→ using $\bar v_B \equiv \bar v(q_B, s_B)$ etc for brevity, this reads

$$\mathcal{M} = \frac{-ie^2 Q_q g_s}{q^2}\, \varepsilon_\lambda^{*a} T^a\, \bar v_B \gamma_\mu u_A \bar u_1 \underbrace{\left\{\gamma^\lambda \frac{\not p_1+\not p_3}{(p_1+p_3)^2}\gamma^\mu + \gamma^\mu \frac{-\not p_2-\not p_3}{(p_2+p_3)^2}\gamma^\lambda\right\}}_{\equiv S^{\lambda\mu}} v_2$$

→ for cross section, need $|\mathcal{M}|^2$

as in §4.1, unpolarized e^+e^- beams ⇒ average over incoming spins

spin-blind detector ⇒ sum over outgoing spins

$$\langle|\mathcal{M}|^2\rangle = \sum_{pol}\frac{1}{4}\sum_{s}\mathcal{M}\mathcal{M}^\dagger$$

$$= \left(\frac{e^2Q_q g_s}{q^2}\right)^2 \underbrace{\sum_{pol}\varepsilon_\lambda^{*a}\varepsilon_\sigma^{b}}_{=-g_{\lambda\sigma}\delta^{ab}} \underbrace{\left(T^a_{ij}\right)\left(T^b_{ij}\right)^\dagger}_{=tr(T^aT^b)=\frac{1}{2}\delta^{ab}} \frac{1}{4}\sum_{s_{AB12}} \bar v_B\gamma_\mu \underbrace{u_A\bar u_A}_{\sum_{s_A}=\not q_A}\gamma_\nu^\dagger v_B\; \bar u_1 S^{\lambda\mu}\underbrace{v_2\bar v_2}_{\sum_{s_2}=\not p_2}\left(S^{\sigma\nu}\right)^\dagger u_1$$

(completeness rel., Fey gauge) (normalization) etc

perform spin-sums via completeness rels, see pg. 44

(T hermitean, γ hermitean)

$$= -\frac{e^4Q_q^2g_s^2}{4q^4}\,\frac{N_c^2-1}{2}\,\underbrace{tr\left(\not q_B\gamma_\mu\not q_A\gamma_\nu\right)}_{\equiv L_{\mu\nu}}\,\underbrace{tr\left(\not p_1 S^{\lambda\mu}\not p_2 S^{\nu}{}_{\lambda}\right)}_{\equiv G^{\mu\nu}}$$

$$L_{\mu\nu} = q_B^\sigma q_A^\rho\, 4\left(g_{\sigma\mu}g_{\rho\nu} - g_{\sigma\rho}g_{\mu\nu} + g_{\sigma\nu}g_{\mu\rho}\right) \quad \text{(see pg. 30)}$$

$$= 4\left(q_{B\mu}q_{A\nu} + q_{A\mu}q_{B\nu} - q_A\cdot q_B\, g_{\mu\nu}\right)$$

→ now, total cross section is (pg 44, $m_i=0$), using dimensional regularization,

$$\sigma_{e^+e^-\to q\bar q g} = \frac{1}{2s}\left(\prod_{i=1}^{3}\int\frac{d^{d-1}p_i}{(2\pi)^{d-1}\,2E_{p_i}}\right)(2\pi)^d\,\delta^{(d)}(p_1+p_2+p_3-q_A-q_B)\,\langle|\mathcal{M}|^2\rangle$$

$$= -\frac{e^4g_s^2\left(\sum_q Q_q^2\right)C_F N_c}{8s\,(q^2)^2\,(2\pi)^{2d-3}}\,L_{\mu\nu}\,\underbrace{\left(\prod_{i=1}^{3}\int\frac{d^{d-1}p_i}{2|\vec p_i|}\right)\delta^{(d)}(p_1+p_2+p_3-q)\,G^{\mu\nu}}_{\equiv I^{\mu\nu}(q)}$$

((used $C_F \equiv \frac{N_c^2-1}{2N_c}$))

→ note that (check?!) $q_\mu I^{\mu\nu}(q) = 0$

$$\Rightarrow \; I^{\mu\nu}(q) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) I(q^2) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) \frac{g_{\sigma\rho} I^{\sigma\rho}(q)}{d-1}$$

follows from contracting the eq. with $g_{\mu\nu}$

$$\Rightarrow \; L_{\mu\nu} I^{\mu\nu}(q) = 4\left[q_A\cdot q_B(1-d+1) - \frac{1}{q^2}\left(2\, q\cdot q_B \; q\cdot q_A - q_A\cdot q_B \; q^2\right)\right] \frac{g_{\sigma\rho} I^{\sigma\rho}(q)}{d-1}$$

use $q_A\cdot q_B = \frac{1}{2}\big(\underbrace{q_A^2}_{=0} + \underbrace{q_B^2}_{=0}\big) + q_A\cdot q_B = \frac{1}{2}(q_A+q_B)^2 = \frac{q^2}{2} = \frac{s}{2}$

