dim: $[m] \equiv 1$)

$[\partial_\mu] = 1$, $[x] = -1$

$\mathcal{L} \ni m \bar\psi \psi \Rightarrow [\psi] = \frac{d-1}{2}$

$\mathcal{L} \ni (\partial^\mu A_\mu)^2 \Rightarrow [A_\mu] = \frac{d-2}{2}$

$\mathcal{L} \ni \bar c \partial^2 c \Rightarrow [c] = \frac{d-2}{2}$

$D_\mu \sim \partial_\mu + g A_\mu \Rightarrow [g] = \frac{4-d}{2}$

→ we had defined $g_B^2 = Z_g^2\, g_R^2$, $Z_g = 1 + \dots$: $[Z_g] = 0$

dim: $(4-d) = (0) + (4-d)$

it is convenient to use integer-dimensional renormalized couplings already in d dimensions

$$\underline{\underline{g_B^2 = Z_g^2\, \hat{g}_R^2\, \mu^{4-d}}} \quad , \quad [\hat{g}_R^2] = 0$$

arbitrary mass-scale, $[\mu] = 1$

⇒ in fact, in all our expressions above, $g^2 \to \hat{g}_R^2$ was understood

$$\left(\left(Z \sim 1 + \underbrace{g_B^2 \int \frac{d^d\ell}{\ell^4}}_{\text{dim-less}} = 1 + \underbrace{Z_g^2}_{\checkmark}\, \underbrace{\hat{g}_R^2}_{\checkmark}\, \underbrace{\mu^{4-d} \int \frac{d^d\ell}{\ell^4}}_{\text{dim-less}} \quad , \quad Z \text{ is fct of dim-less parameters!} \right)\right)$$

• <u>QCD Beta-function</u>

immediate consequence : $\hat{g}_R^2$ is a function of μ

$\mu^2\partial_{\mu^2} \cdot \big|$ $\quad g_B^2 = Z_g^2\, \hat{g}_R^2\, \mu^{4-d}$ $\qquad \hat{g}_R^2 \equiv g^2$ from here on

$$\Rightarrow \quad 0 = \underbrace{\left(\mu^2\partial_{\mu^2} Z_g^2\right)}\, g^2 \mu^{4-d} + Z_g^2 \left(\mu^2\partial_{\mu^2} g^2\right) \mu^{4-d} + Z_g^2 g^2 \frac{4-d}{2} \mu^{4-d}$$

$$= \left(\mu^2\partial_{\mu^2} g^2\right)\left(\partial_g Z_g^2\right) + \left(\mu^2\partial_{\mu^2}\xi\right)\cancel{\left(\partial_\xi Z_g^2\right)}^{0}$$

$$\Leftrightarrow \quad \boxed{\beta(g^2) \equiv \mu^2\partial_{\mu^2} g^2 = \frac{\frac{d-4}{2} g^2}{1 + g^2 (\partial_{g^2} Z_g^2)\, Z_g^{-2}}}$$

in $d = 4 - 2\varepsilon$ dimensions, (pg. 39),

$$Z_g = 1 - \frac{g^2}{\varepsilon}\frac{a}{2} + O(g^4) \qquad = \frac{-\varepsilon g^2}{1 - \frac{g^2}{\varepsilon} a + O(g^4)} = -\varepsilon g^2 \left(1 + \frac{g^2}{\varepsilon} a + O(g^4)\right)$$

where $a \equiv \frac{1}{16\pi^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right) \equiv \frac{\beta_0}{16\pi^2}$ $\qquad \longrightarrow -a g^4 + O(g^6) \quad$ for $\varepsilon \to 0$

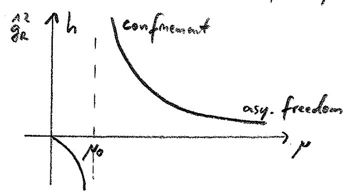
→ more convenient to use $h \equiv \frac{\hat{g}_R^2}{16\pi^2}$:

$$\beta(h) = \mu^2\partial_{\mu^2} h = \frac{\frac{d-4}{2} h}{1 + h (\partial_h Z_g^2)\, Z_g^{-2}} \approx \underbrace{-\beta_0 h^2}_{\text{1-loop}} - \beta_1 h^3 - \beta_2 h^4 \underbrace{- \beta_3 h^5}_{\text{4-loop}} + \underbrace{O(h^6)}_{\text{unknown}}$$

• <u>running coupling</u>

solve the differential equation $\mu^2\partial_{\mu^2} h = -\beta_0 h^2$

soln : $h(\mu) = \dfrac{1}{\text{const.} + \beta_0 \ln(\mu^2)} = \dfrac{1}{\beta_0 \ln\left(\frac{\mu^2}{\mu_0^2}\right)}$



$\beta_0 = \frac{11}{3}N_c - \frac{2}{3}N_f > 0 \quad$ for $N_f < \frac{11}{2}N_c$

(QCD : $6 < \frac{33}{2}$ ✓)

• <u>higher-order systematics</u> of renormalization factors:

(in literature, often find $\gamma(h) = -\gamma_0 h - \gamma_1 h^2 - \gamma_2 h^3 \dots$)

in general (schematically), $Z_i \sim 1 + h\frac{1}{\varepsilon} + h^2\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\right) + h^3\left(\frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\right) + \dots$

furthermore, existence of the limit $\varepsilon \to 0$ in β-fct (or, analogously, in <u>"anomalous dimensions"</u> $\gamma_i \equiv -\mu^2\partial_{\mu^2} \ln Z_i$) fixes the coefficients of poles $\frac{1}{\varepsilon^{n>1}}$ in term of those of $\frac{1}{\varepsilon}$.

<u>e.g.</u> $\quad Z_g = 1 + h\frac{z_{11}}{\varepsilon} + h^2\left(\frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon}\right) + h^3\left(\frac{z_{33}}{\varepsilon^3} + \frac{z_{32}}{\varepsilon^2} + \frac{z_{31}}{\varepsilon}\right) + O(h^4)$

$$\Rightarrow \beta(h) \overset{!}{=} h(-\varepsilon) + h^2(2z_{11}) + h^3\left(\frac{4z_{22} - 6z_{11}^2}{\varepsilon} + 4z_{21}\right) + h^4\left(\frac{2(3z_{33} - 11 z_{22} z_{11} + 9 z_{11}^3)}{\varepsilon^2} + \frac{2(3z_{32} - 11 z_{21} z_{11})}{\varepsilon} + 6 z_{31}\right) + O(h^5)$$

$$\equiv -\beta_0 h^2 - \beta_1 h^3 - \beta_2 h^4 + \dots$$

$\Rightarrow z_{11} = -\frac{\beta_0}{2}$, $z_{21} = -\frac{\beta_1}{4}$, $z_{31} = -\frac{\beta_2}{6}$, ...

and $z_{22} \overset{!}{=} \frac{3}{2} z_{11}^2 = \frac{3}{8}\beta_0^2$; $z_{32} \overset{!}{=} \frac{11}{3} z_{21} z_{11} = \frac{11}{24}\beta_1\beta_0$; $z_{33} \overset{!}{=} -\frac{5}{16}\beta_0^3$; ...

such that $\quad Z_g = 1 + h\left(-\frac{\beta_0}{2\varepsilon}\right) + h^2\left(\frac{3\beta_0^2}{8\varepsilon^2} - \frac{\beta_1}{4\varepsilon}\right) + h^3\left(-\frac{5\beta_0^3}{16\varepsilon^3} + \frac{11\beta_1\beta_0}{24\varepsilon^2} - \frac{\beta_2}{6\varepsilon}\right) + \dots$

i.e., all information is already encoded in the $\frac{1}{\varepsilon}$ poles ✓

• one often needs <u>ε-expansions</u> of Gamma-functions .

(recall pg 31 : $I_n^a(\Delta) \sim \dfrac{\Gamma(a + d/2)\,\Gamma(n - a - d/2)}{\Gamma(d/2)}$)

Mathematica : $\Gamma(1-\varepsilon) \sim \sum \varepsilon^{n+1}$ $\underline{\text{PolyGamma}[n,1]}$ $= \partial_n^2 \ln(\Gamma(n))$

more useful : $\Gamma(1-\varepsilon) = e^{\gamma_E \varepsilon}\, e^{\sum_{n=2}^{\infty} \frac{\varepsilon^n}{n} \zeta(n)}$

Riemann Zeta $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} \quad (n>1)$

Euler-Mascheroni $\gamma_E = -\Gamma'(1) = -\int_0^\infty dx\, e^{-x} \ln(x) = \lim_{N\to\infty}\left(\sum_{k=1}^{N} \frac{1}{k} - \ln(N)\right) \approx 0.5772\dots$

4. QCD in e^+e^- - annihilation

→ want to compare basic properties of (perturbative) QCD with experiment.

→ consider $e^+e^- \to$ hadrons

- total cross section $\sigma \sim \frac{\alpha_{em}^2}{s}$

  calculate α_s corrections, $\sigma \to \sigma \cdot (1+\alpha_s)$

  renormalization scheme dependence enters at α_s^2

  inclusive cross section: $(1+\alpha_s+\alpha_s^2+\alpha_s^3)$ known,

  high precision QCD result!

  non-perturbative corrections expected to be small

  ⇒ used as one of the most precise measurements of α_s.

- QCD predicts "jet" structure for final-state hadrons

  define jet cross sections

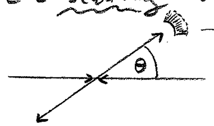
  calculate them, compare with experiment

  ⇒ can also be used to measure α_s,

  and to test/"see" triple-gluon-vertex.

4.1 $e^+e^- \to$ hadrons at leading order

remember: 2→2 scattering in CMS (center of mass system)



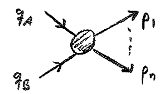
detector measures in

$d\Omega = \sin\theta \, d\theta \, d\varphi$

spherical coords.: $\int d\Omega = \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi = \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\varphi = 4\pi$

total cross section $\sigma = \int d\Omega \left(\frac{d\sigma}{d\Omega}\right)$ ⟸ differential cross section

remember: Fermi's golden rule



$$\sigma_{2\to n} = \frac{1}{F} \int d\Phi_n \, |\mathcal{M}|^2$$

↑ amplitude, e.g. from Feynman diagrs

phase space integral

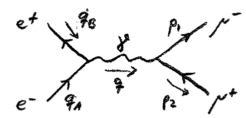
$$\left(\prod_{i=1}^{n} \int \frac{d^3 p_i}{(2\pi)^3 2E_{p_i}}\right)(2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^{n} p_i - q_A - q_B\right)$$

where $F = 4\sqrt{(q_A\cdot q_B)^2 - m_A^2 m_B^2}$

$= 2\sqrt{(s-m_A^2-m_B^2)^2 - 4m_A^2 m_B^2}$, $s = (q_A+q_B)^2$ $(= (E_A+E_B)^2$ in CMS$)$

$= 4(E_A+E_B)|\vec{q}_A|$ in CMS, $q = (E, \vec{q})$

remember: amplitude $e^+e^- \to \mu^+\mu^-$ [see, e.g., Peskin/Schroeder §5.1]



$$= \bar{v}(q_B, s_B)(-ie\gamma^\mu)u(q_A, s_A)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}(p_1, s_1)(-ie\gamma^\nu)v(p_2, s_2)$$

↑ spin: ±½ ↑ Dirac spinor for incoming particle

start with unpolarized beam of e^+, e^-

⇝ average over spin states s_A, s_B : $\frac{1}{4}\sum_{s_A=+,-}\sum_{s_B=+,-}$

detector does not measure spin of final state

⇝ sum over spins s_1, s_2

$$\frac{1}{4}\sum_{s_i=\pm 1/2} |\mathcal{M}|^2 = \frac{1}{4}\sum_s \left|\right.$$  $$\left.\right|^2 = \frac{1}{4}\sum_s \left|\frac{e^2}{q^2}\bar{v}_B\gamma u_A \, \bar{u}_1\gamma v_2\right|^2$$

$$= \frac{e^4}{4q^4}\sum_s \bar{v}_B\gamma u_A \bar{u}_A\gamma v_B \, \bar{u}_1\gamma v_2 \bar{v}_2\gamma u_1$$

do spin sums via completeness rels

$\sum_s u_A\bar{u}_A = \not{q}_A + m_e$, $\sum_s v_2\bar{v}_2 = \not{p}_2 - m_\mu$

$$= \frac{e^4}{4q^4} \operatorname{tr}\left((\not{q}_B - m_e)\gamma^\mu(\not{q}_A+m_e)\gamma^\nu\right)\operatorname{tr}\left((\not{p}_1+m_\mu)\gamma_\mu(\not{p}_2-m_\mu)\gamma_\nu\right)$$

$$= \frac{e^4}{4q^4} 4\left[q_B^\mu q_A^\nu + q_A^\mu q_B^\nu - g^{\mu\nu}(q_A\cdot q_B + m_e^2)\right] 4\left[p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu} - g_{\mu\nu}(p_1\cdot p_2 + m_\mu^2)\right]$$

$$= \frac{8e^4}{q^4}\left(q_A\cdot p_1 \, q_B\cdot p_2 + q_A\cdot p_2 \, q_B\cdot p_1 + m_e^2 \, p_1\cdot p_2 + m_\mu^2 \, q_A\cdot q_B + 2m_e^2 m_\mu^2\right)$$

CMS: $\vec{q}_B = -\vec{q}_A$, $\vec{p}_2 = -\vec{p}_1$; E-cons.: $2E = E_1 + E_2 = 2E_1$

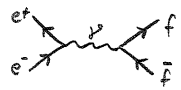
⇒ $E_A = E_B = E_1 = E_2 = E$, $\vec{q}_A^{\,2} + m_e^2 = \vec{p}_1^{\,2} + m_\mu^2$

$$= \frac{e^4}{E^4}\left(E^4 + m_e^2E^2 + m_\mu^2E^2 + (E^2-m_e^2)(E^2-m_\mu^2)\cos^2\theta\right)$$

$\measuredangle(\vec{q}_A, \vec{p}_1)$

45

generalization: amplitude $e^+e^- \to f\bar{f}$



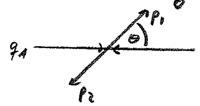
where $f \neq e$, $f \in \left\{ \begin{matrix} \nu_e, \nu_\mu, \nu_\tau, u, c, t \\ \mu, \tau, d, s, b \end{matrix} \right\}$

use charge $Q_f = \left\{ \begin{matrix} 0, 0, 0, +\frac{2}{3}, +\frac{2}{3}, +\frac{2}{3} \\ -1, -1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3} \end{matrix} \right\}$

$$\Rightarrow \langle |\mathcal{M}|^2 \rangle \equiv \sum_{\text{colors}} \sum_f \frac{1}{4} \sum_s |\mathcal{M}|^2 = \sum_{\substack{\text{colors}\\ f}} \frac{Q_f^2 e^4}{E^4} \Big(E^4 + m_e^2 E^2 + m_f^2 E^2 + (E^2 - m_e^2)(E^2 - m_f^2)\cos^2\theta \Big)$$

remember: phase space integration for $2 \to 2$ scattering

$A + B \to 1 + 2$ in CMS:



q_A $\quad q_B = (E_B, -\vec{q}_A)$

$$d\sigma_{2\to 2} = \frac{1}{F} d\Phi_2 |\mathcal{M}|^2 \quad , \quad \text{use } \delta^{(3)} \text{ for } \vec{p}_2\text{-integral}$$

$$= \frac{1}{(8\pi)^2} \frac{|\mathcal{M}|^2}{|\vec{q}_A|(E_A + E_B)} d^3\vec{p}_1 \frac{\delta\left(\sqrt{m_1^2 + \vec{p}_1^2} + \sqrt{m_2^2 + \vec{p}_1^2} - E_A - E_B\right)}{\sqrt{m_1^2 + \vec{p}_1^2}\sqrt{m_2^2 + \vec{p}_1^2}}$$

use that $|\mathcal{M}|^2 = |\mathcal{M}|^2(\vec{q}_A, \vec{q}_B, \vec{p}_1, \vec{p}_2) = |\mathcal{M}|^2(|\vec{q}_A|, |\vec{p}_1|, \cos\theta)$

spherical coords, $\varrho \equiv |\vec{p}_1|$, $d^3\vec{p}_1 = \varrho^2 d\varrho \, \underbrace{\sin\theta \, d\theta \, d\varphi}_{= d\Omega}$

do $\int d\varrho$ using δ fct

$$\frac{d\sigma_{2\to 2}}{d\Omega} = \frac{1}{(8\pi)^2} \frac{|\vec{p}_1|}{|\vec{q}_A|} \frac{|\mathcal{M}|^2(|\vec{q}_A|, |\vec{p}_1|, \cos\theta)}{(E_A + E_B)^2} \cdot \Theta(\sqrt{s} - m_1 - m_2)$$

total cross section

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int_{-1}^{1} d(\cos\theta) \frac{2\pi}{(8\pi)^2} \frac{\sqrt{E^2 - m_f^2}}{\sqrt{E^2 - m_e^2}} \frac{\langle |\mathcal{M}|^2 \rangle}{(2E)^2} \Theta(E - m_f)$$

