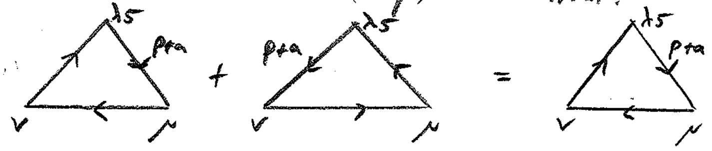
- reason for above result: Δ is linearly divergent!

  $\Rightarrow$ have to make sure (even before calculating $b\Delta$)

  that integral is well-defined ((i.e. its value does not depend on the physicist doing the calculation))

freedom of choice to label internal (loop) momentum:

$\Delta^{\lambda\mu\nu}(a, b_1, b_2) \equiv$  $+ \begin{pmatrix} \mu \leftrightarrow \nu \\ b_1 \leftrightarrow b_2 \end{pmatrix}$

but which a to choose?

only sensible answer: choose a such that $b_{1\mu} \Delta^{\lambda\mu\nu}(a, b_1, b_2) = 0 = b_{2\nu} \Delta^{\lambda\mu\nu}$

compute $\Delta^{\lambda\mu\nu}(a, b_1, b_2) - \Delta^{\lambda\mu\nu}(b_1, b_2)$ with above "careful" way:

use $f(p) = \mathrm{tr}\left(\gamma^\lambda \gamma^5 \frac{1}{\not p - \not b_2} \gamma^\nu \frac{1}{\not p - \not b_1} \gamma^\mu \frac{1}{\not p}\right)$

note $\lim\limits_{P \to \infty} f(P) = \lim\limits_{P \to \infty} \frac{\mathrm{tr}(\gamma^\lambda \gamma^5 \not P \gamma^\nu \not P \gamma^\mu \not P)}{P^6} \overset{\text{(check?!)}}{=} \frac{-4i P^2 P_\sigma \varepsilon^{\sigma\nu\mu\lambda}}{P^6}$

$$\Rightarrow \Delta^{\lambda\mu\nu}(a, b_1, b_2) - \Delta^{\lambda\mu\nu}(b_1, b_2) = \frac{4i}{8\pi^2} \lim_{P \to \infty} \left\langle a^\omega \frac{P_\omega P_\sigma}{P^2} \varepsilon^{\sigma\nu\mu\lambda} + \begin{pmatrix} \mu \leftrightarrow \nu \\ b_1 \leftrightarrow b_2 \end{pmatrix} \right\rangle$$

$$= \frac{i}{8\pi^2} \varepsilon^{\sigma\nu\mu\lambda} a_\sigma + \begin{pmatrix} \mu \leftrightarrow \nu \\ b_1 \leftrightarrow b_2 \end{pmatrix}$$

in general, $a = \alpha(b_1 + b_2) + \beta(b_1 - b_2)$ are all possible shifts

$$\Rightarrow \Delta^{\lambda\mu\nu}(a, b_1, b_2) = \Delta^{\lambda\mu\nu}(b_1, b_2) + \frac{i\beta}{4\pi^2} \varepsilon^{\lambda\mu\nu\sigma} (b_1 - b_2)_\sigma$$

((α dropped out due to antisym of ε))

$$\Rightarrow b_{1\mu} \Delta^{\lambda\mu\nu}(a, b_1, b_2) = \underbrace{b_{1\mu} \Delta^{\lambda\mu\nu}(b_1, b_2)}_{= \frac{i}{8\pi^2} \varepsilon^{\lambda\nu\tau\sigma} b_{1\tau} b_{2\sigma}} + \frac{i\beta}{4\pi^2} \varepsilon^{\lambda\mu\nu\sigma} (b_{1\mu} b_{1\sigma} - b_{1\mu} b_{2\sigma})$$

(see pg. 73)

$$\overset{!}{=} 0 \quad \Rightarrow \quad \beta = -\frac{1}{2}$$

- note: Feynman rules are <u>not</u> sufficient to compute $\langle 0 | T\, J_5^\lambda(0) J^\mu(x_1) J^\nu(x_2) | 0 \rangle$

  have to <u>supplement them</u> by vector current conservation!

