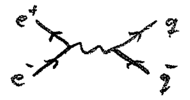
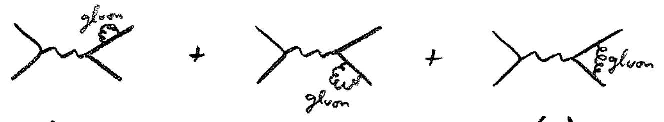
4.3.2 virtual corrections: $\sigma_{e^+e^-\to q\bar q}$ at $\mathcal{O}(\alpha_s)$

structure of cross section computation:

$$|\mathcal{M}_0 + \alpha_s \mathcal{M}_1 + \mathcal{O}(\alpha_s^2)|^2 = |\mathcal{M}_0|^2 + \alpha_s(\mathcal{M}_1\mathcal{M}_0^* + \mathcal{M}_0\mathcal{M}_1^*) + \mathcal{O}(\alpha_s^2)$$

→ need to compute interference term $\mathcal{M}_1\mathcal{M}_0$ only.

recall $\mathcal{M}_0 =$ 

$\alpha_s\mathcal{M}_1 =$ 

- in dimensional regularization,  $\sim \int d^dk \, \frac{1}{k^2}\frac{\not k+\not p}{(k+p)^2} = \not p \, I(p^2)$

  with $\dim[I(p^2)] = m^{d-4}$

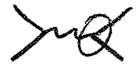
  but $p^2=0$ (due to onshell condition), so no scale ⇒ <u>zero</u>

  $\Rightarrow \mathcal{M}_{1(a)} = 0 = \mathcal{M}_{1(b)}$ in dim. reg.

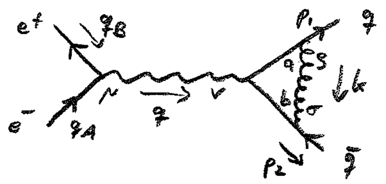
- for diagram (c), need to do some computation

$$\sigma_{e^+e^-\to q\bar q} \overset{\text{pg. 49}}{=} \frac{1}{2s}\left(\prod_{i=1}^{2}\int\frac{d^{d-1}p_i}{(2\pi)^{d-1}2E_{p_i}}\right)(2\pi)^d\,\delta^{(d)}(p_1+p_2-q_A-q_B)\,\langle|\mathcal{M}|^2\rangle$$

structure of $\alpha\mathcal{M}_{1(c)}$ is similar to $\mathcal{M}_0$!

recall (pg 44) $\mathcal{M}_0 =$  $= \frac{e^2Q_q}{q^2}\,\bar v_B\gamma^\mu u_A\,\bar u_1\gamma_\mu v_2$

$$\langle|\mathcal{M}_0|^2\rangle \overset{\text{do spin sums}}{=} \frac{e^4Q_q^2}{4q^4}\,\mathrm{tr}(\not q_B\gamma^\mu\not q_A\gamma^{\nu})^{(*)}\,\mathrm{tr}(\not p_1\gamma_\mu\not p_2\gamma_\nu)^{(*)}$$

now, $\alpha_s\mathcal{M}_1 =$ 

use Feynman gauge $\xi=1$, massless quarks $m_q=0$

$$= \bar v_B(-ie\gamma^\mu)u_A\left(-\frac{ig_{\mu\nu}}{q^2}\right)\bar u_1(ig_s\gamma^\rho T^a)\int\frac{d^dk}{(2\pi)^d}\,\frac{i(\not k+\not p_1)}{(k+p_1)^2}(-ieQ_q\gamma^\nu)\frac{i(\not k-\not p_2)}{(k-p_2)^2} \,*$$

$$*\,(ig_s\gamma^\sigma T^b)\,v_2\left(\frac{-i\delta^{ab}g_{\rho\sigma}}{k^2}\right)$$

Fierz, pg.9

use $T^aT^a = T^a_{ij}T^a_{jk} \overset{\downarrow}{=} \frac{1}{2}\left(\delta_{ik}\delta_{jj} - \frac{1}{N}\delta_{ij}\delta_{jk}\right) = \frac{\delta_{ik}}{2}\left(N - \frac{1}{N}\right) = \mathbb{1}\, C_F$

$$= \frac{e^2 Q_q}{q^2}\,\bar v_B \gamma^\mu u_A\, \bar u_1 \underbrace{g_s^2 C_F \mathbb{1} \int \frac{d^dk}{(2\pi)^d}\, \gamma^\rho \frac{\not k + \not p_1}{(k+p_1)^2}\gamma_\mu \frac{\not k - \not p_2}{(k-p_2)^2}\gamma_\rho \frac{1}{k^2}}_{\equiv \Lambda_\mu} v_2$$

$$\Rightarrow \left\langle \alpha_s\left(M_1 M_0^\dagger + M_0 M_1^\dagger\right)\right\rangle = \frac{e^4 Q_q^2}{4q^4}\, tr\left(\not p_B \gamma^\mu \not p_A \gamma^\nu\right) tr\left(\not p_1 \Lambda_\mu \not p_2 \gamma_\nu\right) + c.c.$$