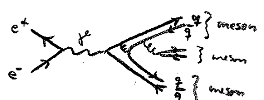
$$= \sum_{\substack{\text{colors}\\ f}} \frac{\pi}{3} \frac{Q_f^2 \alpha_{em}^2}{E^2} \frac{\sqrt{1 - m_f^2/E^2}}{\sqrt{1 - m_e^2/E^2}} \left(1 + \frac{m_e^2}{2E^2}\right)\left(1 + \frac{m_f^2}{2E^2}\right) \Theta(E - m_f)$$

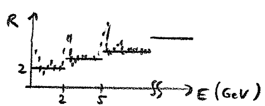
$$\approx \frac{\pi}{3} \frac{\alpha_{em}^2}{E^2} \cdot \sum_{\substack{\text{colors}\\ f}} Q_f^2 \, \Theta(E - m_f), \text{ for } E \gg m_f \gg m_e$$

$$\Rightarrow R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \frac{\sum_q \sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} \approx N_c \sum_q Q_q^2 \, \Theta(E - m_q)$$

$\sum_{\text{colors}}$

$$\approx N_c \left\{ \left(\tfrac{2}{3}\right)_u^2 + \left(-\tfrac{1}{3}\right)_d^2 + \left(-\tfrac{1}{3}\right)_s^2 + \left(\tfrac{2}{3}\right)_c^2 + \left(-\tfrac{1}{3}\right)_b^2 + \left(\tfrac{2}{3}\right)_t^2 \right\} = N_c \left\{ \tfrac{4}{9}, \tfrac{5}{9}, \underset{uds}{\tfrac{2}{3}}, \underset{+c}{\tfrac{10}{9}}, \underset{+b}{\tfrac{11}{9}}, \underset{+t}{\tfrac{5}{3}} \right\}$$





$\Rightarrow N_c \stackrel{!}{=} 3$

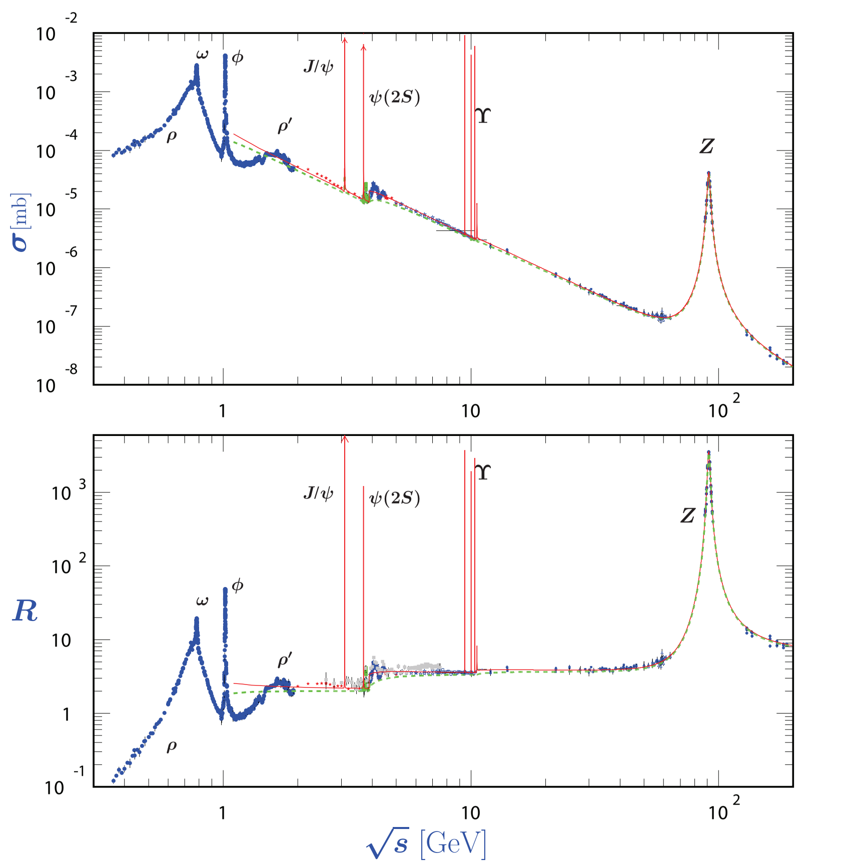


Figure 41.6: World data on the total cross section of $e^+e^- \to hadrons$ and the ratio $R(s) = \sigma(e^+e^- \to hadrons, s)/\sigma(e^+e^- \to \mu^+\mu^-, s)$. $\sigma(e^+e^- \to hadrons, s)$ is the experimental cross section corrected for initial state radiation and electron-positron vertex loops, $\sigma(e^+e^- \to \mu^+\mu^-, s) = 4\pi\alpha^2(s)/3s$. Data errors are total below 2 GeV and statistical above 2 GeV. The curves are an educative guide: the broken one (green) is a naive quark-parton model prediction, and the solid one (red) is 3-loop pQCD prediction (see "Quantum Chromodynamics" section of this *Review*, Eq. (9.7) or, for more details, K. G. Chetyrkin *et al.*, Nucl. Phys. **B586**, 56 (2000) (Erratum *ibid.* **B634**, 413 (2002)). Breit-Wigner parameterizations of J/ψ, $\psi(2S)$, and $\Upsilon(nS), n = 1, 2, 3, 4$ are also shown. The full list of references to the original data and the details of the R ratio extraction from them can be found in [arXiv:hep-ph/0312114]. Corresponding computer-readable data files are available at http://pdg.lbl.gov/current/xsect/. (Courtesy of the COMPAS (Protvino) and HEPDATA (Durham) Groups, May 2010.) See full-color version on color pages at end of book.

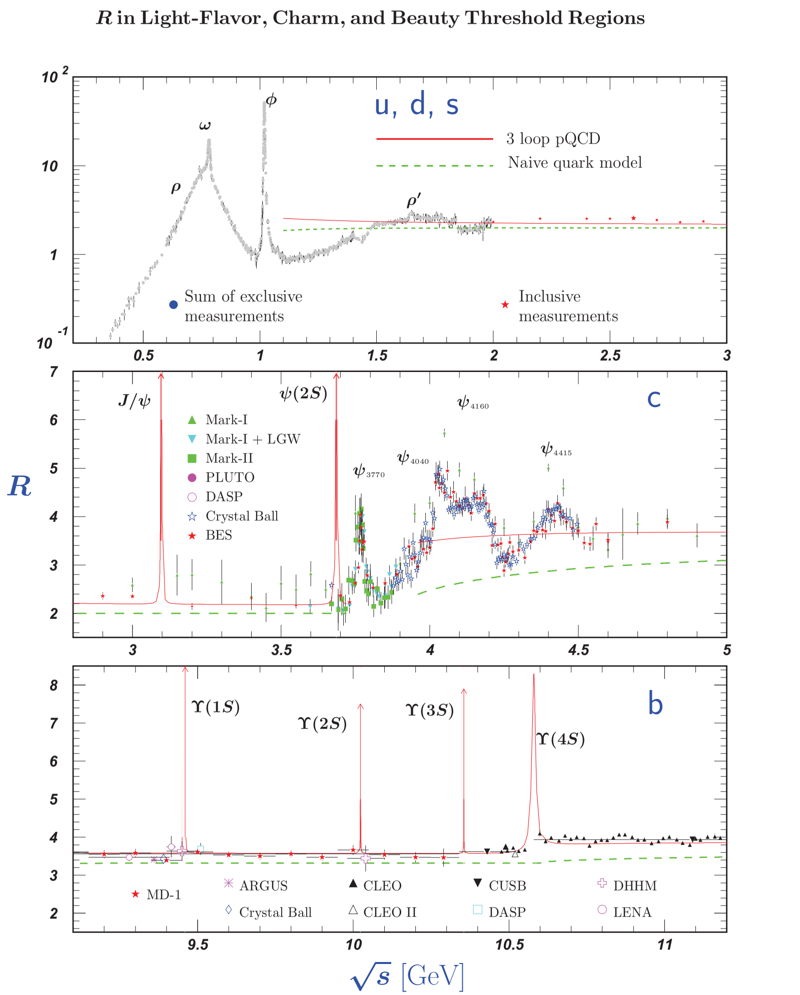
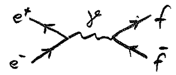


Figure 41.7: R in the light-flavor, charm, and beauty threshold regions. Data errors are total below 2 GeV and statistical above 2 GeV. The curves are the same as in Fig. 41.6. **Note:** CLEO data above $\Upsilon(4S)$ were not fully corrected for radiative effects, and we retain them on the plot only for illustrative purposes with a normalization factor of 0.8. The full list of references to the original data and the details of the R ratio extraction from them can be found in [arXiv:hep-ph/0312114]. The computer-readable data are available at http://pdg.lbl.gov/current/xsect/. (Courtesy of the COMPAS (Protvino) and HEPDATA (Durham) Groups, May 2010.) See full-color version on color pages at end of book.

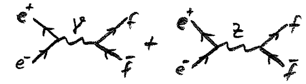
4.2 The Z-peak in R(s)

- in §4.1, studied tree level  , $f \neq e$

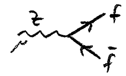
$$\frac{d\sigma_{e\bar{e}\to f\bar{f}}}{d\cos\theta} \overset{\text{pg. 45}}{=} \frac{2\pi}{(8\pi)^2}\frac{\boxed{\langle|\mathcal{M}|^2\rangle}}{(2E)^2}\,\Theta(E-m_f) \qquad = \sum_{\substack{\text{colors}\\ f}} Q_f^2 e^4\left\{1+\frac{m_e^2+m_f^2}{E^2}+\left(1-\frac{m_e^2}{E^2}\right)\left(1-\frac{m_f^2}{E^2}\right)\cos^2\theta\right\}$$

use $\alpha = \frac{e^2}{4\pi}$, $(2E)^2 = s$ in CMS

$$= \frac{\pi\alpha^2}{2s}\sum_{c,f} Q_f^2\left\{1+4\,\frac{m_e^2+m_f^2}{s}+\left(1-\frac{4m_e^2}{s}\right)\left(1-\frac{4m_f^2}{s}\right)\cos^2\theta\right\}\Theta(\sqrt{s}-2m_f)$$

- generalization in Standard Model : 

[see, e.g., Ellis/Stirling/Webber §3.1]

use SM Feynman rules :  $= -\frac{ie}{2\sin\theta_w\cos\theta_w}\gamma^\mu(V_f - A_f\gamma^5)$

where $\sin^2\theta_w \approx 0.23$ is the weak mixing angle

$A_f = \pm\frac{1}{2}$ for $f \in \left\{\begin{matrix} \nu_e, \nu_\mu, \nu_\tau, u, c, t \\ e, \mu, \tau, d, s, b\end{matrix}\right\}$ is the axial $fZ\bar{f}$ coupling

$V_f = A_f - 2Q_f\sin^2\theta_w$ vector $fZ\bar{f}$ coupling

now, taking $|\mathcal{M}|^2$, get interference term as well

take high E limit, $\sqrt{s} \gg m_f \gg m_e$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s}\sum_{c,f}\left\{\underbrace{(1+\cos^2\theta)}_{\to 2+\frac{2}{3}}\left(\overset{\gamma^2}{Q_f^2} - 2Q_f V_e V_f\,\kappa\,\overset{\gamma Z \text{ interference}}{\frac{s(s-m_Z^2)}{(s-m_Z^2)^2+\Gamma_Z^2 m_Z^2}} + (A_e^2+V_e^2)(A_f^2+V_f^2)\,\kappa^2\,\overset{Z^2}{\frac{s^2}{(s-m_Z^2)^2+\Gamma_Z^2 m_Z^2}}\right)\right.$$

$$\left. + \frac{\overset{\int_{-1}^{1} d\cos\theta \,\to\, 0}{\cos\theta}}{(s-m_Z^2)^2+\Gamma_Z^2 m_Z^2}\left(-4Q_f A_e A_f\,\kappa\, s(s-m_Z^2) + 8A_e V_e A_f V_f\,\kappa^2 s^2\right)\right\}$$

where $\kappa \equiv \frac{\sqrt{2}\,G_F\, m_Z^2}{16\pi\alpha}$, Fermi const $G_F = \frac{1}{\sqrt{2}v^2} \approx 1.166\cdot 10^{-5}\,\text{GeV}^{-2}$

$m_Z \approx 91.1876$ GeV , Z decay width $\Gamma_Z \approx 2.5$ GeV

- at small s/m_Z^2, the additional weak effects are small
  $\Rightarrow$ neglecting them, get back result of pg. 45: $\sigma_{s \ll m_Z^2} = \frac{4\pi\alpha^2}{3s} \sum_{c,f} Q_f^2$
- on Z pole, $s = m_Z^2$, 2^{nd} line dominates

$$\Rightarrow \sigma_{s=m_Z^2} = \frac{4\pi\alpha^2}{3\cancel{m_Z^2}} \sum_{c,f} \left(A_e^2+V_e^2\right)\left(A_f^2+V_f^2\right) \kappa^2 \frac{m_Z^2 \cancel{m_Z^2}}{\Gamma_Z^2 \cancel{m_Z^2}}$$

$$\Rightarrow R_{Z\,peak} = \frac{\sigma(e^+e^- \to Z \to hadrons)}{\sigma(e^+e^- \to Z \to \mu^+\mu^-)} = \frac{N_c \sum_q \left(A_q^2+V_q^2\right)}{A_\mu^2+V_\mu^2}$$

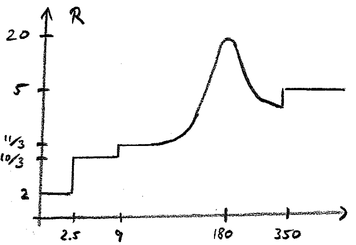
only 5 quarks lighter than Z (($m_{top} \approx 172\,GeV$))

$\rightsquigarrow \sum_q$ goes over $q \in \left\{ \begin{matrix} u, c \\ d, s, b \end{matrix} \right\}$

$$= \frac{3\left[2\left(\frac{1}{4}+\left(\frac{1}{2}-2\left(\frac{2}{3}\right)\sin^2\theta_W\right)^2\right)+3\left(\frac{1}{4}+\left(-\frac{1}{2}-2\left(-\frac{1}{3}\right)\sin^2\theta_W\right)^2\right)\right]}{\frac{1}{4}+\left(-\frac{1}{2}-2(-1)\sin^2\theta_W\right)^2} \xrightarrow{N_c=3,\ \sin^2\theta_W=0.23} \approx 20.095$$

((adding the γ channel, value changes to 19.984))

- so our result would look like this: step functions from γ exchange + broad peak from Z exchange



- comparison with experiment:

((note that PDG-plot was for $\frac{\sigma(e^+e^- \to hadrons)}{\frac{4\pi\alpha^2}{3s}}$, not $\frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}$))

LEP measured $R_{Z\,peak} = 20.767 \pm 0.025$

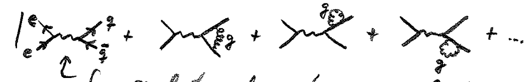
which is $\sim 3.5\%$ higher than the above lowest-order prediction

$\Rightarrow$ discrepancy is (mostly) due to higher-order QCD-corrections!

$\Rightarrow$ compute those (($\sigma \to \sigma(1+\alpha_s+\ldots)$)),
then use experiment to determine α_s.

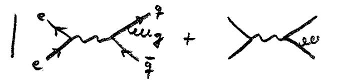
4.3 QCD corrections to $R(s)$

$\rightsquigarrow$ goal: compute $\sigma(e^+e^- \to hadrons) = \sigma_0\left(1 + c\,\alpha_s + \mathcal{O}(\alpha_s^2)\right)$, $c = ?$

- $\sigma_{e^+e^- \to q\bar{q}} \sim |\mathcal{M}|^2 = \Big|$  $+ \ldots \Big|^2$
  for simplicity, only γ here; gluon

  $\Rightarrow$ the $c_1\alpha_s$ term gets contributions from interference of tree + 1loop amplitudes. "virtual correction" c_V

- note that there is another class of diagrams, contributing to the same order:

  $\sigma_{e^+e^- \to q\bar{q}g} \sim \Big|$  $\Big|^2$

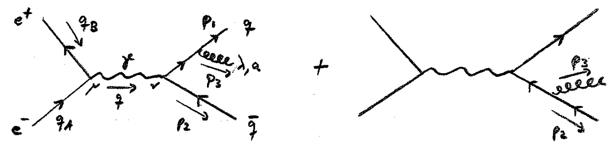
  $\Rightarrow$ "real correction" c_R

$\rightsquigarrow$ total cross section to produce (any number of) partons ($\to$ hadrons) is sum of $\sigma_{e^+e^- \to q\bar{q}} + \sigma_{e^+e^- \to q\bar{q}g} + \ldots$

$\rightsquigarrow$ in fact, both will turn out to be (infrared) divergent, only their sum is finite, hence physical.