  $\rightsquigarrow$ amplitude $\langle 0 | T\, J_5 J J | 0 \rangle$ is <u>defined</u> by $\Delta^{\lambda\mu\nu}(\alpha(b_1 + b_2) - \frac{1}{2}(b_1 - b_2), b_1, b_2)$

6.2 Axial current non-conservation

→ in §6.1, learned how to properly define $\Delta^{\lambda\mu\nu}$;

now, check axial current conservation by computing $q_\lambda \Delta^{\lambda\mu\nu}(\overset{\text{(fixed by above)}}{a}, b_1, b_2)$

$$\bullet \quad q_\lambda \Delta^{\lambda\mu\nu}(\overset{(= -\frac{1}{2}(b_1-b_2))}{a}, b_1, b_2) = \underbrace{q_\lambda \Delta^{\lambda\mu\nu}(b_1,b_2)} \underbrace{- \frac{i}{8\pi^2} \varepsilon^{\lambda\mu\nu\sigma} (b_1+b_2)_\lambda (b_1-b_2)_\sigma}_{= + \frac{i}{4\pi^2} \varepsilon^{\mu\nu\lambda\sigma} b_{1\lambda} b_{2\sigma}}$$

$$= i \int \frac{d^4p}{(2\pi)^4} \operatorname{tr}\left(\underbrace{\not q}_{= \not p - (\not p - \not q)} \gamma^5 \frac{1}{\not p - \not q} \gamma^\nu \frac{1}{\not p - \not b_1} \gamma^\mu \frac{1}{\not p} + \left(\begin{smallmatrix}\mu \leftrightarrow \nu \\ b_1 \leftrightarrow b_2\end{smallmatrix}\right) \right)$$

; use cyclicity of trace; $\{\gamma^\mu, \gamma^5\} = 0$

$$= i \int \frac{d^4p}{(2\pi)^4} \operatorname{tr}\left(\gamma^\mu \gamma^5 \frac{1}{\not p - \not q} \gamma^\nu \frac{1}{\not p - \not b_1} - \gamma^\nu \gamma^5 \frac{1}{\not p - \not b_1} \gamma^\mu \frac{1}{\not p} + \left(\begin{smallmatrix}\mu \leftrightarrow \nu \\ b_1 \leftrightarrow b_2\end{smallmatrix}\right) - \left(\begin{smallmatrix}\mu \leftrightarrow \nu \\ b_1 \leftrightarrow b_2\end{smallmatrix}\right) \right)$$

(pg. 73)

$$= b_{1\sigma} \Delta^{\mu\sigma\nu}(b_1, b_2) + \left(\begin{smallmatrix}\mu \leftrightarrow \nu \\ b_1 \leftrightarrow b_2\end{smallmatrix}\right) = \frac{i}{8\pi^2} \varepsilon^{\mu\nu\tau\sigma} b_{1\tau} b_{2\sigma} \cdot 2$$

$$= \frac{i}{2\pi^2} \varepsilon^{\mu\nu\lambda\sigma} b_{1\lambda} b_{2\sigma}$$

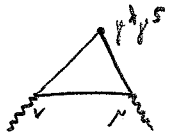
⇒ axial current is <u>not</u> conserved!

this is known as (axial/chiral) anomaly:

quantum fluctuations destroyed the (classical) axial current conservation.

Consequences / remarks (w/o derivations)

• <u>gauge</u> our theory $\mathcal{L}_1$: $\boxed{\mathcal{L}_2 \equiv \bar\psi i\gamma^\mu(\partial_\mu - ie\overset{\text{"photon"}}{A_\mu})\psi}$



$$\rightarrow \quad \partial_\mu j_5^\mu = \begin{cases} 0 & \text{classically} \\ \frac{e^2}{(4\pi)^2} \varepsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma} & \text{quantum} \end{cases}$$

((check ?! : $b \to \partial$ in $F = \partial A - \partial A$))

historically important! decay $\overset{\text{(massless)}}{\pi^0} \to \gamma + \gamma$ <u>forbidden</u> via (wrong) $\partial_\mu j_5^\mu = 0$,

but decay is <u>observed</u> experimentally, as predicted by (correct) $\partial_\mu j_5^\mu \neq 0$.

