4.3.3 Result

let $\sigma_0 \equiv \sigma_{e^+e^- \to q\bar{q}}$ (tree level) denote the leading order (LO) result
(see pg. 45; d-dim result on pg 51)

then, in dimensional regularization, real and virtual QCD-corrections are

LO (pg. 51) $$\sigma_{e^+e^- \to q\bar{q}g} = \sigma_0 \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{s}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} - \pi^2 + O(\varepsilon) \right) + O(\alpha_s^2)$$

NLO (pg. 54) $$\sigma_{e^+e^- \to q\bar{q}} = \sigma_0 \left\{ 1 + \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{s}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \pi^2 + O(\varepsilon) \right) + O(\alpha_s^2) \right\}$$

the sum, needed for the hadronic cross section, is finite;
can take $\varepsilon \to 0$

NLO $$\Rightarrow \sigma_{e^+e^- \to \text{hadrons}} = \sigma_0 \left\{ 1 + \frac{\alpha_s C_F}{2\pi} \frac{3}{2} + O(\alpha_s^2) \right\} \overset{N_c=3}{=} \sigma_0 \left\{ 1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2) \right\}$$

in QCD, $N_c = 3$, $C_F = \frac{N_c^2-1}{2N} = \frac{4}{3}$

Before using this results, a couple of remarks:

- cancellation of soft and collinear divergences between the real and virtual gluon diagrams is not accidental.
  They are in fact guaranteed by theorems (Bloch/Nordsieck, Kinoshita/Lee/Nauenberg (KLN)): suitably defined inclusive quantities will be IR safe in the massless limit.
  (($\sigma_{e^+e^- \to \text{hadrons}}$ is such a quantity; $\sigma_{e^+e^- \to q\bar{q}}$ is not))
  → proof in QCD: see e.g. [Collins/Soper, Ann Rev Nucl Sci 37(1987) 383]
- our result would be worthless if it depended on our choice of regularization procedure, dim. reg.
  Proof of independence is beyond this lecture; but demonstrate it by comparing with gluon mass regularization scheme (m_g = gluon mass)

$$\sigma_{e^+e^- \to q\bar{q}g} = \sigma_0 \frac{\alpha_s C_F}{2\pi} \left(\ln^2 \frac{s}{m_g^2} - 3 \ln \frac{s}{m_g^2} + 7 - \frac{\pi^2}{3} + O(\varepsilon) \right)$$

$$\sigma_{e^+e^- \to q\bar{q}} = \sigma_0 \left\{ 1 + \frac{\alpha_s C_F}{2\pi} \left(-\ln^2 \frac{s}{m_g^2} + 3 \ln \frac{s}{m_g^2} - \frac{11}{2} + \frac{\pi^2}{3} + O(\varepsilon) \right) \right.$$

$$\Rightarrow \sigma_{e^+e^- \to \text{hadrons}} = \sigma_0 \left\{ 1 + \frac{\alpha_s C_F}{2\pi} \frac{3}{2} + O(\alpha_s^2) \right\}$$

→ individual cross sections completely different; sum scheme independent!

- at this order (NLO), and for $m_q = 0$, our computation of the QCD correction is independent of the nature of the exchanged weak boson (we took the photon only, cf pg. 48).

→ generalization: the $(1 + \frac{\alpha_s}{\pi})$ result is valid also for $R_{Z\,peak}$, see pg. 47

ratios: $R(s) = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c\left(\sum_f Q_f^2\right)\left(1 + \frac{\alpha_s C_F}{2\pi}\frac{3}{2} + O(\alpha_s^2)\right)$

$\quad s \ll m_Z^2$; otherwise see § 4.2

$$R_{Z\,peak} = \left.\frac{\sigma(e^+e^- \to Z \to hadrons)}{\sigma(e^+e^- \to Z \to \mu^+\mu^-)}\right|_{s=m_Z^2} = 19.984\left(1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2)\right)$$

note that NLO correction is positive.

α_s :
- comparing with the (correctly scaled) experimental result

$R_{Z\,peak}(LEP) = 20.767 \pm 0.025$ (see pg. 47)

⇒ our first measurement of α_s : $\alpha_s(m_Z) = 0.123 \pm 0.004$

((compare with "world average" from PDG (2010): 0.1184 ± 0.0007))

- as another determination of α_s, let us compare to data taken by PETRA (DESY), at $\sqrt{s} \approx 34$ GeV

$R(s \approx (34\,GeV)^2, PETRA) = 3.88 \pm 0.03$

we would predict (udscb – t too heavy)

$\quad$ (b: 4.5 GeV, t: 172 GeV)

$$R((34 GeV)^2) = \underbrace{3\left(2\left(\tfrac{2}{3}\right)^2 + 3\left(-\tfrac{1}{3}\right)^2\right)}_{=\frac{11}{3} \approx 3.667}\left(1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2)\right)$$

or, including Z, $R((34 GeV)^2) = 3.69\left(1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2)\right)$

⇒ our second measurement of α_s : $\alpha_s(34 GeV) = 0.162 \pm 0.026$

- recall (§ 3.4, pg. 42) that α_s is running!

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln(\mu^2/\mu_0^2)}, \quad \text{with } \beta_0 = \frac{11}{3}N_c - \frac{2}{3}N_f \overset{N_c=3}{=} 11 - \frac{2}{3}N_f$$

$$\Leftrightarrow \frac{1}{\alpha_s(\mu_1)} - \frac{1}{\alpha_s(\mu_2)} = \frac{\beta_0}{4\pi}\ln\left(\frac{\mu_1^2}{\mu_2^2}\right)$$

⇒ 2^{nd} measurement translates into $\alpha_s(m_Z = 91.2\,GeV) = 0.135 \pm 0.018$

→ so far, our NLO correction to R shows correct qualitative features.