$\Rightarrow$ in practice, pick a regularization scheme (again $d = 4-2\varepsilon$) and remove regulator in the end ($\varepsilon \to 0$)

4.3.1 real corrections : $\sigma_{e^+e^- \to q\bar{q}g}$



(see pg. 44, plus Fy rules from pp 24,25)

spin $= \pm\frac{1}{2}$; Dirac spinor for incoming particle; polarization vector for outgoing gluon

$$= \bar{v}(q_B,s_B)(-ie\gamma^\mu)u(q_A,s_A)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\varepsilon_\lambda^{*a} \ *$$

$$* \ \bar{u}(p_1,s_1)\left\{(ig_s\gamma^\lambda T^a)\frac{i(\not{p}_1+\not{p}_3)}{(p_1+p_3)^2}(-ieQ_q\gamma^\nu) + (-ieQ_q\gamma^\nu)\frac{i(-\not{p}_2-\not{p}_3)}{(p_2+p_3)^2}(ig_s\gamma^\lambda T^a)\right\}v(p_2,s_2)$$

note: have used Feynman gauge here, $\xi = 1$
have used massless quarks, $m_q = 0$

→ using $\bar{v}_B \equiv \bar{v}(q_B, s_B)$ etc for brevity, this reads

$$\mathcal{M} = \frac{-ie^2 Q_q g_s}{q^2} \, \varepsilon_\lambda^{*a} T^a \; \bar{v}_B \gamma_\mu u_A \bar{u}_1 \underbrace{\left\{ \gamma^\lambda \frac{\not{p}_1 + \not{p}_3}{(p_1+p_3)^2} \gamma^\mu + \gamma^\mu \frac{-\not{p}_2 - \not{p}_3}{(p_2+p_3)^2} \gamma^\lambda \right\}}_{\equiv S^{\lambda\mu}} v_2$$

→ for cross section, need $|\mathcal{M}|^2$

as in §4.1, unpolarized e^+, e^- beams ⇒ average over incoming spins

spin-blind detector ⇒ sum over outgoing spins

$$\langle |\mathcal{M}|^2 \rangle = \sum_{pol} \frac{1}{4} \sum_s \mathcal{M}\mathcal{M}^\dagger$$

$$= \left(\frac{e^2 Q_q g_s}{q^2}\right)^2 \underbrace{\sum_{pol} \varepsilon_\lambda^{*a} \varepsilon_\sigma^b}_{= -g_{\lambda\sigma}\delta^{ab}} \underbrace{(T^a_{ij})(T^b_{ij})^\dagger}_{= tr(T^a T^b) = \frac{1}{2}\delta^{ab}} \frac{1}{4} \sum_{s_{AB12}} \bar{v}_B \gamma_\mu \underbrace{u_A \bar{u}_A}_{\sum_{s_A} = \not{q}_A} \gamma_\nu^\dagger v_B \; \bar{u}_1 S^{\lambda\mu} \underbrace{v_2 \bar{v}_2}_{\sum_{s_2} = \not{p}_2} (S^{\sigma\nu})^\dagger u_1$$

(completeness rel., Fey gauge) (normalization) perform spin-sums via completeness rels, see pg. 44 etc

(T hermitean, γ hermitean)

$$= -\frac{e^4 Q_q^2 g_s^2}{4 q^4} \frac{N_c^2 - 1}{2} \underbrace{tr(\not{q}_B \gamma_\mu \not{q}_A \gamma_\nu)}_{= L_{\mu\nu}} \underbrace{tr(\not{p}_1 S^{\lambda\mu} \not{p}_2 S^\nu{}_\lambda)}_{\equiv G^{\mu\nu}}$$

$$L_{\mu\nu} = q_B^\sigma q_A^\beta \, 4 (g_{\sigma\mu} g_{\beta\nu} - g_{\sigma\beta} g_{\mu\nu} + g_{\sigma\nu} g_{\mu\beta}) \qquad \text{(see pg. 30)}$$

$$= 4 (q_{B\mu} q_{A\nu} + q_{A\mu} q_{B\nu} - q_A \cdot q_B \, g_{\mu\nu})$$

→ now, total cross section is (pg 44, $m_i = 0$), using <u>dimensional regularization</u>,

$$\sigma_{e^+e^- \to q\bar{q}g} = \frac{1}{2s} \left(\prod_{i=1}^3 \int \frac{d^{d-1} p_i}{(2\pi)^{d-1} 2E_{p_i}} \right) (2\pi)^d \delta^{(d)}(p_1 + p_2 + p_3 - q_A - q_B) \langle |\mathcal{M}|^2 \rangle$$

$$= -\frac{e^4 g_s^2 \left(\sum_q Q_q^2\right) C_F N_c}{8s (q^2)^2 (2\pi)^{2d-3}} L_{\mu\nu} \underbrace{\left(\prod_{i=1}^3 \int \frac{d^{d-1} p_i}{2|\vec{p}_i|} \right) \delta^{(d)}(p_1+p_2+p_3-q)}_{\equiv I^{\mu\nu}(q)} G^{\mu\nu}$$

(used $C_F \equiv \frac{N_c^2 - 1}{2N_c}$)

→ note that (check ?!) $q_\mu I^{\mu\nu}(q) = 0$

$$\Rightarrow I^{\mu\nu}(q) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) I(q^2) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) \frac{g_{\sigma\beta} I^{\sigma\beta}(q)}{d-1}$$

follows from contracting the eq. with $g_{\mu\nu}$

$$\Rightarrow L_{\mu\nu} I^{\mu\nu}(q) = 4 \left[q_A \cdot q_B (1 - d + 1) - \frac{1}{q^2} \left(2\, q \cdot q_B \, q \cdot q_A - q_A \cdot q_B \, q^2 \right) \right] \frac{g_{\sigma\beta} I^{\sigma\beta}(q)}{d-1}$$

use $q_A \cdot q_B = \frac{1}{2}(\underbrace{q_A^2}_{=0} + \underbrace{q_B^2}_{=0}) + q_A \cdot q_B = \frac{1}{2}(q_A + q_B)^2 = \frac{q^2}{2} = \frac{s}{2}$ (onshell)

$$= 2q^2 (2-d) \frac{g_{\mu\nu} I^{\mu\nu}(q)}{d-1}$$

→ in $\sigma_{ee \to q\bar{q}g}$, need now

$$g_{\mu\nu} G^{\mu\nu} = tr(\not{p}_1 S^{\lambda\mu} \not{p}_2 S_{\mu\lambda}) = \text{(check ?!)} = 4(d-2) \frac{x_1^2 + x_2^2 + \frac{d-4}{2} x_3^2}{(1-x_1)(1-x_2)}$$

where $x_i \equiv \frac{2 p_i \cdot q}{q^2}$

→ plug these elements into cross section:

$$\sigma_{e^+e^- \to q\bar{q}g} = + \frac{e^4 g_s^2 \left(\sum_q Q_q^2\right) C_F N_c}{s^2 (2\pi)^{2d-3}} \frac{(d-2)^2}{d-1} \underbrace{\left(\prod_{i=1}^3 \int \frac{d^{d-1} p_i}{2|\vec{p}_i|} \right) \delta^{(d)}(p_{123} - q)} \frac{x_1^2 + x_2^2 + \frac{d-4}{2} x_3^2}{(1-x_1)(1-x_2)}$$

rewrite phase space integral in terms of x_i [see eg. Muta §5.1]

$$= \frac{\pi (\pi s)^{d-3}}{4 \Gamma(d-2)} \left(\prod_{i=1}^3 \int_0^1 \frac{dx_i}{(1-x_i)^{\frac{4-d}{2}}} \right) \delta(2 - x_1 - x_2 - x_3)$$

$(g_s^2 = 4\pi \mu^{2\varepsilon} \alpha_s)$

$d = 4 - 2\varepsilon$

$$= \alpha^2 \alpha_s \left(\sum_q Q_q^2\right) C_F N_c \frac{2}{s} \left(\frac{4\pi\mu}{s}\right)^{2\varepsilon} \frac{(1-\varepsilon)^2}{(3-2\varepsilon)\Gamma(2-2\varepsilon)} \cdot I$$

where

$$I \equiv \left(\prod_{i=1}^3 \int_0^1 \frac{dx_i}{(1-x_i)^\varepsilon} \right) \delta(2 - x_1 - x_2 - x_3) \frac{x_1^2 + x_2^2 - \varepsilon x_3^2}{(1-x_1)(1-x_2)}$$

$$= \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \frac{x_1^2 + x_2^2 - \varepsilon (2 - x_1 - x_2)^2}{(1-x_1)^{1+\varepsilon} (1-x_2)^{1+\varepsilon} (x_1 + x_2 - 1)^\varepsilon}$$

eg Mathematica

$$= \frac{2(2 - 6\varepsilon + 5\varepsilon^2 - 2\varepsilon^3)\Gamma^3(1-\varepsilon)}{\varepsilon^2 \Gamma(3 - 3\varepsilon)} \approx \frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \left(\frac{19}{2} - \pi^2\right) + O(\varepsilon)$$

→ compare result with $\sigma_{e^+e^-\to q\bar q}$ in <u>same</u> regularization:

repeat calculation of $\left|\cdots\right|^2$ (see § 4.1) in d-dimensions,

$$\sigma^{d=4-2\varepsilon}_{e^+e^-\to q\bar q} = \frac{4\pi\alpha^2}{3s}\left(\sum_q Q_q^2\right) N_c \left(\frac{4\pi}{s}\right)^\varepsilon \frac{3(1-\varepsilon)\Gamma(2-\varepsilon)}{(3-2\varepsilon)\Gamma(2-2\varepsilon)}$$

$$\Rightarrow \sigma_{e^+e^-\to q\bar q g} = \sigma_{e^+e^-\to q\bar q}\cdot \frac{\alpha_s C_F}{2\pi}\left(\frac{4\pi\mu^2}{s}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)}\, I$$

- note that I is <u>divergent</u> for $\varepsilon\to 0$.

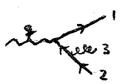
not a desaster (see pg 48): $\lim_{\varepsilon\to 0}$ (real + virtual corr.) should exist.

↳ compute next.

- physical <u>origin</u> of divergences:

in a 'naive' calculation $(d=4)$, $I = \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$

→ divergences come from $x_1 = 1$ and $x_2 = 1$



but $1-x_1 = 1-\frac{2p_1\cdot q}{q^2} = \frac{(q-p_1)^2-p_1^2}{q^2} = \frac{(p_2+p_3)^2-p_1^2}{q^2} = \frac{2p_2\cdot p_3 + p_2^2+p_3^2-p_1^2}{q^2}$

onshell: $p_i^2 = 0 \Rightarrow p_i = E_i(1,\vec e_i)$

$$= \frac{2E_2E_3(1-\vec e_2\vec e_3)}{q^2}$$

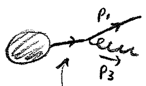
→ divergences originate from $\begin{cases}\vec e_1\vec e_3 \to 1 & : q, g\\ \vec e_2\vec e_3 \to 1 & : \bar q, g\end{cases}$ <u>collinear</u> (~~when~~ $\theta_{13}\to 0$)

and from $E_3\to 0$: g <u>soft</u>

both are referred to as <u>infrared</u> singularities.

another view of the same fact:

divergences originate from diverging propagators



$$\sim \frac{1}{(p_1+p_3)^2} = \frac{1}{2p_1\cdot p_3} = \frac{1}{2E_1E_3(1-\vec e_1\vec e_3)} = \frac{1}{2E_1E_3(1-\cos\theta_{13})}$$

collinear limit, $\theta\to 0$: $\approx \frac{1}{2E_1E_3\theta^2} \Rightarrow |\mathcal{M}|^2 \sim \frac{\theta^2}{\theta^4}$ ← from numerators

soft limit, $E_3\to 0$: interference term in $|\cdots + \cdots|^2$

gives $|\mathcal{M}|^2 \sim \frac{p_1\cdot p_2}{p_1\cdot p_3\; p_2\cdot p_3} \approx \frac{1}{E_3^2}$

in phase space integral, $\frac{d^3p_3}{2E_3} = \frac{1}{2E_3}E_3^2 dE_3 \sin\theta\, d\theta\, d\varphi \sim E_3\, dE_3\, \theta\, d\theta$

⇒ logarithmic singularities in both limits.

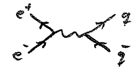
4.3.2 virtual corrections: $\sigma_{e^+e^-\to q\bar q}$ at $\mathcal{O}(\alpha_s)$

structure of cross section computation:

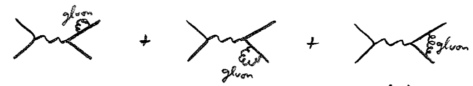
$$|\mathcal{M}_0 + \alpha_s\mathcal{M}_1 + \mathcal{O}(\alpha_s^2)|^2 = |\mathcal{M}_0|^2 + \alpha_s\left(\mathcal{M}_1\mathcal{M}_0^* + \mathcal{M}_0\mathcal{M}_1^*\right) + \mathcal{O}(\alpha_s^2)$$

→ need to compute interference term $\mathcal{M}_1\mathcal{M}_0$ only.

recall $\mathcal{M}_0 =$



$\alpha_s\mathcal{M}_1 =$



(a) (b) (c)

- in dimensional regularization,

$\sim \int d^dk \frac{1}{k^2}\frac{\not k+\not p}{(k+p)^2} = \not p\, I(p^2)$

with $\dim[I(p^2)] = m^{d-4}$

but $p^2 = 0$ (due to onshell condition), so no scale ⇒ <u>zero</u>

$\Rightarrow \mathcal{M}_{1(a)} = 0 = \mathcal{M}_{1(b)}$ in dim. reg.

- for diagram (c), need to do some computation

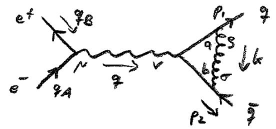
$$\sigma_{e^+e^-\to q\bar q} \overset{\text{pg. 49}}{=} \frac{1}{2s}\left(\prod_{i=1}^2 \int \frac{d^{d-1}p_i}{(2\pi)^{d-1}2E_{p_i}}\right)(2\pi)^d\,\delta^{(d)}(p_1+p_2-q_A-q_B)\,\langle|\mathcal{M}|^2\rangle$$

structure of $\alpha\mathcal{M}_{1(c)}$ is similar to $\mathcal{M}_0$!

recall (pg 44) $\mathcal{M}_0 = \cdots = \frac{e^2Q_q}{q^2}\bar v_B\gamma^\mu u_A\,\bar u_1\gamma_\mu v_2$

$$\langle|\mathcal{M}_0|^2\rangle \overset{\text{do spin sums}}{=} \frac{e^4Q_q^2}{4q^4}\,\mathrm{tr}\left(\not q_B\gamma^\mu\not q_A\gamma^{\nu(4)}\right)\mathrm{tr}\left(\not p_1\gamma_\mu\not p_2\gamma_\nu^{(4)}\right)$$

now, $\alpha_s\mathcal{M}_1 =$



use Feynman gauge $\xi = 1$, massless quarks $m_q = 0$

$$= \bar v_B(-ie\gamma^\mu)u_A\left(-\frac{ig_{\mu\nu}}{q^2}\right)\bar u_1(ig_s\gamma^\rho T^a)\int\frac{d^dk}{(2\pi)^d}\frac{i(\not k+\not p_1)}{(k+p_1)^2}(-ieQ_q\gamma^\nu)\frac{i(\not k-\not p_2)}{(k-p_2)^2}\,*$$

$$*\,(ig_s\gamma^\sigma T^b)v_2\left(\frac{-i\delta^{ab}g_{\rho\sigma}}{k^2}\right)$$

use $T^a T^a = T^a_{ij} T^a_{jk} \overset{\text{Fierz, pg. 9}}{=} \frac{1}{2}\left(\delta_{ik}\delta_{jj} - \frac{1}{N}\delta_{ij}\delta_{jk}\right) = \frac{\delta_{ik}}{2}\left(N - \frac{1}{N}\right) = \mathbb{1} C_F$

$$= \frac{e^2 Q_q}{q^2} \bar v_B \gamma^\mu u_A \; \bar u_1 \underbrace{g_s^2 C_F \mathbb{1} \int \frac{d^d k}{(2\pi)^d} \gamma^3 \frac{\not k + \not p_1}{(k+p_1)^2} \gamma_\mu \frac{\not k - \not p_2}{(k-p_2)^2} \gamma_3 \frac{1}{k^2}}_{\equiv \Lambda_\mu} v_2$$

$$\Rightarrow \left\langle \alpha_s \left(\mathcal{M}_1 \mathcal{M}_0^+ + \mathcal{M}_0 \mathcal{M}_1^+\right)\right\rangle = \frac{e^4 Q_q^2}{4 q^4} \operatorname{tr}\left(\not p_B \gamma^\mu \not p_A \gamma^\nu\right) \operatorname{tr}\left(\not p_1 \Lambda_\mu \not p_2 \gamma_\nu\right) + \text{c.c.}$$

- note that IF the contribution of Λ_μ inside the 2^{nd} trace can be reduced to $\Lambda_\mu = \gamma_\mu \cdot \alpha_s C_F \frac{\Lambda(q^2)}{2}$

  THEN $\left\langle \alpha_s \left(\mathcal{M}_1 \mathcal{M}_0^+ + \mathcal{M}_0 \mathcal{M}_1^+\right)\right\rangle = \left\langle |\mathcal{M}_0|^2 \right\rangle \cdot \alpha_s C_F \operatorname{Re}\left[\Lambda(q^2)\right]$

  and σ follows without additional work!