- note that IF the contribution of Λ_μ inside the 2nd trace can be reduced to $\Lambda_\mu = \gamma_\mu \cdot \alpha_s C_F \frac{\Lambda(q^2)}{2}$

  THEN $\left\langle \alpha_s\left(M_1 M_0^\dagger + M_0 M_1^\dagger\right)\right\rangle = \langle |M_0|^2\rangle \cdot \alpha_s C_F\, Re\left[\Lambda(q^2)\right]$

  and σ follows without additional work!

→ the idea is to profit from the onshell conditions, such that if there is a term, say, $\not p_1 \gamma_\mu \not p_1$ in Λ_μ it gets killed by $\not p_1 \Lambda_\mu = \not p_1 \not p_1 \gamma_\mu \not p_1 = \frac{1}{2}\{\gamma_\rho, \gamma_\nu\} p_1^\rho p_1^\nu \gamma_\mu \not p_1 = p_1^2 \gamma_\mu \not p_1 = 0$

- $$\Lambda_\mu = 4\pi\mu^{2\varepsilon}\alpha_s C_F \underbrace{\gamma^\rho{}_{\sigma\mu\nu\rho}} \underbrace{\int \frac{d^dk}{(2\pi)^d} \frac{(k+p_1)^\sigma (k-p_2)^\nu}{(k+p_1)^2 (k-p_2)^2 k^2}}_{\equiv I^{\sigma\nu}}$$

pg.35

$\overset{\text{pg.35}}{=} -2\gamma_{\nu\mu\sigma} + (4-d)\gamma_{\sigma\mu\nu}$ ①

or $= -4\left(\gamma_\nu g_{\mu\sigma} - \gamma_\mu g_{\nu\sigma} + \gamma_\sigma g_{\nu\mu}\right) + (6-d)\gamma_{\sigma\mu\nu}$ ②

or $= 2(4-d)\left(\gamma_\sigma g_{\mu\nu} - \gamma_\mu g_{\nu\sigma} + \gamma_\nu g_{\sigma\mu}\right) - (6-d)\gamma_{\nu\mu\sigma}$ ③

- solve the loop integral $I^{\sigma\nu}$ via Feynman parameters (see pg.30)

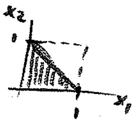
$(q^2 = (p_1+p_2)^2 = 2p_1p_2)$

$$I^{\sigma\nu} = \Gamma(3)\int_0^1 dx_1 \ldots dx_3\, \delta(1-x_1-x_2-x_3) \int \frac{d^dk}{(2\pi)^d} \frac{(k+p_1)^\sigma (k-p_2)^\nu}{\left[x_1(k^2 + 2kp_1 + \overset{0}{p_1^2}) + x_2(k^2 - 2kp_2 + \overset{0}{p_2^2}) + x_3 k^2\right]^3}$$

denominator $[..] = \left[(k + x_1p_1 - x_2p_2)^2 + x_1x_2 q^2\right]$ (used δ fct for k^2-prefactor)

shift $k \to k - x_1p_1 + x_2p_2$, linear terms in k integrate to zero

$$= \Gamma(3)\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^dk}{(2\pi)^d} \frac{k^\sigma k^\nu - \left[(1-x_1)p_1 + x_2p_2\right]^\sigma \left[x_1p_1 + (1-x_2)p_2\right]^\nu}{(k^2 + x_1x_2 q^2)^3}$$



momentum integrals are standard (see pg 31),
after $k^\sigma k^\nu \to \frac{g^{\sigma\nu} k^2}{d}$ and Wick rotation : $I_3^1(-x_1x_2q^2)$, $I_3^0(-x_1x_2q^2)$

$$= \frac{1}{(4\pi)^{d/2}} \frac{1}{(-q^2)^{2-d/2}} \left\{ \frac{g^{\sigma\nu}}{d} \frac{d}{2} \Gamma(2-\tfrac{d}{2}) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{(x_1x_2)^{2-d/2}} + \Gamma(3-\tfrac{d}{2}) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{[\,]^\sigma [\,]^\nu}{(x_1x_2)^{3-d/2}} \frac{1}{(-q^2)} \right\}$$