→ but what about higher orders ?

4.3.4 Higher-order QCD corrections to R(s)

write $R(s) = 3\left(\sum_f Q_f^2\right) \cdot K_{QCD}$

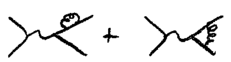
where $K_{QCD} = 1 + 1 \cdot \frac{\alpha_s(\mu)}{\pi} + \sum_{n \geq 2} C_n\left(\frac{s}{\mu^2}\right) \cdot \left(\frac{\alpha_s(\mu)}{\pi}\right)^n$

↑ result of our computation;
from { tree-level $q\bar{q}g$, one-loop $q\bar{q}$ } final state

the functions $C_n\left(\frac{s}{\mu^2}\right)$ follow from higher-order computations:

e.g. C_2 from { tree-level $q\bar{q}gg$, $q\bar{q}q\bar{q}$; one-loop $q\bar{q}g$; two-loop $q\bar{q}$ } final states

etc ...

→ note that in our computation, there were no UV divergences ((in fact, those in  cancel exactly)), so we did not need to renormalize, hence our coefficient did not depend on the renormalization scale μ: $C_1\left(\frac{s}{\mu^2}\right) = 1$.

→ in higher orders, we will encounter UV divs, hence $C_{n \geq 2}$ are renormalization scheme dependent.

If we could sum the whole series, it would be μ-indep.

In a truncated series, μ-dependence is of higher order.

⇒ μ-dependence of $C_n\left(\frac{s}{\mu^2}\right)$ is fixed by knowing μ-dependence of $\alpha_s(\mu)$ (pg 38: $\mu^2 \partial_{\mu^2} \alpha_s = -\frac{\beta_0}{4\pi}\alpha_s^2 - \frac{\beta_1}{(4\pi)^2}\alpha_s^3 - \dots$)

$$\Rightarrow C_2\left(\frac{s}{\mu^2}\right) = C_2(1) + C_1(1) \boxed{\frac{\beta_0}{4} \ln\left(\frac{\mu^2}{s}\right)} \equiv L$$

$$C_3\left(\frac{s}{\mu^2}\right) = C_3(1) + C_1(1) L^2 + \left[C_1(1)\frac{\beta_1}{4\beta_0} + 2C_2(1)\right] L$$

etc. (check ?!)

(($\beta_1 = \frac{2}{3}\left(17 N_c^2 - 5 N_c N_f - 3 C_F N_f\right) \overset{N_c = 3}{=} 102 - \frac{38}{3} N_f$))

- C_2 and C_3 have been computed [Samuel/Surguladze, PRL 66 (1991) 560] [Gorishny/Kataev/Larin, PL B 259 (1991) 144]

(here $N_c = 3$, ren. scale set to $\mu = \sqrt{s}$, $\overline{MS}$ scheme)

$$C_2(1) = \left(\frac{365}{24} - 11\zeta(3)\right) + \left(-\frac{11}{12} + \frac{2}{3}\zeta(3)\right)N_f$$
$$\approx 1.986 - 0.115\, N_f$$

$$C_3(1) = \left(\frac{87029}{288} - \frac{1103}{4}\zeta(3) + \frac{275}{6}\zeta(5)\right) + \left(-\frac{7847}{216} + \frac{262}{9}\zeta(3) - \frac{25}{9}\zeta(5)\right)N_f$$
$$+ \left(\frac{151}{162} - \frac{19}{27}\zeta(3)\right)N_f^2 - \frac{\pi^2}{432}(33-2N_f)^2 + \eta\left(\frac{55}{72} - \frac{5}{3}\zeta(3)\right)$$
$$\approx -6.637 - 1.200\, N_f - 0.005\, N_f^2 - 1.240\, \eta$$

where $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ is the Riemann Zeta function $\quad \zeta(3) \approx 1.202057$, $\zeta(5) \approx 1.036928$

and $\eta = \dfrac{\left(\sum_q Q_q\right)^2}{3\left(\sum_q Q_q^2\right)}$, $\sum_q$ over all quarks with $m_q \ll \sqrt{s}$ (effectively massless)

((for $R_{Z\,peak}$, QCD corrections are again the same, except that $\eta \to \dfrac{\left(\sum_q V_q\right)^2}{3\sum_q (V_q^2 + A_q^2)}$))

- having a few orders of the perturbative series, can now discuss convergence
  - → coefficients are scheme-dependent, so can try to find an "optimal" scheme (from the point of view of convergence), examples are:
    - FAC (fastest apparent convergence): choose scale $\mu = \mu_{FAC}$ such that $R^{(1)}(\mu_{FAC}) = R^{(n)}(\mu_{FAC})$
    - PMS (principle of minimal sensitivity): choose $\mu = \mu_{PMS}$ such that $\mu\partial_\mu R^{(n)}(\mu)\Big|_{\mu_{PMS}} = 0$ [Stevenson, PLB 100 (1981) 61]
    - BLM ([Brodsky/Lepage/Mackenzie, PR D28 (1983) 228]): absorb all N_f-terms into α_s via β fct
    - etc ...

  for $R^{(3)}(s)$, these are $\{\mu_{FAC}, \mu_{PMS}, \mu_{BLM}\} \approx \{0.692, 0.587, 0.708\}\sqrt{s}$ ($N_f = 5$ massless flavors)

  → μ-variation does get weaker, comparing $R^{(1)}(s)$, $R^{(2)}(s)$, $R^{(3)}(s)$