  → the idea is to profit from the onshell conditions, such that if there is a term, say, $\not p_1 \gamma_\mu \not p_1$ in Λ_μ it gets killed by $\not p_1 \Lambda_\mu = \not p_1 \not p_1 \gamma_\mu \not p_1 = \frac{1}{2}\{\gamma_\rho, \gamma_\nu\} p_1^\rho p_1^\nu \gamma_\mu \not p_1 = p_1^2 \gamma_\mu \not p_1 = 0$

- $$\Lambda_\mu = 4\pi\mu^{2\varepsilon} \alpha_s C_F \underbrace{\gamma^3 \sigma \mu \nu 3}_{} \underbrace{\int \frac{d^4 k}{(2\pi)^d} \frac{(k+p_1)^\sigma (k-p_2)^\nu}{(k+p_1)^2 (k-p_2)^2 k^2}}_{\equiv I^{\sigma\nu}}$$

  $$\overset{\text{pg. 35}}{=} -2\gamma_{\nu\mu\sigma} + (4-d)\gamma_{\sigma\mu\nu} \qquad ①$$

  $$\text{or} = -4\left(\gamma_\nu g_{\mu\sigma} - \gamma_\mu g_{\nu\sigma} + \gamma_\sigma g_{\nu\mu}\right) + (6-d)\gamma_{\sigma\mu\nu} \qquad ②$$

  $$\text{or} = 2(4-d)\left(\gamma_\sigma g_{\mu\nu} - \gamma_\mu g_{\nu\sigma} + \gamma_\nu g_{\sigma\mu}\right) - (6-d)\gamma_{\nu\mu\sigma} \qquad ③$$

- solve the loop integral $I^{\sigma\nu}$ via Feynman parameters (see pg. 30)

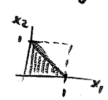
$$I^{\sigma\nu} = \Gamma(3)\int_0^1 dx_1 \dots dx_3\, \delta(1-x_1-x_2-x_3) \int \frac{d^d k}{(2\pi)^d} \frac{(k+p_1)^\sigma (k-p_2)^\nu}{\left[x_1\left(k^2+2kp_1+p_1^2\right) + x_2\left(k^2-2kp_2+p_2^2\right) + x_3 k^2\right]^3}$$

$\left(q^2 = (p_1+p_2)^2 = 2p_1p_2\right)$

denominator $[\dots] = \left[(k + x_1 p_1 - x_2 p_2)^2 + x_1 x_2 q^2\right]$ (used δ fct for k^2-prefactor)

shift $k \to k - x_1 p_1 + x_2 p_2$, linear terms in k integrate to zero

$$= \Gamma(3)\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^d k}{(2\pi)^d} \frac{k^\sigma k^\nu - \left[(1-x_1)p_1 + x_2 p_2\right]^\sigma \left[x_1 p_1 + (1-x_2)p_2\right]^\nu}{(k^2 + x_1 x_2 q^2)^3}$$



momentum integrals are standard (see pg 31), after $k^\sigma k^\nu \to \frac{g^{\sigma\nu} k^2}{d}$ and Wick rotation : $I_3^1(-x_1 x_2 q^2)$, $I_3^0(-x_1 x_2 q^2)$

$$= \frac{1}{(4\pi)^{d/2}} \frac{1}{(-q^2)^{2-d/2}} \left\{ \frac{g^{\sigma\nu}}{d} \frac{d}{2} \Gamma(2-\tfrac{d}{2}) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{(x_1 x_2)^{2-d/2}} + \Gamma(3-\tfrac{d}{2}) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{[\;]^\sigma [\;]^\nu}{(x_1 x_2)^{3-d/2}} \frac{1}{(-q^2)} \right\}$$

perform the integrals over Fy parameters, e.g. via Mathematica, for $\operatorname{Re}(d) > 4$

$$= \frac{1}{(4\pi)^{d/2}} \frac{1}{(-q^2)^{2-d/2}} \left\{ g^{\sigma\nu} \frac{1}{2} \Gamma(2-\tfrac{d}{2}) \frac{\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}{(\frac{d}{2}-1)\Gamma(d-1)} + \Gamma(3-\tfrac{d}{2}) \frac{\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}{(\frac{d}{2}-2)\Gamma(d-1)} \left[p_1^\sigma p_1^\nu + p_2^\sigma p_2^\nu + \frac{8-4d+d^2}{(d-4)(d-2)} p_1^\sigma p_2^\nu + \frac{d-4}{d-2} p_2^\sigma p_1^\nu \right] \frac{1}{(-q^2)} \right\}$$

- $$\Lambda_\mu = 4\pi\mu^{2\varepsilon} \alpha_s C_F \underline{\gamma^3 \sigma \mu \nu 3} \overbrace{\left\{ I_1 \underline{g^{\sigma\nu}} + I_2 \underline{p_1^\sigma p_1^\nu} + I_3 \underline{p_2^\sigma p_2^\nu} + I_4 \underline{p_1^\sigma p_2^\nu} + I_5 \underline{p_2^\sigma p_1^\nu} \right\}}^{I^{\sigma\nu}\text{ ; coeffs } I_i \text{ from above}}$$

  ① $\to (2-d)\gamma^\nu{}_{\mu\nu} \overset{\text{pg 34}}{=} (2-d)^2 \gamma_\mu$; ① $\to 0$; ① $\to 0$; ② $\to 4\gamma_\mu p_1 p_2$; ③ $\to -2(4-d)\gamma_\mu p_2 \cdot p_1$

  have now specified terms that survive in product $\not p_1 \Lambda_\mu \not p_2$ (in particular, $0 = \{\not p_1 \gamma_\mu \not p_1, \not p_2 \gamma_\mu \not p_2, \not p_1 \gamma_\mu \not p_2, \not p_2 p_{1\mu}, \not p_1 p_{2\mu}\}$)

$$= 4\pi\mu^{2\varepsilon} \alpha_s C_F \frac{1}{(4\pi)^{d/2}} \frac{1}{(-q^2)^{2-d/2}} \left\{ \frac{1}{2}\Gamma(2-\tfrac{d}{2}) \frac{\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}{(\frac{d}{2}-1)\Gamma(d-1)} (2-d)^2 \gamma_\mu + \right.$$

$$\left. + \Gamma(3-\tfrac{d}{2}) \frac{\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}{(\frac{d}{2}-2)\Gamma(d-1)} \left[0 + 0 + \frac{8-4d+d^2}{(d-4)(d-2)} 2\gamma_\mu \boxed{\frac{2p_1 p_2}{-q^2}}^{-1} + \frac{d-4}{d-2}(d-4)\gamma_\mu \boxed{\frac{2p_1p_2}{-q^2}}^{-1} \right] \right\}$$

$$= \frac{1}{2}\gamma_\mu \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{-q^2}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \left\{ \left(\frac{1}{\varepsilon} + 1 + \mathcal{O}(\varepsilon)\right) + 0 + 0 + \left(-\frac{2}{\varepsilon^2} - \frac{4}{\varepsilon} - 9 + \mathcal{O}(\varepsilon)\right) + \mathcal{O}(\varepsilon) \right\}$$

$$\overset{d=4-2\varepsilon}{=} \frac{1}{2}\gamma_\mu \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{s}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \pi^2 + \mathcal{O}(\varepsilon) + i\pi\left(-\frac{2}{\varepsilon} - 3 + \mathcal{O}(\varepsilon)\right) \right)$$

↑ from $(-1)^\varepsilon$; but irrelevant

$\left((-)^\varepsilon = e^{i\pi\varepsilon} \quad \operatorname{Re}[(-)^\varepsilon] = \cos(\pi\varepsilon) \right)$

- collecting, we finally have

$$\sigma_{e^+e^- \to q\bar q} = \sigma^{(\text{tree})}_{e^+e^- \to q\bar q} \cdot \left\{ 1 + \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{s}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \pi^2 + \mathcal{O}(\varepsilon) \right) + \mathcal{O}(\alpha_s^2) \right\}$$

4.3.3 Result

let $\sigma_0 \equiv \sigma_{e^+e^- \to q\bar{q}}$ (tree level) denote the leading order (LO) result
(see pg. 45; d-dim result on pg 51)

then, in dimensional regularization, real and virtual QCD-corrections are

LO (pg. 51) $\sigma_{e^+e^- \to q\bar{q}g} = \sigma_0 \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{s}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} - \pi^2 + O(\varepsilon) \right) + O(\alpha_s^2)$

NLO (pg. 54) $\sigma_{e^+e^- \to q\bar{q}} = \sigma_0 \left\{ 1 + \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{s}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \pi^2 + O(\varepsilon) \right) + O(\alpha_s^2) \right\}$

the sum, needed for the hadronic cross section, is finite;
can take $\varepsilon \to 0$

NLO $\Rightarrow \sigma_{e^+e^- \to \text{hadrons}} = \sigma_0 \left\{ 1 + \frac{\alpha_s C_F}{2\pi} \frac{3}{2} + O(\alpha_s^2) \right\} \underset{N_c=3}{=} \sigma_0 \left\{ 1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2) \right\}$

in QCD, $N_c = 3$, $C_F = \frac{N_c^2-1}{2N} = \frac{4}{3}$

Before using this results, a couple of remarks:

- cancellation of soft and collinear divergences between the real and virtual gluon diagrams is not accidental.
  They are in fact guaranteed by theorems (Bloch/Nordsieck, Kinoshita/Lee/Nauenberg (KLN)): suitably defined inclusive quantities will be IR safe in the massless limit.
  (($\sigma_{e^+e^- \to \text{hadrons}}$ is such a quantity; $\sigma_{e^+e^- \to q\bar{q}}$ is not))
  → proof in QCD: see e.g. [Collins/Soper, Ann Rev Nucl Sci 37 (1987) 383]
- our result would be worthless if it depended on our choice of regularization procedure, dim. reg.
  Proof of independence is beyond this lecture; but demonstrate it by comparing with gluon mass regularization scheme (m_g = gluon mass)

$$\sigma_{e^+e^- \to q\bar{q}g} = \sigma_0 \frac{\alpha_s C_F}{2\pi} \left(\ln^2 \frac{s}{m_g^2} - 3 \ln \frac{s}{m_g^2} + 7 - \frac{\pi^2}{3} + O(\varepsilon) \right)$$

$$\sigma_{e^+e^- \to q\bar{q}} = \sigma_0 \left\{ 1 + \frac{\alpha_s C_F}{2\pi} \left(-\ln^2 \frac{s}{m_g^2} + 3 \ln \frac{s}{m_g^2} - \frac{11}{2} + \frac{\pi^2}{3} + O(\varepsilon) \right) \right.$$

$$\Rightarrow \sigma_{e^+e^- \to \text{hadrons}} = \sigma_0 \left\{ 1 + \frac{\alpha_s C_F}{2\pi} \frac{3}{2} + O(\alpha_s^2) \right\}$$

→ individual cross sections completely different; sum scheme independent!

- at this order (NLO), and for $m_q = 0$, our computation of the QCD correction is independent of the nature of the exchanged weak boson (we took the photon only, cf pg. 48).
  → generalization: the $\left(1 + \frac{\alpha_s}{\pi}\right)$ result is valid also for $R_{Z\,\text{peak}}$, see pg. 47

ratios: $R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \underbrace{N_c\left(\sum_f Q_f^2\right)}_{s \ll m_Z^2} \left(1 + \frac{\alpha_s C_F}{2\pi} \frac{3}{2} + O(\alpha_s^2) \right)$; otherwise see § 4.2

$$R_{Z\,\text{peak}} = \left. \frac{\sigma(e^+e^- \to Z \to \text{hadrons})}{\sigma(e^+e^- \to Z \to \mu^+\mu^-)} \right|_{s = m_Z^2} = 19.984 \left(1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2) \right)$$

note that NLO correction is positive.

α_s:
- comparing with the (correctly scaled) experimental result
  $R_{Z\,\text{peak}}(\text{LEP}) = 20.767 \pm 0.025$ (see pg. 47)
  ⇒ our first measurement of α_s: $\alpha_s(m_Z) = 0.123 \pm 0.004$
  ((compare with "world average" from PDG (2010): 0.1184 ± 0.0007))
- as another determination of α_s, let us compare to data taken by PETRA (DESY), at $\sqrt{s} \approx 34$ GeV
  $R(s \approx (34\,\text{GeV})^2, \text{PETRA}) = 3.88 \pm 0.03$
  we would predict ($u\,d\,s\,c\,\underset{4.5\,\text{GeV}}{b}$ − $\underset{172\,\text{GeV}}{t}$ too heavy)
  $R((34\,\text{GeV})^2) = \underbrace{3\left(2\left(\tfrac{2}{3}\right)^2 + 3\left(-\tfrac{1}{3}\right)^2\right)}_{= \frac{11}{3} \approx 3.667} \left(1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2) \right)$
  or, including Z, $R((34\,\text{GeV})^2) = 3.69 \left(1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2) \right)$
  ⇒ our second measurement of α_s: $\alpha_s(34\,\text{GeV}) = 0.162 \pm 0.026$
- recall (§ 3.4, pg. 42) that α_s is running!
  $\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln\left(\mu^2/\mu_0^2\right)}$, with $\beta_0 = \frac{11}{3} N_c - \frac{2}{3} N_f \overset{N_c=3}{=} 11 - \frac{2}{3} N_f$
  $\Leftrightarrow \frac{1}{\alpha_s(\mu_1)} - \frac{1}{\alpha_s(\mu_2)} = \frac{\beta_0}{4\pi} \ln\left(\frac{\mu_1^2}{\mu_2^2}\right)$
  ⇒ 2nd measurement translates into $\alpha_s(m_Z = 91.2\,\text{GeV}) = 0.135 \pm 0.018$

→ so far, our NLO correction to R shows correct qualitative features.

→ but what about higher orders?

4.3.4 Higher-order QCD corrections to $R(s)$

write $R(s) = 3\left(\sum_f Q_f^2\right) \cdot K_{QCD}$

where $K_{QCD} = 1 + 1 \cdot \frac{\alpha_s(\mu)}{\pi} + \sum_{n\geq 2} \underbrace{C_n\left(\frac{s}{\mu^2}\right)} \cdot \left(\frac{\alpha_s(\mu)}{\pi}\right)^n$

↑ result of our computation;
from $\left\{\begin{array}{l}\text{tree-level } q\bar{q}g \\ \text{one-loop } q\bar{q}\end{array}\right\}$ final state

the functions $\underbrace{C_n\left(\frac{s}{\mu^2}\right)}$ follow from higher-order computations:

e.g. C_2 from $\left\{\begin{array}{l}\text{tree-level } q\bar{q}gg,\ q\bar{q}q\bar{q} \\ \text{one-loop } q\bar{q}g \\ \text{two-loop } q\bar{q}\end{array}\right\}$ final states

etc ...

→ note that in our computation, there were no UV divergences
((in fact, those in + cancel exactly)),
so we did not need to renormalize, hence our coefficient did not depend on the renormalization scale μ: $C_1\left(\frac{s}{\mu^2}\right) = 1$.

→ in higher orders, we will encounter UV divs, hence $C_{n\geq 2}$ are renormalization scheme dependent.
If we could sum the whole series, it would be μ-indep.
In a truncated series, μ-dependence is of higher order.

⇒ μ-dependence of $C_n\left(\frac{s}{\mu^2}\right)$ is fixed by knowing μ-dependence of $\alpha_s(\mu)$ (pg 38: $\mu^2\partial_{\mu^2}\alpha_s = -\frac{\beta_0}{4\pi}\alpha_s^2 - \frac{\beta_1}{(4\pi)^2}\alpha_s^3 - \ldots$)

⇒ $C_2\left(\frac{s}{\mu^2}\right) = C_2(1) + C_1(1)\boxed{\frac{\beta_0}{4}\ln\left(\frac{\mu^2}{s}\right)} \equiv L$

$C_3\left(\frac{s}{\mu^2}\right) = C_3(1) + C_1(1)\,L^2 + \left[C_1(1)\frac{\beta_1}{4\beta_0} + 2C_2(1)\right]L$

etc. (check ?!)