($\Gamma_i/\Gamma \approx 98.8\,\%$, see PDG

- re-write $\mathcal{L}_2$ with $\psi_{R/L} \equiv \frac{1 \pm \gamma^5}{2} \psi$,

  introduce left- and right-handed currents $J^\mu_{R/L} \equiv \bar\psi_{R/L} \gamma^\mu \psi_{R/L}$

  $$\Rightarrow \partial_\mu J^\mu_{R/L} = \pm \frac{1}{2} \frac{e^2}{(4\pi)^2} \varepsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma}$$

  ((that's why the anomaly is called "chiral"))

- add a fermion mass to $\mathcal{L}_2$: $\boxed{\mathcal{L}_3 \equiv \bar\psi \left[i\gamma^\mu(\partial_\mu - ieA_\mu) - m \right] \psi}$

  $\Rightarrow$ invariance under $\psi \to e^{i\theta\gamma^5}\psi$ broken by $m \neq 0$.

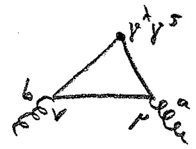
  classically, $\partial_\mu J^\mu_5 = 2m\, \bar\psi i\gamma^5 \psi$, axial current not conserved.

  $\rightsquigarrow$ anomaly dictates an additional term (generated by quantum fluct.),

  $$\partial_\mu J^\mu_5 = 2m\, \bar\psi i \gamma^5 \psi + \frac{e^2}{(4\pi)^2} \varepsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma}$$

- generalize to non-abelian case : $\boxed{\mathcal{L}_4 = \bar\psi\, i\gamma^\mu \left(\partial_\mu - i g A^a_\mu T^a\right) \psi}$

  calculation as before; vertex "μ" gets T^a
  vertex "ν" gets T^b



  summing over fermions in the loop gives $Tr(T^a T^b)$

  $$\Rightarrow \partial_\mu J^\mu_5 = \frac{g^2}{(4\pi)^2} \varepsilon^{\mu\nu\lambda\sigma}\, Tr\big(\underbrace{F_{\mu\nu}}_{F^a_{\mu\nu} T^a} F_{\lambda\sigma} \big)$$

  , see §2.2, pg. 17

  $\rightsquigarrow$ since $FF \sim A^2, A^3, A^4$, non-abelian symmetry immediately tells us there are triangle / square / pentagon anomalies in QCD :

- higher orders? e.g. 3 loop

  etc.

  expect correction $\sim * \left[1 + fct(e, g, \dots) \right]$ $\leftarrow$ all couplings of theory

  anomaly nonrenormalization theorem : $fct(e, g, \dots) = 0$ (!)

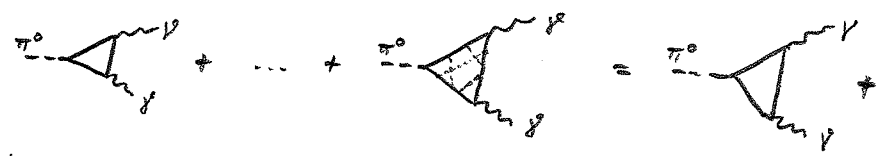
  for a proof see [Adler / Bardeen, Phys. Rev. 182 (1969) 1517]
  [Collins, Renormalization, pg. 352]

→ we can heuristically understand this:

before integrating over momenta of internal propagators, the integrand has $\gtrsim 5$ fermion propags

⇝ sufficiently convergent, so we can shift momenta naively (cf. p. 73)

→ historically, nonrenormalization of the anomaly important for developing concept of color:

$\pi^0 \to \gamma + \gamma \sim$  $+ 0 + .. + 0$

process could be computed with confidence from one diagram, decay amplitude does not depend on details of strong interactions; result was factor of 3 too small ⇒ 3 types of quarks!

- beyond the Standard Model (BSM) - considerations:

  are quarks / leptons composed of more fundamental fermions (preons)?

  ⇝ nonrenormalization theorem severely constrains possible preon models/theories (as long as they are formulated via QFT as we know it): anomaly at preon level must be the same as at quark/lepton level.

  → anomaly matching conditions

  see e.g. ['t Hooft, Recent developments in gauge theories, Plenum Press 1980]
  [Zee, Phys. Lett. B 95 (1980) 290]

- a last historic note:

  after discovery of chiral anomaly, there were claims that path integral is wrong!

  → is $\int \mathcal{D}\bar\psi\, \mathcal{D}\psi\; e^{i\int d^4x\, \bar\psi i\gamma^\mu(\partial_\mu - ieA_\mu)\psi}$ unable to tell us that it is not invariant under chiral trafo $\psi \to e^{i\theta\gamma^5}\psi$?!

  → it does tell us: action invariant, measure changes ("Jacobian")

  see [Fujikawa, Phys. Rev. Lett. 42 (1979) 1195]