perform the integrals over Fy parameters, e.g. via Mathematica, for Re(d) > 4

$$= \frac{1}{(4\pi)^{d/2}} \frac{1}{(-q^2)^{2-d/2}} \left\{ g^{\sigma\nu} \frac{1}{2} \Gamma(2-\tfrac{d}{2}) \frac{\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}{(\frac{d}{2}-1)\,\Gamma(d-1)} + \Gamma(3-\tfrac{d}{2}) \frac{\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}{(\frac{d}{2}-2)\,\Gamma(d-1)} \left[p_1^\sigma p_1^\nu + p_2^\sigma p_2^\nu + \frac{8-4d+d^2}{(d-4)(d-2)} p_1^\sigma p_2^\nu + \frac{d-4}{d-2} p_2^\sigma p_1^\nu \right] \frac{1}{(-q^2)} \right\}$$

$I^{\sigma\nu}$; coeffs I_i from above

- $$\Lambda_\mu = (4\pi\mu^{2\varepsilon}\alpha_s C_F\, \underline{\gamma^\rho \sigma_{\mu\nu\rho}} \left\{ \underbrace{I_1 g^{\sigma\nu}}_{①} + \underbrace{I_2 p_1^\sigma p_1^\nu}_{①\,\to 0} + \underbrace{I_3 p_2^\sigma p_2^\nu}_{①\,\to 0} + \underbrace{I_4 p_1^\sigma p_2^\nu}_{②\,\to 4\gamma_\mu p_1 p_2} + \underbrace{I_5 p_2^\sigma p_1^\nu}_{③\,\to -2(4-d)\gamma_\mu p_2\cdot p_1} \right\}$$

$$① \to (2-d)\gamma^\nu{}_{\mu\nu} \overset{\text{pg } 34}{=} (2-d)^2 \gamma_\mu$$

have now specified terms that survive in product $\not{p}_1 \Lambda_\mu \not{p}_2$
(in particular, $0 = \{ \not{p}_1\gamma_\mu\not{p}_1, \not{p}_2\gamma_\mu\not{p}_2, \not{p}_1\gamma_\mu\not{p}_2, \not{p}_2 p_{1\mu}, \not{p}_1 p_{2\mu} \}$)

$$= (4\pi\mu^{2\varepsilon}\alpha_s C_F \frac{1}{(4\pi)^{d/2}} \frac{1}{(-q^2)^{2-d/2}} \left\{ \frac{1}{2}\Gamma(2-\tfrac{d}{2}) \frac{\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}{(\frac{d}{2}-1)\,\Gamma(d-1)} (2-d)^2 \gamma_\mu + \right.$$
$$\left. + \Gamma(3-\tfrac{d}{2}) \frac{\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}{(\frac{d}{2}-2)\,\Gamma(d-1)} \left[0 + 0 + \frac{8-4d+d^2}{(d-4)(d-2)} 2\gamma_\mu \boxed{\frac{2p_1\cdot p_2}{-q^2}}^{-1} + \frac{d-4}{d-2} (d-4)\gamma_\mu \boxed{\frac{2p_1\cdot p_2}{-q^2}}^{-1} \right] \right\}$$

$$= \frac{1}{2}\gamma_\mu \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{-q^2}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \left\{ \left(\frac{1}{\varepsilon} + 1 + O(\varepsilon)\right) + 0 + 0 + \left(-\frac{2}{\varepsilon^2} - \frac{4}{\varepsilon} - 9 + O(\varepsilon)\right) + O(\varepsilon) \right\}$$

$d = 4-2\varepsilon$

$$= \frac{1}{2}\gamma_\mu \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{s}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \pi^2 + O(\varepsilon) + i\pi\left(-\frac{2}{\varepsilon} - 3 + O(\varepsilon)\right) \right)$$

from $(-1)^\varepsilon$; but irrelevant

($(-)^\varepsilon = e^{i\pi\varepsilon}$, $Re[(-1)^\varepsilon] = \cos(\pi\varepsilon)$)

- collecting , we finally have

$$\sigma_{e^+e^-\to q\bar{q}} = \sigma^{(\text{tree})}_{e^+e^-\to q\bar{q}} \cdot \left\{ 1 + \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{s}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \pi^2 + O(\varepsilon) \right) + O(\alpha_s^2) \right\}$$