(($\beta_1 = \frac{2}{3}\left(17N_c^2 - 5N_cN_f - 3C_FN_f\right) \overset{N_c=3}{\downarrow}= 102 - \frac{38}{3}N_f$))

• C_2 and C_3 have been computed [Samuel/Surguladze, PRL 66(1991) 560] [Gorishny/Kataev/Larin, PL B 259 (1991) 144]
(here $N_c = 3$, ren. scale set to $\mu = \sqrt{s}$, $\overline{MS}$ scheme)

$$C_2(1) = \left(\frac{365}{24} - 11\zeta(3)\right) + \left(-\frac{11}{12} + \frac{2}{3}\zeta(3)\right)N_f \approx 1.986 - 0.115\,N_f$$

$$C_3(1) = \left(\frac{87029}{288} - \frac{1103}{4}\zeta(3) + \frac{275}{6}\zeta(5)\right) + \left(-\frac{7847}{216} + \frac{262}{9}\zeta(3) - \frac{25}{9}\zeta(5)\right)N_f$$
$$+ \left(\frac{151}{162} - \frac{19}{27}\zeta(3)\right)N_f^2 - \frac{\pi^2}{432}(33-2N_f)^2 + \eta\left(\frac{55}{72} - \frac{5}{3}\zeta(3)\right)$$
$$\approx -6.637 - 1.200\,N_f - 0.005\,N_f^2 - 1.240\,\eta$$

where $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ is the Riemann Zeta function $\zeta(3) \approx 1.202057$, $\zeta(5) \approx 1.036928$

and $\eta = \frac{\left(\sum_q Q_q\right)^2}{3\left(\sum_q Q_q^2\right)}$, $\sum_q$ over all quarks with $m_q \ll \sqrt{s}$ (effectively massless)

((for $R_{Z\text{peak}}$, QCD corrections are again the same, except that $\eta \to \frac{\left(\sum_q V_q\right)^2}{3\sum_q\left(V_q^2 + A_q^2\right)}$))

• having a few orders of the perturbative series, can now discuss convergence

→ coefficients are scheme-dependent, so can try to find an "optimal" scheme (from the point of view of convergence),

examples are:
FAC (fastest apparent convergence)
choose scale $\mu = \mu_{FAC}$ such that $R^{(1)}(\mu_{FAC}) = R^{(n)}(\mu_{FAC})$

PMS (principle of minimal sensitivity)
choose $\mu = \mu_{PMS}$ such that $\mu\partial_\mu R^{(n)}(\mu)\big|_{\mu_{PMS}} = 0$
[Stevenson, PLB 100 ((1981) 61]

BLM ([Brodsky/Lepage/Mackenzie, PR D28 (1983) 228])
absorb all N_f-terms into α_s via β fct

etc ...

for $R^{(3)}(s)$, these are $\{\mu_{FAC}, \mu_{PMS}, \mu_{BLM}\} \overset{(N_f = 5 \text{ massless flavors})}{\approx} \{0.692, 0.587, 0.708\}\sqrt{s}$

→ μ-variation does get weaker, comparing $R^{(1)}(s)$, $R^{(2)}(s)$, $R^{(3)}(s)$

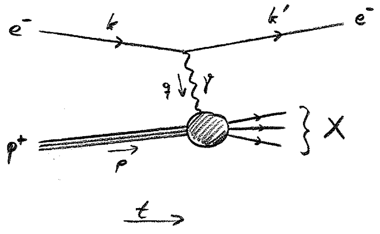
5. Deep inelastic scattering (DIS)

We now (temporarily) go back to tree-level phenomenology.

Q.: are quarks real physical constituents of hadrons,
or just a mathematical convenience for describing the hadron's wavefct?

$\rightsquigarrow$ DIS gives information on internal proton structure

$\rightsquigarrow$ structure functions, Bjorken scaling, parton distribution fcts



photon virtuality $Q^2 = -q^2$ $\left(\left(Q^2 \sim \frac{1}{\lambda^2}\right)\right)$
controls resolving power of photon

$Q^2 \ll \frac{1}{R^2}$ $\to$ elastic e-p scattering
(R: proton radius)

$Q^2 \gg \frac{1}{R^2}$ $\to$ resolve p constituents,
elastic e-q scattering

we are interested in deep ($Q^2 \gg M_p^2$) inelastic ($(p+q)^2 \gg M_p^2$) scattering

5.1 Structure functions

- want to describe process with Lorentz-invariant variables

  center-of-mass energy $s \equiv (k+p)^2$

  at fixed s, scattered e^- has 2 non-trivial variables (E, θ);

  use $Q^2 \equiv -q^2$, $x \equiv \frac{Q^2}{2p\cdot q}$

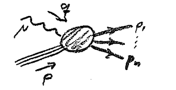
  ((other choices: $W^2 \equiv (p+q)^2 = Q^2\,\frac{1-x}{x}$ (invariant mass of γp-system)

  $y \equiv \frac{p\cdot q}{p\cdot k} = \frac{Q^2}{xs}$))

  kinematic limits: $Q^2 < s$, $x > Q^2/s$

  ((of course $\underset{\approx 0}{k^2 = m_e^2}$, $\underset{\approx 0}{p^2 = M_p^2}$; for $Q^2 \gtrsim M_Z^2$, also Z^0 exchange))

- know nothing about detailed structure of proton

  $\rightsquigarrow$ parametrize $\mathcal{M} =$  $= e\,T_\mu(p,q;\{p_X\})$

  $\Rightarrow \frac{1}{4}\Big|$  $\Big|^2 = \frac{1}{4}\frac{e^4}{Q^4}\,\underbrace{\mathrm{tr}\left(\not{k}\gamma^\mu\not{k}'\gamma^\nu\right)}_{\equiv L^{\mu\nu} = 4\left(k^\mu k'^\nu + k^\nu k'^\mu - k\cdot k' g^{\mu\nu}\right)}\,T_\mu(p,q,\{p_X\})\,T_\nu^*(p,q,\{p_X\})$ (see p. 49)

  in complete analogy to $e^+e^- \to \mu^+\mu^-,\ q\bar{q}$ etc.

- for total cross section, need to integrate over phase space

  $$\int d\Phi_{X+1} = \underbrace{\int dQ^2\,dx\,\frac{Q^2}{16\pi^2 s x^2}}_{\text{electron kinematics}}\ \underbrace{\int d\Phi_X}_{\text{n-body phase space for } X}$$

  for an inclusive process (don't measure $X \to$ sum over them),

  $$\frac{d^2\sigma}{dQ^2\,dx} = \frac{1}{2s}\,\frac{Q^2}{16\pi^2 s x^2}\,\underbrace{\sum_X\int d\Phi_X\,\frac{1}{4}|\mathcal{M}|^2}_{=\frac{1}{4}\frac{e^4}{Q^4}L^{\mu\nu}\underbrace{\sum_X\int d\Phi_X\,T_\mu(p,q,\{p_X\})\,T_\nu^*(p,q,\{p_X\})}_{\equiv H_{\mu\nu}}}$$

- consider $H_{\mu\nu}$: have summed and integrated all X dependence,

  so $H_{\mu\nu}(p,q)$, must be symm. in μ,ν ((parity cons. in QED, QCD))

  $$\Rightarrow H_{\mu\nu} = -H_1 g_{\mu\nu} + H_2\frac{p_\mu p_\nu}{Q^2} + H_3\frac{q_\mu q_\nu}{Q^2} + H_4\frac{p_\mu q_\nu + q_\mu p_\nu}{Q^2}$$

  where H_i are scalar fcts, $H_i\left(q^2 = -Q^2,\ p\cdot q = \frac{Q^2}{2x},\ \cancel{p^2 = M_p^2}\right)$
  neglect in DIS

  ((including Z^0 exchange, $\ldots + H_5\,\varepsilon_{\mu\nu\rho\sigma}\frac{p^\rho q^\sigma}{Q^2}$))

  $$\Rightarrow L^{\mu\nu}H_{\mu\nu} = 8(kk')H_1 + 8\frac{(pk)(pk')}{Q^2}H_2 + \underset{\text{(check?!)}}{0\cdot H_3 + 0\cdot H_4}$$

  used $d=4$, neglected M_p^2

  now $Q^2 = -q^2 = -(k-k')^2 = 2kk' - \cancel{2m_e^2}$

  $s = (p+k)^2 = 2pk + \cancel{M_p^2} + \cancel{m_e^2}$

  $pk' = p(k-q) = pk\left(1 - \frac{pq}{pk}\right) = pk\,(1-y)$

  $$= 4Q^2 H_1 + 2\frac{s^2}{Q^2}(1-y)\,H_2$$

- $$\frac{d^2\sigma}{dQ^2\,dx} = \frac{1}{2s}\,\frac{Q^2}{16\pi^2 s x^2}\,\frac{1}{4}\,\frac{\alpha^2 16\pi^2}{Q^4}\left(4Q^2 H_1 + 2\frac{s^2}{Q^2}(1-y)H_2\right) \; ; \; \text{def} \quad H_1 = 8\pi F_1, \; H_2 = 16\pi x F_2$$
$$= \frac{4\pi\alpha^2}{xQ^4}\left\{ x y^2 F_1(x,Q^2) + (1-y)F_2(x,Q^2)\right\}$$

((recall $y = Q^2/xs$))

→ amazing: w/o knowing ep interaction, derived s-dependence of σ !

→ the F's are called "structure functions" of the proton

Sometimes, see $F_T(x,Q^2) = 2x F_1(x,Q^2)$ "transverse"

$F_L(x,Q^2) = F_2(x,Q^2) - 2xF_1(x,Q^2)$ "longitudinal"

so $$\frac{d^2\sigma}{dQ^2dx} = \frac{2\pi\alpha^2}{xQ^4}\left\{\left(1+(1-y)^2\right)F_T(x,Q^2) + 2(1-y)F_L(x,Q^2)\right\}$$
$$= \frac{2\pi\alpha^2}{xQ^4}\left\{\left(1+(1-y)^2\right)F_2(x,Q^2) - y^2 F_L(x,Q^2)\right\}$$

useful since (for most current data) $y^2 \ll 1$

- have isolated all non-trivial x, Q^2 dependence into F_2, F_L ;
but still don't know anything about these fcts.

Assumption: interaction of γ with innards of proton does not involve any dimensionful scale

⇒ dim-less F's can not depend on dimensionful parameter Q^2

$$\Rightarrow \frac{d^2\sigma}{dQ^2dx} = \frac{2\pi\alpha^2}{xQ^4}\left\{\left(1+(1-y)^2\right)F_2(x) - y^2F_L(x)\right\} \qquad \text{Bjorken scaling}$$

→ experimentally, this is true (to a pretty good approximation);
but proton consists of quarks, bound at distance scale $\sim 1/M_p$,
so how can the interaction possibly be M_p-indep. ?!

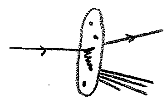
⇝ answer via parton model

5.2 Parton distribution functions

- description of process in Lorentz-invariant ;
parton model most easily formulated in "Breit-frame" : $E_\gamma = 0$, $E_p = \frac{Q}{2x}$

proton in its rest frame:  $2R$ ⇝ Breit frame:  $2R$ L-contraction

$\frac{4RxM_p}{Q} \ll 2R$

DIS in Breit frame: 

transverse size of photon $\sim 1/Q \ll 2R$

⇒ photon interacts with tiny fraction of disk

→ if quarks sufficiently dilute in p, photon does not resolve q interactions ; incoherent γq collisions !

→ since quarks act as if they don't interact,
their interaction does not introduce a dimensionful scale ⇒ Bjorken scaling

- more precisely :
proton ≡ bundle of comoving partons, carrying the proton's momentum p.
parton of type q carries fraction between $(\eta, \eta+d\eta)$ of p
during a fraction $d\eta \cdot \underbrace{f_q(\eta)}_{\text{pdf}}$ of the time
pdf, probability / parton distrib. fct.

assume partons pointlike ($r^2 \ll 1/Q^2$) and dilute ($f_q(\eta) \ll Q^2R^2$)

→ incoherent γ-parton-scattering

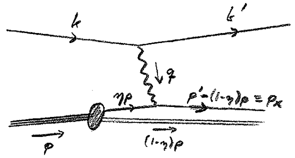
with $$\frac{d^2\sigma(e+p(p))}{dQ^2\,dx} = \sum_q \int_0^1 d\eta\, f_q(\eta)\, \underbrace{\frac{d^2\sigma(e+q(\eta p))}{dQ^2\,dx}}_{\text{partonic cross section}}$$

note: In partonic cross section, elastic scattering
→ outgoing parton is on mass-shell.
$2\to2$ scattering → only 1 non-triv kinematical variable (see pg. 45),
so $\frac{d^2\sigma}{dQ^2dx} \sim \delta$, where δ fct fixes one of the variables x, Q^2.
For massless partons, $(q+\eta p)^2 = 2\eta(pq) - Q^2 \overset{!}{=} 0 \Leftrightarrow \eta = x$

note: partons = quarks = fermions ⇒ (helicity cons.) $F_L = 0$ Callan-Gross relation

((partons = scalars ⇒ $F_T = 0$))

- to obtain the parton model prediction for structure fcts, need to calculate partonic cross section.
  → need matrix el. for $eq \to eq$
  get by "crossing symmetry" from $e^+e^- \to q\bar{q}$



$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4}\frac{1}{N_c} N_c \sum_s \left| \; \right|^2$$

← from sum over outgoing color
← from average over incoming color

(see pg. 44, $\mu \to q$, $m=0$)

$$= \frac{8 e^2 (Q_q e)^2}{(q^2)^2} \left[(\eta p \cdot k)^2 + ((p' - (1-\eta)p)\cdot k)^2 \right]$$

convert to our kinematic invariants

$Q^2 = -q^2$, $s = (p+k)^2 = 2p\cdot k + \cancel{m_p^2} + \cancel{m_e^2}$, $p' k = (p+q)\cdot k = \frac{s}{2} + q\cdot k$

$q\cdot k = (k-k')\cdot k = \frac{1}{2}(k-k')^2 + \frac{1}{2}\cancel{m_e^2} - \frac{1}{2}\cancel{m_e^2} = -Q^2/2$

$$\Rightarrow [\ldots] = \left[\left(\frac{\eta s}{2}\right)^2 + \left(\frac{s}{2} - \frac{Q^2}{2} - (1-\eta)\frac{s}{2}\right)^2 \right] = \left(\frac{\eta s}{2}\right)^2 \left[1 + \left(1 - \frac{Q^2}{\eta s}\right)^2 \right]$$

$$= 8 (4\pi\alpha)^2 Q_q^2 \; \frac{1 + \left(1 - \frac{Q^2}{\eta s}\right)^2}{4\left(\frac{Q^2}{\eta s}\right)^2}$$

→ phase space integration, see pg. 60

$$\int d\Phi_{X+1} = \underbrace{\int dQ^2 dx \frac{Q^2}{16\pi^2 s x^2}}_{\text{electron kinematics}} \int d\Phi_X$$

here: X consists of one massless parton

$$\int d\Phi_X = \int \frac{d^4 p_X}{(2\pi)^3} \delta(p_X^2)\,(2\pi)^4 \delta^4(\eta p + q - p_X)$$
$$= (2\pi)\,\delta\big((\eta p + q)^2\big) = (2\pi)\,\delta(\cancel{\eta^2 p_p^2} + 2\eta p q + q^2)$$
$$= (2\pi)\frac{1}{|2pq|}\,\delta\!\left(\eta - \frac{Q^2}{2pq}\right) = (2\pi)\frac{x}{Q^2}\,\delta(\eta - x)$$

→ partonic cross section

$$\sigma(e + q(\eta p)) = \frac{1}{2\eta s}\int d\Phi_{X+1} \langle |\mathcal{M}|^2 \rangle$$

$$\frac{d^2\sigma(e+q(\eta p))}{dQ^2\,dx} = \frac{1}{2\eta s}\frac{Q^2}{16\pi^2 s x^2}\frac{2\pi x}{Q^2}\,\delta(\eta - x)\,\langle |\mathcal{M}|^2\rangle \overset{y = \frac{Q^2}{xs}}{=} \frac{y^2}{16\pi Q^4}\,\delta(\eta - x)\, 2 (4\pi\alpha)^2 Q_q^2\,\frac{1 + (1-y)^2}{y^2}$$
$$= \frac{2\pi\alpha^2 Q_q^2}{Q^4}\,\delta(\eta - x)\left[1 + (1-y)^2\right]$$

- finally get cross section for $e+p$ (see pg. 62)

$$\frac{d\sigma(e+p(p))}{dQ^2\,dx} = \sum_q \int_0^1 d\eta\, f_q(\eta)\cdot \frac{2\pi\alpha^2 Q_q^2}{Q^4}\,\delta(\eta - x)\left[1 + (1-y)^2\right]$$
$$= \frac{2\pi\alpha^2}{x Q^4}\left[1 + (1-y)^2\right] \sum_q Q_q^2\, x\, f_q(x)$$

⤳ comparing with §5.1 ((result in terms of structure fcts F_2, F_L (pg. 61)))

$$\Rightarrow F_2(x, Q^2) = \sum_q Q_q^2\, x\, f_q(x) \quad , \quad F_L(x, Q^2) = 0$$

note: F_2 is Q^2-independent: Bjorken scaling!
$F_L = 0$ was the Callan-Gross relation.

⤳ we will see that QCD corrections do violate Bjorken scaling;
in experimental data, however, it is satisfied pretty well → Figure 1

- in practice, measure F_2 from different data sets and extract the f_q's.

$$F_2^{ep} = x\left[\tfrac{1}{9}(f_d + f_{\bar d} + f_s + f_{\bar s}) + \tfrac{4}{9}(f_u + f_{\bar u} + f_c + f_{\bar c}) \right]$$

since the pdf's contribute differently in different experiments
(e.g. F_2^{en}, $F_2^{\nu p}$, $F_2^{\bar\nu p}$, ...)
can do global fits to extract them. Typical results → Figure 2

⤳ useful checks via sum rules: $\int_0^1 dx\, f_{u_v}(x) = 2$, $\int_0^1 dx\, f_{d_v}(x) = 1$, etc.

5.3 QCD corrections in DIS

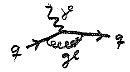
⤳ α_s is not small, so our above LO treatment of DIS might get important corrections.

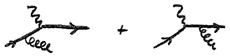
⤳ how does the parton model emerge from QCD ?

⤳ structure fcts will (slowly: logarithmically) depend on Q^2, leading to violation of Bjorken scaling

⤳ have to compute NLO corrections to DIS ;
divergences, splitting fcts, factorization, (DGLAP) evolution eqs , data

- three sources of NLO corrections

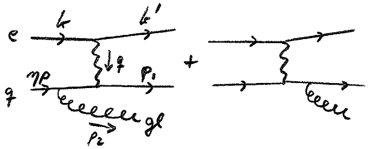
  (1) $eq \to eq$ at 1-loop ("virtual corr.") 

  (2) $eq \to eqg$ at tree-level 

  (3) $eg \to eq\bar{q}$ at tree-level  ← completely new structure, not present in parton model

5.3.1 DIS @ NLO : $eq \to eqg$

- 2 diagrams, get $\langle |\mathcal{M}|^2 \rangle$ by "crossing" from $e^+e^- \to q\bar{q}g$ (see §4.3.1)



$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4}\frac{1}{N_c} N_c \frac{8 C_F e^4 Q_q^2 g_s^2}{(kk')(p_1 p_2)(\eta p \cdot p_2)} \left[(p_1 \cdot k)^2 + (\eta p \cdot k)^2 + (p_1 \cdot k')^2 + (\eta p \cdot k')^2 \right]$$

→ phase space $\displaystyle\int d\Phi_3 = \underbrace{\int dQ^2 dx \frac{Q^2}{16\pi^2 s x^2}}_{\text{electron}} \int d\Phi_X$ here: X consists of 2 partons → non-triv.

$$\int d\Phi_X = \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta) \frac{1}{32\pi^2} = \int_0^{2\pi} d\varphi \int_0^1 dz \frac{1}{16\pi^2}$$

where (φ, θ) refer to direction of p_1 in CMS system of $\eta p + q$

alternatively, use Lorentz-invar. variable $z \equiv \frac{p_1 \cdot p}{q \cdot p} = \frac{1}{2}(1 - \cos\theta)$

→ partonic cross section

$$\frac{d^2\sigma(e+q)}{dQ^2 dx} = \frac{1}{2\eta s} \frac{Q^2}{16\pi^2 s x^2} \int_0^{2\pi} d\varphi \int_0^1 dz \frac{1}{16\pi^2} \frac{2 C_F e^4 Q_q^2 g_s^2}{(kk')(p_1p_2)(\eta p \cdot p_2)} \left[\text{see} \uparrow \right]$$

$$= \frac{Q^2 C_F \alpha^2 Q_q^2 \alpha_s}{2\eta x^2 s^2} \int_0^1 dz \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{[\ldots]}{(kk')(p_1p_2)(\eta p \cdot p_2)}$$

rewrite scalar products in terms of kinematic variables, perform φ-integration, use $\bar{x} \equiv x/\eta$

$$= \frac{Q^2 C_F \alpha^2 Q_q^2 \alpha_s}{2\eta x^2 s^2} \int_0^1 dz \frac{1}{y^2 Q^2} \left\{ \left[1 + (1-y)^2\right] \underbrace{\left(\frac{1+\bar{x}^2}{1-\bar{x}} \frac{1+z^2}{1-z} + 3 - z - \bar{x} + 11 \bar{x} z \right)}_{\equiv f_2(\bar{x}, z)} - y^2 \left(8 \bar{x} z \right) \right\}$$

will give a non-zero F_L

will give divergence in F_2 due to $\int dz$

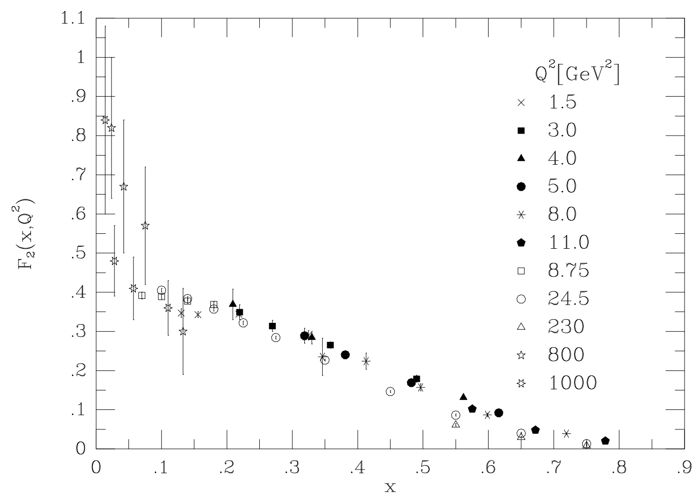


Figure 1: The structure function F_2 as a function of x for various Q^2 values, exhibiting Bjorken scaling, taken from [Ellis/Stirling/Webber]

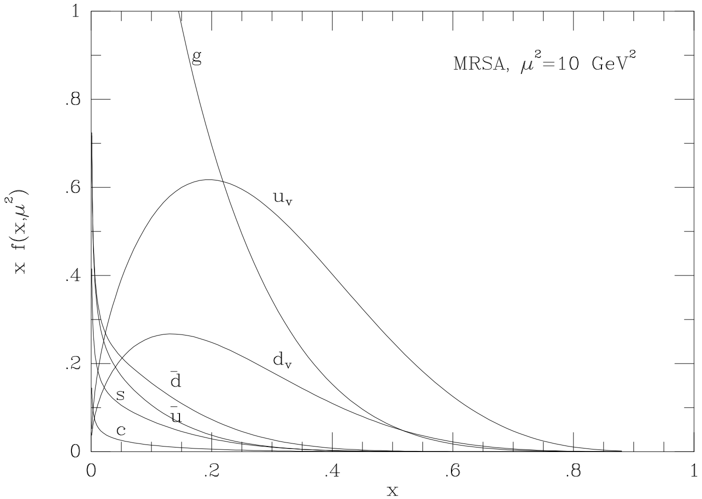


Figure 2: Parton distribution function set A from the Martin-Roberts-Stirling group, taken from [Ellis/Stirling/Webber]. Note that this uses the common notation of defining valence quark distributions, $f_{u_v} \equiv f_u - f_{\bar{u}}$, $f_{d_v} \equiv f_d - f_{\bar{d}}$.

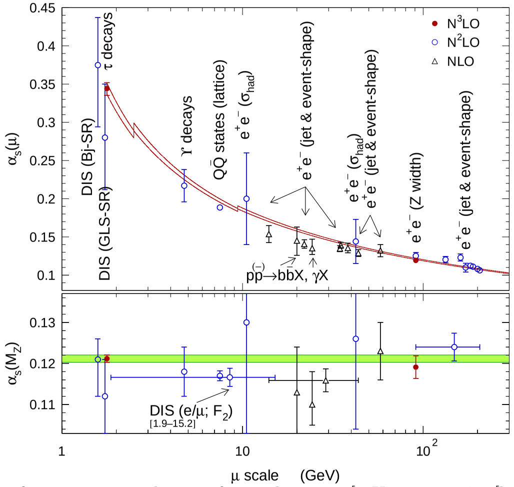


Figure 3: Results of a recent compilation of α_S values, see [arXiv:0803.0979 [hep-ph], arXiv:hep-ex/0606035]. The scale dependence shows excellent agreement with the predictions of perturbative QCD over a wide energy range. When translated into measurements of $\alpha_S(M_z)$, the separate measurements cluster strongly around the average value, $\alpha_S(M_z) = 0.1204 \pm 0.0009$

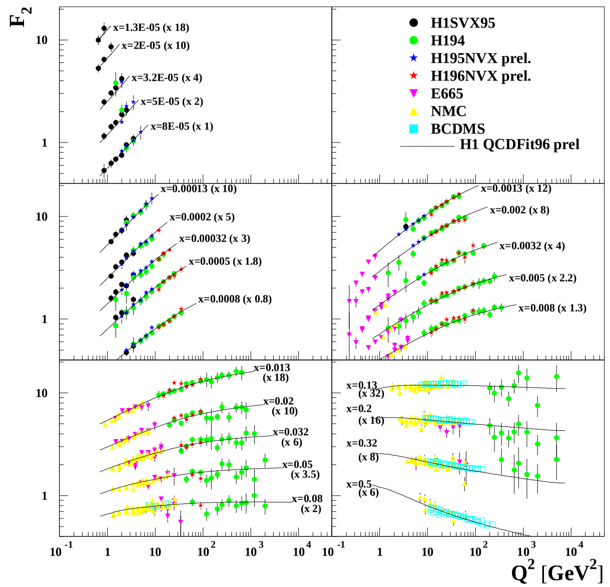


Figure 4: Fit to the F_2 data over a wide range of Q^2 values, exhibiting violation of Bjorken scaling

- from cross section $\dfrac{d^2\sigma(e+p)}{dQ^2\,dx} \overset{\text{(pg. 62)}}{=} \sum_q \int_0^1 d\eta\, f_q(\eta)\, \dfrac{d^2\sigma(e+q(\eta p))}{dQ^2\,dx}$

  $\overset{\text{(pg. 61)}}{=} \dfrac{2\pi\alpha^2}{xQ^4}\left\{\left[1+(1-y)^2\right]F_2(x,Q^2) - y^2 F_L(x,Q^2)\right\}$

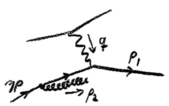
  read off $F_2(x,Q^2) = \dfrac{C_F\alpha_s}{2\pi}\sum_q \int_0^1 d\eta\, f_q(\eta)\, Q_q^2\, \boxed{\dfrac{xQ^4}{2\eta x^2 s^2 y^2}}_{=\frac{x}{2\eta}} \int_0^1 dz\, \bar{F}_2\left(\frac{x}{\eta}, z\right)$

  $\left(\bar{x} \equiv \dfrac{x}{\eta}\right)$ $\left(\dfrac{d\eta}{\eta} = -\dfrac{d\bar{x}}{\bar{x}}\right)$

  $= \dfrac{C_F\alpha_s}{2\pi}\sum_q \dfrac{1}{2}\int_x^\infty d\bar{x}\, f_q\left(\frac{x}{\bar{x}}\right)\frac{x}{\bar{x}}\, Q_q^2 \int_0^1 dz \left(\dfrac{1+\bar{x}^2}{1-\bar{x}}\,\dfrac{1+z^2}{1-z} + 3 - z - \bar{x} + 11\bar{x}z\right)$

- consider the divergence in F_2

  comes from $z \to 1$, where $z = \dfrac{p_1 \cdot p}{q \cdot p}$

  $\to$ outgoing gluon collinear with incoming quark: $\eta p \cdot p_2 = \eta p \cdot (\eta p + q - p_1) = \cancel{\eta^2 p_p^2} + \eta p \cdot q - \eta p \cdot p_1 = \eta p q (1-z)$

  $\rightsquigarrow$ internal line becomes on-shell, causing the divergence:

  $(\eta p - p_2)^2 = \cancel{\eta^2 p_p^2} - 2\eta p \cdot p_2 + \cancel{p_2^2}^{\,0} = -2\eta p \cdot q\,(1-z)$, quark propagator $\sim \dfrac{1}{1-z}$

  Note also: coefficient of divergence $\sim \dfrac{1}{1-\bar{x}}$, diverges at $\bar{x} = 1$,

  when gluon is infinitely soft

- regulate divergence

  consider transverse momentum $k_\perp$ of outgoing quark in CMS system of $(\eta p + q)$;

  ((it turns out that $k_\perp^2 = Q^2\left(\frac{\eta}{x} - 1\right) z (1-z)$))

  $z \to 1$ means $k_\perp^2 \to 0$, so restricting $k_\perp^2 > \mu^2$ (with $\mu^2 \ll Q^2$)

  regulates the divergence at $z \to 1$ ($\mu \to 0$ gives full result)

  $\rightsquigarrow \int_0^1 dz \to \int_{z_-}^{z_+} dz$, where $z_{+/-} = \frac{1}{2}\left(1 \pm \sqrt{1 - 4\frac{\mu^2}{Q^2(\frac{\eta}{x}-1)}}\right)$; $\int_0^1 dz \approx \int_0^{1-\frac{\mu^2}{Q^2}} dz$

  $\Rightarrow F_2(x,Q^2) = \dfrac{\alpha_s}{2\pi}\sum_q \dfrac{1}{2}\int_x^\infty d\bar{x}\, f_q\left(\frac{x}{\bar{x}}\right)\frac{x}{\bar{x}}\, Q_q^2 \left(2\hat{P}(\bar{x}) \ln\dfrac{Q^2}{\mu^2} + R(\bar{x})\right)$

  ↑ divergence ↑ finite; $\mu \to 0$ done

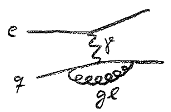
  with "(unregularized) splitting function" $\hat{P}(x) \equiv C_F \dfrac{1+x^2}{1-x}$

  describes probability distribution of outgoing quark in $\overset{p}{\longrightarrow}$ xp

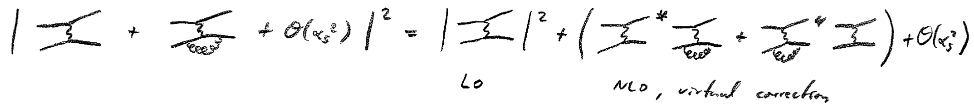
  note: have not (yet) removed the divergence,

  only regulated it.

5.3.2 DIS @ NLO: 1-loop $eq \to eq$

diagram:  $\sim \alpha_s$

in $|\mathcal{M}|^2$, only need interference term with tree level diagram



can again take everything from $e^+e^- \to q\bar{q}$ (§ 4.3.2) via crossing

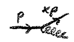
→ here, recall only some of the features of that calculation, to illustrate the physics; interested in divergences.

- external particles same as at LO → kinematics is the same
  → $\int d\Phi \sim \delta(y - x)$ (see § 5.2, pg. 63)
- as in $e^+e^- \to q\bar{q}$, interference term is divergent (and negative);
  divergence comes from same kinematic region as in § 5.3.1:
  if gluon is soft, or collinear with either of the quarks.
- result: same form as $F_2(x, Q^2)$ above, with (unregularized)
  splitting fct $\hat{P}(\bar{x})$ replaced by $\tilde{P}(\bar{x})$ ((and different $R(\bar{x})$))
  $\tilde{P}(\bar{x}) \sim \delta(1-\bar{x})$ from $\delta(y-x)$

  $\Rightarrow$ real + virtual $\sim P(x) \equiv \hat{P}(x) + \tilde{P}(x)$

  mathematical trick: plus-distribution
  given $f(x)$, well-defined for $0 \le x < 1$
  define distribution $f(x)_+$ on $0 \le x \le 1$ as:
  $$f(x)_+ \equiv f(x) - \delta(1-x) \int_0^1 dx' f(x')$$
  ((most useful for $f(x)$ which diverge at $x \to 1$))
  $\Rightarrow$ for any test function $g(x)$ which is smooth at $x = 1$,
  $$\int_0^1 dx\, f(x)_+\, g(x) = \int_0^1 dx\, f(x) \left[g(x) - g(1) \right]$$
  ((the latter integral being finite if $g(x) \to g(1)$ sufficiently quickly))

- splitting fct $P(x) = C_F \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x) \right]$

  note: – inserting $P(x)$ into $F_2(x, Q^2)$, the divergence at $\bar{x} \to 1$ cancels;
  but the divergence at $z \to 1$, parametrized by $\ln \frac{Q^2}{\mu^2}$, remains.
  – in fact, $P(x)$ is the first correction to a fct
  that describes the momentum distribution of quarks within quarks,
  $$\mathcal{P}(x) \equiv \delta(1-x) + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{Q_0^2} P(x) + \mathcal{O}\left(\alpha_s^2 \ln^2\left(\frac{Q^2}{Q_0^2}\right)\right)$$
  ((see pg. 66:  etc; "pure quark" at scale Q_0^2))

5.3.3 Factorization, evolution

⇝ understand why results are still divergent!
compare to $e^+e^- \to q\bar{q}$ case: there, real + virtual = finite
here, main difference is: introduced pdf's f_q

- $z \to 1$ divergence comes from internal quark propagator $\sim \frac{1}{1-z}$ (see pg. 66);
  roughly, vanishingly small virtuality $\hat{=}$ arbitrarily long time scales (uncertainty princ.)
  ⇝ contradiction to assumptions of parton model: rapid snapshot of proton
- the actual problem is overcounting:
  pdf's $\hat{=}$ internal proton structure $\hat{=}$ result of QCD interactions
  our calculation: QCD corrections to q scattering
  integrated over all final states (hence all E scales)
  ⇝ but are these QCD corr. already somehow in the pdf's?

⇒ to resolve overcounting: separate ("factorize") different types of physics at different energy scales!

introduce factorization scale μ.

physics at scales $< \mu$ ≡ part of hadron wave fct ≡ pdf's
$> \mu$ ≡ part of partonic scattering cross section ≡ coefficient fcts

note: the cutoff introduced in § 5.3.2, $eq \to egq$, was hence correct.

((• kinematic limits: note that $\int_x^\infty d\bar{x} \to \int_x^1 d\bar{x}$ (cf pg. 66);

outgoing quark(p_1) and gluon (p_2) are <u>real</u> particles

$0 \le (p_1+p_2)^2 = (\eta p + q)^2 = \cancel{\eta^2 p^2} + 2\eta p\cdot q + q^2 = 2p\cdot q\left(\eta - \frac{Q^2}{2p\cdot q}\right) = 2p\cdot q(\eta - x) = 2p\cdot q\,\eta(1-\bar{x})$))

• since pdf's contain physics below μ only, they <u>depend</u> on μ

$$F_2(x,Q^2) = \sum_q Q_q^2 \int_x^1 d\bar{x}\, f_q\left(\frac{x}{\bar{x}},\mu^2\right)\frac{x}{\bar{x}}\left\{\delta(1-\bar{x}) + \frac{\alpha_s}{2\pi}\left(P(\bar{x})\ln\frac{Q^2}{\mu^2} + R(\bar{x})\right) + O(\alpha_s^2)\right\}$$

$\Rightarrow$ structure fct depends on Q^2 now, <u>violates</u> Bjorken scaling!

$\to$ μ^2-dependence ?!↯ ad-hoc theoretical construct

• physical cross sections cannot depend on μ^2

$\Rightarrow \partial_{\mu^2} F_2(x,Q^2) \overset{!}{=} 0$

or, at least, in our calculation $\mu^2\partial_{\mu^2} F_2(x,Q^2) = O(\alpha_s^2)$

$$\Leftrightarrow \quad \mu^2\partial_{\mu^2} f_q(x,\mu^2) = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\bar{x}}{\bar{x}} f_q\left(\frac{x}{\bar{x}},\mu^2\right)P(\bar{x}) + O(\alpha_s^2)$$

<u>Dobshitzer - Gribov - Lipatov - Altarelli - Parisi</u> eqn (DGLAP, GLAP, AP)

• try to understand <u>physical content</u> of DGLAP eqn

$$\left(\frac{1+x^2}{1-x}\right)_+ \overset{(pg\,67)}{=} \frac{1+x^2}{1-x} - \delta(1-x)\int_0^1 dx' \frac{1+x'^2}{1-x'}$$

$\frac{1+x'^2}{1-x'} = 2 - (1-x')(1+x')$

$\uparrow = 1+x^2$ due to δ fct

$$= (1+x^2)\left\{\frac{1}{1-x} - \delta(1-x)\int_0^1 dx' \frac{1}{1-x}\right\} + \delta(1-x)\underbrace{\int_0^1 dx'(1+x')}_{=3/2}$$

$$= \frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)$$

$\Rightarrow P(x) = C_F\left(\frac{1+x^2}{1-x}\right)_+$, (cf. pg 68)

$$\text{now,}\quad \mu^2\partial_{\mu^2} f_q(x,\mu^2) = \underbrace{\frac{\alpha_s}{2\pi}\int_x^1\frac{d\bar{x}}{\bar{x}} f_q\left(\frac{x}{\bar{x}},\mu^2\right)C_F\frac{1+\bar{x}^2}{1-\bar{x}}}_{\text{pdf increase}} \underbrace{- \frac{\alpha_s}{2\pi}\int_x^1\frac{d\bar{x}}{\bar{x}} f_q\left(\frac{x}{\bar{x}},\mu^2\right)C_F\delta(1-\bar{x})\int_0^1 dx'\frac{1+x'^2}{1-x'}}_{\overset{!}{=} -\frac{\alpha_s}{2\pi}f_q(x,\mu^2)C_F\int_0^1 d\bar{x}\frac{1+\bar{x}^2}{1-\bar{x}}\ \text{pdf decrease}}$$

so at given x, pdf $\left\{\begin{matrix}\text{increase}\\ \text{decrease}\end{matrix}\right\}$ from $\left\{\begin{matrix}\text{higher-}x\text{ quarks}\\ \text{quarks at }x\end{matrix}\right\}$ reducing their momentum fraction by radiating off gluons.

$\rightsquigarrow$ both pieces are <u>divergent</u> at $\bar{x}\to 1$ (due to infinitesimally soft gluon radiation); div's <u>cancel</u> because # gained quarks = # lost quarks

• <u>solve</u> DGLAP eqn ?

in practice: done numerically

• <u>scheme/scale dependence</u>

the above factorization (of structure fct F in $f_{\text{non-pert}} * \text{coeff.fct}_{\text{pert.}}$) may look pretty arbitrary; can be proven to all orders in pert. theory;

$\to$ instead of transverse momentum cutoff μ^2 above, use <u>dim. reg.</u>

$\Rightarrow$ NLO cross section in d dimensions: divergence is pole $\frac{1}{\varepsilon}$ now

$$F_2(x,Q^2) = \sum_q Q_q^2\int_x^1 d\bar{x}\,\bar{f}_q\left(\frac{x}{\bar{x}}\right)\frac{x}{\bar{x}}\left\{\delta(1-\bar{x}) + \frac{\alpha_s}{2\pi}\left(P(\bar{x})\left(\frac{4\pi\mu^2}{Q^2}\right)^\varepsilon\left(-\frac{1}{\varepsilon}\right) + \bar{R}(\bar{x}) + O(\varepsilon)\right) + O(\alpha_s^2)\right\}$$

(from $\frac{g^2_{\text{bare}}}{4\pi} = Z_g^2\alpha_s\mu^{4-d}$)

↑ "bare" pdf ↑ <u>same</u> P as above ↑ different from R

now (cf §3.3), since pdf's are not physical observables, define modified set of pdf's as

$$x\bar{f}_q(x) = \int_x^1 d\bar{x}\, f_q\left(\frac{x}{\bar{x}},\mu_F^2\right)\frac{x}{\bar{x}}\left\{\delta(1-\bar{x}) - \frac{\alpha_s}{2\pi}\left(P(\bar{x})\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^\varepsilon\left(-\frac{1}{\varepsilon}\right) + K(\bar{x})\right)\right\}$$

↑ again, arbitrary "factorization" scale ↑ arbitrary finite fct

(check?!)

$$\Rightarrow F_2(x,Q^2) = \sum_q Q_q^2\int_x^1 d\bar{x}\, f_q\left(\frac{x}{\bar{x}},\mu_F^2\right)\frac{x}{\bar{x}}\left\{\delta(1-\bar{x}) + \frac{\alpha_s}{2\pi}\left(P(\bar{x})\ln\frac{Q^2}{\mu_F^2} + \bar{R}(\bar{x}) - K(\bar{x})\right) + O(\alpha_s^2)\right\}$$

note: • same form as pg. 69 (top), except for finite piece

• μ-dependence has cancelled

• for $\mu_{(F)}$-evolution of pdf, DGLAP eqn is valid

• f_q also depends on <u>choice</u> of K, "scheme dependence";

factorization theorem proves that

- for any physical quantity, all K and μ_F-dependence cancels
- scheme- and scale-dependent pdf's $f_q(x,\mu^2)$ are <u>universal</u> (i.e. process-independent)

• common choices: $\overline{MS}$ scheme ($K(\bar{x}) \equiv 0$)

DIS scheme ($K(\bar{x}) \equiv \bar{R}(\bar{x})$)

note: • dependence on scheme and scale cancels in physical quantities after calculating infinitely many orders in pert. theory ...

→ at finite order: residual dependence

→ need a procedure to <u>choose value for μ_F</u>

<u>n^{th} order</u> of pert. theory contains terms $\sim \alpha_s^n \ln^m \frac{Q^2}{\mu_F^2}$ ($m \leq n$)

so for reasons of <u>convergence</u>, take μ_F "not too far from" Q^2.

5.3.4 DIS @ NLO : $eg \to eq\bar{q}$

→ have so far not looked at process (3), see pg. 65

• most of § 5.3.1 – 3 carries over

again have a collinear singularity, coming from internal quark going "onshell".

• singularity can again be absorbed into factorized universal gluon-pdf f_g

$$\Rightarrow F_2(x,Q^2) \ni \sum_q Q_q^2 \int_x^1 d\bar{x}\, f_g\left(\frac{x}{\bar{x}},\mu_F^2\right) \frac{x}{\bar{x}} \left\{ \frac{\alpha_s}{2\pi}\left(P_{qg}(\bar{x}) \ln \frac{Q^2}{\mu_F^2} + R_g(\bar{x}) - K_{qg}(\bar{x}) \right) + O(\alpha_s^2) \right\}$$

with splitting fct $P_{qg}(x) = \frac{1}{2}\left[x^2 + (1-x)^2 \right]$ ((so far, had $P = P_{qq}$))

• in general, DGLAP eqn is a set of coupled eqs

$$\mu^2 \partial_{\mu^2} f_a(x,\mu^2) = \sum_b \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\bar{x}}{\bar{x}} f_b\left(\frac{x}{\bar{x}},\mu^2\right) P_{ab}(\bar{x}) + O(\alpha_s^2)$$

(at higher orders, also → $P_{gg}(x)$ contributes)

((systematics of labelling the splitting fcts: $a \to b$, z ← internal line $\sim \frac{1}{1-z}$))

• Q^2-dependence of $F_2(x,Q^2)$ is entirely driven by μ_F^2-dependence of pdfs, which is predicted by DGLAP evolution eqs;

⇒ structure fct data over a wide range of Q^2 provide a stringent test of perturbative QCD → <u>Figure 4</u>

6. "Anomalies"

[Peskin/Schroeder §19 ; Zee § IV.7]

~ Q.: symmetry of classical physics $\hat{=}$ symmetry of quantum physics ?

(action $S[\phi]$ invariant under $\phi \to \phi + \delta\phi$) (path integral $\int \mathcal{D}\varphi\, e^{iS[\phi]}$ invariant)

→ A.: not always; measure $\mathcal{D}\varphi$ may or may not be invariant.

example: rotational invariance of QM

would be strange if quantum fluctuations preferred a specific direction!

but: quantum fluctuation <u>can</u> break (some) classical symmetries;

this phenomenon is called "anomaly";

conceptually clear: change of integration variables → don't forget Jacobian

important subject in QFT → many ways of looking at it

here: see a class. symmetry vanishing, by explicit Feynman diag calc.

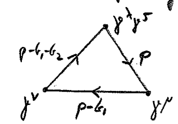
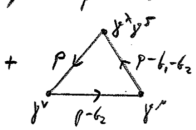
• theory of one massless fermion: $\boxed{\mathcal{L}_1 \equiv \bar{\psi} i \gamma^\mu \partial_\mu \psi}$

invariant under $\psi \to e^{i\theta}\psi$ and $\psi \to e^{i\theta\gamma^5}\psi$

conserved current $j^\mu \equiv \bar{\psi}\gamma^\mu\psi$ (vector) and $j_5^\mu \equiv \bar{\psi}\gamma^\mu\gamma^5\psi$ (axial)

((check: $\partial_\mu j^\mu = 0 = \partial_\mu j_5^\mu$ via class. eqn. of motion $i\gamma^\mu\partial_\mu\psi = 0$))

• calculate (Fourier transform of) amplitude $\langle 0 | T\, j_5^\lambda(0)\, j^\mu(x_1)\, j^\nu(x_2) | 0 \rangle$

$\Delta^{\lambda\mu\nu}(k_1,k_2) =$  +  , $q \equiv k_1 + k_2$

$$= (-1)\int \frac{d^4p}{(2\pi)^4} \mathrm{tr}\left(\gamma^\lambda\gamma^5 \frac{i}{\not{p}-\not{q}} \gamma^\nu \frac{i}{\not{p}-\not{k}_1} \gamma^\mu \frac{i}{\not{p}} + \gamma^\lambda\gamma^5 \frac{i}{\not{p}-\not{q}} \gamma^\mu \frac{i}{\not{p}-\not{k}_2} \gamma^\nu \frac{i}{\not{p}} \right)$$

if (in QFT) $\partial_\mu j^\mu = 0$ holds, then $k_{1\mu}\Delta^{\lambda\mu\nu} = 0$ and $k_{2\nu}\Delta^{\lambda\mu\nu} = 0$

$\prime\prime$ $\partial_\mu j_5^\mu = 0$ holds, then $q_\lambda \Delta^{\lambda\mu\nu} = 0$

→ can check this by explicit computation.

6.1 Vector current conservation

- would non-conservation of either current be a disaster?

  - j^μ : charge $Q = \int d^3x\, j^0$ counts # of fermions

    → would be difficult to interpret if not conserved!

    couple photon to ψ, photon line coming into vertex γ^μ

    would have propag $\frac{-i}{k_1^2}\left(g_{\mu\xi} - (1-\xi)\frac{k_{1\mu}k_{1\xi}}{k_1^2}\right)$

    → gauge dependence falls out if $k_{1\mu}\Delta^{\lambda\mu\nu} = 0$

  - j_5^μ : who care if axial charge $Q^5 = \int d^3x\, j_5^0$ changes in time?

- naive calculation

$$k_{1\mu}\Delta^{\lambda\mu\nu}(k_1,k_2) = i\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\Big(\gamma^\lambda\gamma^5\frac{1}{\not p-\not q}\gamma^\nu\frac{1}{\not p-\not k_1}\underbrace{\not k_1}_{=\not p-(\not p-\not k_1)}\frac{1}{\not p} + \gamma^\lambda\gamma^5\frac{1}{\not p-\not q}\underbrace{\not k_1}_{=(\not p-\not k_2)-(\not p-\not q)}\frac{1}{\not p-\not k_2}\gamma^\nu\frac{1}{\not p}\Big)$$

$$= i\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\Big(\gamma^\lambda\gamma^5\frac{1}{\not p-\not q}\gamma^\nu\frac{1}{\not p-\not k_1} - \underbrace{\gamma^\lambda\gamma^5\frac{1}{\not p-\not k_2}\gamma^\nu\frac{1}{\not p}}_{\text{shift } p\to p-k_1}\Big)$$

$$= 0$$

- more careful calculation

  when is it OK to shift integration variables?

  1 dim: $\int_{-\infty}^{\infty}dp\,[f(p+a)-f(p)] = \int_{-\infty}^{\infty}dp\,[a\,\partial_p f(p) + \mathcal{O}(a^2)] = a\big(f(\infty)-f(-\infty)\big) + \mathcal{O}(a^2)$

  d dim: $\int d^dp_E\,[f(p+a)-f(p)] = \int d^dp_E\,[a^\mu\partial_\mu f(p) + \dots] \overset{\text{Gauss}}{=} \lim_{P\to\infty}\left\langle a^\mu\left(\frac{P_\mu}{P}\right)f(P)\,S_{d-1}(P)\right\rangle$

  $\left(\!\left(\frac{2\pi^{d/2}}{\Gamma(d/2)}P^{d-1}\right)\!\right)$ "surface" of d dim sphere, radius P

  for our 4 dim Minkowskian integral, ← from Wick rot.

$$\int d^4p\,[f(p+a)-f(p)] = \lim_{P\to\infty}\left\langle i\,a^\mu\left(\frac{P_\mu}{P}\right)f(P)\,(2\pi^2P^3)\right\rangle \leftarrow \text{angular average}$$

  use this for $a = -k_1$

  and $f(p) = \mathrm{tr}\left(\gamma^\lambda\gamma^5\frac{1}{\not p-\not k_2}\gamma^\nu\frac{1}{\not p}\right) = \frac{\mathrm{tr}\big(\gamma^5(\not p-\not k_2)\gamma^\nu\not p\gamma^\lambda\big)}{(p-k_2)^2p^2} \overset{\text{(check?!)}}{=} \frac{4i\,\varepsilon^{\tau\nu\sigma\lambda}k_{2\tau}p_\sigma}{(p-k_2)^2p^2}$

$$\Rightarrow k_{1\mu}\Delta^{\lambda\mu\nu}(k_1,k_2) = \frac{i}{(2\pi)^4}\lim_{P\to\infty}\left\langle i(-k_1)^\mu\frac{P_\mu}{P}\,\frac{4i\,\varepsilon^{\tau\nu\sigma\lambda}k_{2\tau}P_\sigma}{(P-k_2)^2P^2}\,2\pi^2P^3\right\rangle$$

$$\left(\langle P_\mu P_\sigma\rangle = \frac{g_{\mu\sigma}}{4}P^2\right) = \frac{i}{8\pi^2}k_1^\mu\,\varepsilon^{\tau\nu\sigma\lambda}g_{\mu\sigma}k_{2\tau} = \frac{i}{8\pi^2}\varepsilon^{\lambda\nu\tau\sigma}k_{1\tau}k_{2\sigma}$$

$\left(\!\left(\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\,\varepsilon^{\mu\nu\rho\sigma}\right)\!\right)$

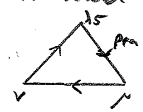
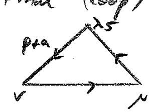
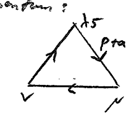
$\rightsquigarrow k_{1\mu}\Delta^{\lambda\mu\nu} \neq 0$?! ↯ fermion # not conserved, we disintegrate

- reason for above result: Δ is linearly divergent!

  ⇒ have to make sure (even before calculating $k\Delta$)

  that integral is well-defined ((i.e. its value does not depend on the physicist doing the calculation))

  freedom of choice to label internal (loop) momentum:

  $\Delta^{\lambda\mu\nu}(a,k_1,k_2) \equiv$  +  =  $+ \begin{pmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{pmatrix}$

  but which a to choose?

  only sensible answer: choose a such that $k_{1\mu}\Delta^{\lambda\mu\nu}(a,k_1,k_2) = 0 = k_{2\nu}\Delta^{\lambda\mu\nu}$

  compute $\Delta^{\lambda\mu\nu}(a,k_1,k_2) - \Delta^{\lambda\mu\nu}(k_1,k_2)$ with above "careful" way:

  use $f(p) = \mathrm{tr}\left(\gamma^\lambda\gamma^5\frac{1}{\not p-\not q}\gamma^\nu\frac{1}{\not p-\not k_1}\gamma^\mu\frac{1}{\not p}\right)$

  note $\lim_{P\to\infty}f(P) = \lim_{P\to\infty}\frac{\mathrm{tr}(\gamma^\lambda\gamma^5\not P\gamma^\nu\not P\gamma^\mu\not P)}{P^6} \overset{\text{(check?!)}}{=} \frac{-4i\,P^2P_\sigma\,\varepsilon^{\sigma\nu\mu\lambda}}{P^6}$

$$\Rightarrow \Delta^{\lambda\mu\nu}(a,k_1,k_2) - \Delta^{\lambda\mu\nu}(k_1,k_2) = \frac{4i}{8\pi^2}\lim_{P\to\infty}\left\langle a^\omega\frac{P_\omega P_\sigma}{P^2}\varepsilon^{\sigma\nu\mu\lambda} + \begin{pmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{pmatrix}\right\rangle$$

$$= \frac{i}{8\pi^2}\varepsilon^{\sigma\nu\mu\lambda}a_\sigma + \begin{pmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{pmatrix}$$

  in general, $a = \alpha(k_1+k_2) + \beta(k_1-k_2)$ are all possible shifts

$$\Rightarrow \Delta^{\lambda\mu\nu}(a,k_1,k_2) = \Delta^{\lambda\mu\nu}(k_1,k_2) + \frac{i\beta}{4\pi^2}\varepsilon^{\lambda\mu\nu\sigma}(k_1-k_2)_\sigma$$

  ((α dropped out due to antisy of ε))

$$\Rightarrow k_{1\mu}\Delta^{\lambda\mu\nu}(a,k_1,k_2) = \underbrace{k_{1\mu}\Delta^{\lambda\mu\nu}(k_1,k_2)}_{=\frac{i}{8\pi^2}\varepsilon^{\lambda\nu\tau\sigma}k_{1\tau}k_{2\sigma}\ \text{(see p. 73)}} + \frac{i\beta}{4\pi^2}\varepsilon^{\lambda\mu\nu\sigma}(k_{1\mu}k_{1\sigma} - k_{1\mu}k_{2\sigma}) \overset{!}{=} 0 \Rightarrow \beta = -\frac{1}{2}$$

- note: Feynman rules are not sufficient to compute $\langle 0|T\, j_5^\lambda(0)j^\mu(x_1)j^\nu(x_2)|0\rangle$

  have to supplement them by vector current conservation!

  $\rightsquigarrow$ amplitude $\langle 0|T\,j_5 j j|0\rangle$ is defined by $\Delta^{\lambda\mu\nu}\big(\alpha(k_1+k_2) - \frac{1}{2}(k_1-k_2),k_1,k_2\big)$

6.2 Axial current non-conservation

$\rightsquigarrow$ in §6.1, learned how to properly define $\Delta^{\lambda\mu\nu}$;

now, check axial current conservation by computing $q_\lambda \Delta^{\lambda\mu\nu}(a, k_1, k_2)$ (fixed by above)

$\left(\left(a = -\tfrac{1}{2}(k_1 - k_2)\right)\right)$

- $$q_\lambda \Delta^{\lambda\mu\nu}(a, k_1, k_2) = \underbrace{q_\lambda \Delta^{\lambda\mu\nu}(k_1,k_2)}_{} - \underbrace{\frac{i}{8\pi^2}\varepsilon^{\lambda\mu\nu\sigma}(k_1+k_2)_\lambda (k_1-k_2)_\sigma}_{= +\frac{i}{4\pi^2}\varepsilon^{\mu\nu\lambda\sigma}k_{1\lambda}k_{2\sigma}}$$

$$= i\int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\left(\not q \gamma^5 \frac{1}{\not p - \not q}\gamma^\nu \frac{1}{\not p - \not k_1}\gamma^\mu \frac{1}{\not p} + \begin{pmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{pmatrix}\right)$$

$\not q = \not p - (\not p - \not q)$; use cyclicity of trace; $\{\gamma^\mu, \gamma^5\} = 0$

$$= i\int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\left(\underbrace{\gamma^\mu\gamma^5 \frac{1}{\not p - \not q}\gamma^\nu \frac{1}{\not p - \not k_1}} - \gamma^\nu\gamma^5 \frac{1}{\not p - \not k_1}\gamma^\mu \frac{1}{\not p} + \begin{pmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{pmatrix} - \underbrace{\begin{pmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{pmatrix}}\right)$$

(pg. 73)

$$= \underbrace{k_{1\sigma}\Delta^{\mu\sigma\nu}(k_1,k_2)} + \begin{pmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{pmatrix} = \frac{i}{8\pi^2}\varepsilon^{\mu\nu\tau\sigma}k_{1\tau}k_{2\sigma} \cdot 2$$

$$= \frac{i}{2\pi^2}\varepsilon^{\mu\nu\lambda\sigma}k_{1\lambda}k_{2\sigma}$$

$\Rightarrow$ axial current is not conserved!

this is known as (axial/chiral) anomaly:

quantum fluctuations destroyed the (classical) axial current conservation.

Consequences / remarks (w/o derivations)

- gauge our theory $\mathcal{L}_1$: $\boxed{\mathcal{L}_2 \equiv \bar\psi\, i\gamma^\mu(\partial_\mu - ie\underbrace{A_\mu}_{\text{"photon"}})\psi}$

$$\rightsquigarrow \partial_\mu j_5^\mu = \begin{cases} 0 & \text{classically} \\ \frac{e^2}{(4\pi)^2}\varepsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma} & \text{quantum}\end{cases}$$

((check?! : $k \to \partial$ in $F = \partial A - \partial A$))

historically important! decay $\pi^0 \to \gamma + \gamma$ (massless) forbidden via (wrong) $\partial_\mu j_5^\mu = 0$,

but decay is observed experimentally, as predicted by (correct) $\partial_\mu j_5^\mu \neq 0$.

($\Gamma_i/\Gamma \approx 98.8\,\%$, see PDG

- re-write $\mathcal{L}_2$ with $\psi_{R/L} \equiv \frac{1 \pm \gamma^5}{2}\psi$,

introduce left- and right-handed currents $j_{R/L}^\mu \equiv \bar\psi_{R/L}\gamma^\mu \psi_{R/L}$

$$\Rightarrow \partial_\mu j_{R/L}^\mu = \pm\frac{1}{2}\frac{e^2}{(4\pi)^2}\varepsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}$$

((that's why the anomaly is called "chiral"))

- add a fermion mass to $\mathcal{L}_2$: $\boxed{\mathcal{L}_3 \equiv \bar\psi\left[i\gamma^\mu(\partial_\mu - ieA_\mu) - m\right]\psi}$

$\Rightarrow$ invariance under $\psi \to e^{i\theta\gamma^5}\psi$ broken by $m \neq 0$.

classically, $\partial_\mu j_5^\mu = 2m\,\bar\psi\, i\gamma^5\psi$, axial current not conserved.

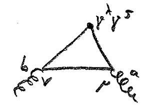
$\rightsquigarrow$ anomaly dictates an additional term (generated by quantum fluct.),

$$\partial_\mu j_5^\mu = 2m\,\bar\psi\, i\gamma^5\psi + \frac{e^2}{(4\pi)^2}\varepsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}$$

- generalize to non-abelian case: $\boxed{\mathcal{L}_4 = \bar\psi\, i\gamma^\mu(\partial_\mu - igA_\mu^a T^a)\psi}$

calculation as before; vertex "μ" gets T^a

vertex "ν" gets T^b



Summing over fermions in the loop gives $\mathrm{Tr}(T^aT^b)$

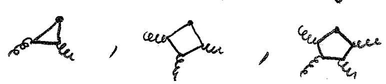
$$\Rightarrow \partial_\mu j_5^\mu = \frac{g^2}{(4\pi)^2}\varepsilon^{\mu\nu\lambda\sigma}\,\mathrm{Tr}\left(F_{\mu\nu}F_{\lambda\sigma}\right)$$

$\uparrow F_{\mu\nu}^a T^a$, see §2.2, pg. 17

$\rightsquigarrow$ since $FF \sim A^2, A^3, A^4$, non-abelian symmetry immediately

tells us there are triangle/square/pentagon anomalies

in QCD:



- higher orders? e.g. 3 loop  etc.

expect correction $\sim * \left[1 + \mathrm{fct}(e, g, \dots)\right]$ $\leftarrow$ all couplings of theory

anomaly nonrenormalization theorem: $\mathrm{fct}(e, g, \dots) = 0$ (!)

for a proof see [Adler/Bardeen, Phys. Rev. 182 (1969) 1517]

[Collins, Renormalization, pg. 352]

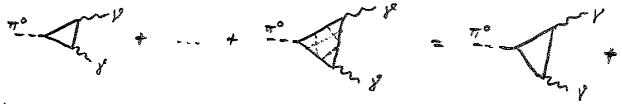
→ we can heuristically <u>understand</u> this:

before integrating over momenta of internal propagators,

the integrand has ≥ 5 fermion propags

⇝ sufficiently <u>convergent</u>, so we can <u>shift</u> momenta naively (cf. p 73)

→ historically, nonrenormalization of the anomaly important for developing concept of <u>color</u>:

$\pi^0 \to \gamma + \gamma \sim$  $+ 0 + \ldots + 0$

process could be computed with confidence from <u>one</u> diagram,

decay amplitude does <u>not</u> depend on details of strong interactions;

result was factor of 3 too small ⇒ 3 types of quarks!

- beyond the Standard Model (BSM) - considerations:

are quarks / leptons composed of more fundamental fermions (preons)?

⇝ nonrenormalization theorem severely constrains possible preon models/theories (as long as they are formulated via QFT as we know it): anomaly at preon level must be the <u>same</u> as at quark/lepton level.

→ <u>anomaly matching conditions</u>

see e.g. ['t Hooft, Recent developments in gauge theories, Plenum Press 1980]

[Zee, Phys. Lett. B 95 (1980) 290]

- a last historic note:

after discovery of chiral anomaly, there were claims that path integral is wrong!

→ is $\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\; e^{i\int d^4x\,\bar{\psi}\, i\gamma^\mu(\partial_\mu - ieA_\mu)\psi}$ unable to tell us

that it is <u>not</u> invariant under chiral trafo $\psi \to e^{i\theta\gamma^5}\psi$?!

→ it <u>does</u> tell us: action invariant, measure changes ("Jacobian")

see [Fujikawa, Phys. Rev. Lett. 42 (1979) 1195]

7. Outlook

⇝ have discussed aspects of a fascinating theory (QCD), with structures like non-Abelian gauge fields, coupling constant renormalization etc.

⇝ have seen in some examples that these abstract mathematical structures actually correspond to experimental observations

⇝ have so far mostly discussed perturbative QCD;

how about non-perturbative phenomena / techniques?

- can one actually <u>solve QCD</u> analytically?

"holy grail" for field theorists!

prize money: \$1 M ; see www.claymath.org/millennium

as a first step: try to solve pure YM (symm's could help!)

((or, even more symmetric, supersymmetric YM (SYM)))

(one) goal: take "idealized" QCD, u+d+gl, all massless

$$\mathcal{L} = \bar{u}\, i\not{D} u + \bar{d}\, i\not{D} d - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

calculate e.g. $\frac{m_{\text{g-Meson}}}{m_{\text{Proton}}} \sim \{\frac{1}{2}, \pi, \ln 2, \zeta(3), \ldots\}$

(($\mathcal{L}$ has dimless parameter g^2; but $g^2(\mu)$ due to renormalization; dimensionful scale Λ_{QCD} where $\frac{g^2(\Lambda_{QCD})}{4\pi} \sim 1$; so each $m \sim \Lambda_{QCD}$))

- note that <u>perturbation theory</u> is somewhat <u>unnatural</u> for solving a highly symmetric gauge theory such as YM:

split $F^a_{\mu\nu} F^{a\mu\nu} \to \underbrace{(\partial A - \partial A)^2}_{\text{gauge invar.}} + \underbrace{A^3 + A^4}_{\text{not g.i.}}$

$= \underbrace{\text{harmonic osc.}}_{\text{not g.i.}} + \text{rest} = \mathcal{L}_0 + \mathcal{L}_{\text{interaction}}$

⇒ clearly not optimal; solvable/integrable systems <u>need</u> symmetry!

- can one work with QCD in the regime where the strong coupling is actually strong ?

  → big open question: <u>confinement</u>

  would like to derive the complete force between quarks from $\mathcal{L}_{QCD}$

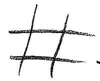
  weak coupling limit: Coulomb potential w/ running coupling ✓✓

  strong coupling limit: linear potential, confines color

  (("string" ?)) derivation needs <u>new tools</u> :

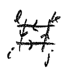
- [Wilson, Phys. Rev. D 10 (1974) 2445] : <u>lattice gauge theory</u>

  violate Lorentz-invariance, not gauge invariance

  formulate theory on 4d Euclidean spacetime <u>lattice</u>  $\}a$,

  perform continuum limit $a \to 0$ in the end, to recover 4d rotational invariance and (after Wick rot.) Lorentz invar.

  fundamental variables: line elements $\overset{U_{ij}}{x_i \longrightarrow x_j}$; U_{ij} = unitary, $N \times N$ matrix

  gauge invariant quantity: <u>plaquette</u>  $tr\,(U_{ij}U_{jk}U_{kl}U_{li})$

  invariant under <u>local</u> trafo: $U_{ij} \to V_i^+ U_{ij} V_j$

  def $\quad Z = \int \mathcal{D}U \; e^{\frac{1}{2g^2}\sum_{plaq} Re\, tr\,(UUUU)} \quad$ (Wilson)

  → is this <u>equivalent to YM</u> ? yes, after $a \to 0$:

  def $U_{ij} = V_i^+ \, e^{ia A_\mu(x)} V_j$, where $x \equiv \frac{x_i + x_j}{2}$, $\mu \equiv i \to j$ direction

  ((along $\hat{\mu} = \frac{x_j - x_i}{a}$))

  then $UUUU = e^{ia^2 F_{\mu\nu} + O(a^3)}$

  and $Re\, tr\,(UUUU) = Re\, tr \{ 1 + ia^2 F_{\mu\nu} - \frac{1}{2} a^4 F_{\mu\nu}F_{\mu\nu} + \dots \}$

  $= tr\,1 - \frac{a^4}{2} tr\, F_{\mu\nu}F_{\mu\nu} + \dots$

  ⇝ beautiful formulation: no gauge fixing, no ghosts

  ⇝ as a challenge, try to <u>incorporate fermions</u> !

  highly nontrivial problem, ongoing research, ...

- for practical purposes, lattice gauge theory allows for <u>numerical computations</u> .

  ⇝ lattice Monte Carlo methods;

  huge world-wide efforts, development of algorithms and computers.

  ⇝ principal theoretical tool for quantitative calculations in hadron physics .

  get e.g. mass spectrum of low-lying mesons + baryons to ~10%

  ((as we have seen in §1.2))

- marriage of relativity + QM ⇒ QFT

  + statistical physics ⇒ <u>thermal QFT</u>

  ⇝ <u>hot QCD</u> is very interesting (phase transitions, ...),

  relevant (early universe, ...) ,

  conceptually clear (hadrons melt → <u>quark-gluon plasma</u>)

  ⇒ can be treated analytically (weak-coupling)

  numerically (lattice Monte Carlo)

  experimentally (heavy-ion collisions)

- as in other QFT's , QCD allows for interesting non-perturbative objects (exact sol'n's of eom; solitons, vortices, monopoles, instantons, ...) and non-perturbative methods (large-N expansion, ...)

- what is next ?

  → master's thesis , ask <u>everyone</u> , get valuable insight into Bielefeld research

  → lectures in WS 11: lattice gauge theory (Karsch)

  supersymmetry (Kögerler)

  electroweak physics (Brauner)