(onshell)

$$= 2q^2(2-d)\,\frac{g_{\mu\nu} I^{\mu\nu}(q)}{d-1}$$

→ in $\sigma_{ee\to q\bar q g}$, need now

$$g_{\mu\nu} G^{\mu\nu} = \mathrm{tr}\left(\not p_1 S^{\lambda\mu} \not p_2 S_{\mu\lambda}\right) = \text{(check?!)} = 4(d-2)\,\frac{x_1^2 + x_2^2 + \frac{d-4}{2} x_3^2}{(1-x_1)(1-x_2)}$$

where $x_i \equiv \dfrac{2 p_i\cdot q}{q^2}$

→ plug these elements into cross section:

$$\sigma_{e^+e^-\to q\bar q g} = + \frac{e^4 g_s^2 \left(\sum_q Q_q^2\right) C_F N_c}{s^2 (2\pi)^{2d-3}} \; \frac{(d-2)^2}{d-1} \; \underbrace{\left(\prod_{i=1}^{3} \int \frac{d^{d-1} p_i}{2|\vec p_i|}\right) \delta^{(d)}(p_{123}-q)} \; \frac{x_1^2 + x_2^2 + \frac{d-4}{2} x_3^2}{(1-x_1)(1-x_2)}$$

rewrite phase space integral in terms of x_i [see eg. Muta §5.1]

$$= \frac{\pi(\pi s)^{d-3}}{4\Gamma(d-2)} \left(\prod_{i=1}^{3} \int_0^1 \frac{dx_i}{(1-x_i)^{\frac{4-d}{2}}}\right) \delta(2-x_1-x_2-x_3)$$

$d = 4-2\varepsilon$

$(g_s^2 = 4\pi \mu^{2\varepsilon} \alpha_s)$

$$= \alpha^2 \alpha_s \left(\sum_q Q_q^2\right) C_F N_c \; \frac{2}{s} \left(\frac{4\pi\mu}{s}\right)^{2\varepsilon} \frac{(1-\varepsilon)^2}{(3-2\varepsilon)\,\Gamma(2-2\varepsilon)} \cdot I$$

where

$$I = \left(\prod_{i=1}^{3} \int_0^1 \frac{dx_i}{(1-x_i)^{\varepsilon}}\right) \delta(2-x_1-x_2-x_3) \; \frac{x_1^2 + x_2^2 - \varepsilon x_3^2}{(1-x_1)(1-x_2)}$$

$$= \int_0^1 dx_1 \int_{1-x_1}^{1} dx_2 \; \frac{x_1^2 + x_2^2 - \varepsilon(2-x_1-x_2)^2}{(1-x_1)^{1+\varepsilon}(1-x_2)^{1+\varepsilon}(x_1+x_2-1)^{\varepsilon}}$$

eg Mathematica

$$= \frac{2(2-6\varepsilon+5\varepsilon^2-2\varepsilon^3)\,\Gamma^3(1-\varepsilon)}{\varepsilon^2\, \Gamma(3-3\varepsilon)} \approx \frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \left(\frac{19}{2} - \pi^2\right) + O(\varepsilon)$$

→ compare result with $\sigma_{e^+e^-\to q\bar{q}}$ in <u>same</u> regularization:

repeat calculation of $\Big|$  $\Big|^2$ (see § 4.1) in d-dimensions,

$$\sigma_{e^+e^-\to q\bar{q}} \overset{d=4-2\varepsilon}{=} \frac{4\pi\alpha^2}{3s}\Big(\sum_q Q_q^2\Big) N_c \left(\frac{4\pi}{s}\right)^{\varepsilon} \frac{3(1-\varepsilon)\,\Gamma(2-\varepsilon)}{(3-2\varepsilon)\,\Gamma(2-2\varepsilon)}$$

$$\Rightarrow \sigma_{e^+e^-\to q\bar{q}g} = \sigma_{e^+e^-\to q\bar{q}} \cdot \frac{\alpha_s C_F}{2\pi}\left(\frac{4\pi\mu^2}{s}\right)^{\varepsilon}\frac{1}{\Gamma(1-\varepsilon)}\, I$$

- note that I is <u>divergent</u> for $\varepsilon \to 0$.

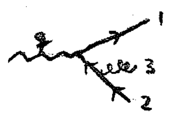
  not a desaster (see pg 48): $\lim\limits_{\varepsilon\to 0}$ (real + virtual corr.) should exist.

  ↳ compute next.

- physical <u>origin</u> of divergences:

  in a 'naive' calculation ($d=4$), $I = \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \,\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$

  ⇝ divergences come from $x_1 = 1$ and $x_2 = 1$



  but $1-x_1 = 1 - \frac{2p_1\cdot q}{q^2} = \frac{(q-p_1)^2 - p_1^2}{q^2} = \frac{(p_2+p_3)^2 - p_1^2}{q^2} = \frac{2p_2\cdot p_3 + p_2^2 + p_3^2 - p_1^2}{q^2}$

  onshell: $p_i^2 = 0 \Rightarrow p_i = E_i(1, \vec{e}_i)$

  $= \frac{2E_2E_3(1-\vec{e}_2\vec{e}_3)}{q^2}$

  $\to$ ~~when~~ $\updownarrow \theta_{13}\to 0$

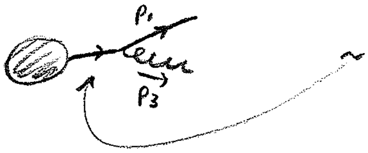
  ⇝ divergences originate from $\begin{cases}\vec{e}_1\cdot\vec{e}_3 \to 1 & : q, g \\ \vec{e}_2\cdot\vec{e}_3 \to 1 & : \bar{q}, g\end{cases}$ <u>collinear</u>

  and from $E_3 \to 0$: g <u>soft</u>

  } both are referred to as <u>infrared</u> singularities.

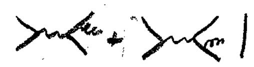
another view of the same fact:

divergences originate from diverging propagators



$$\sim \frac{1}{(p_1+p_3)^2} = \frac{1}{2p_1\cdot p_3} = \frac{1}{2E_1E_3(1-\vec{e}_1\vec{e}_3)} = \frac{1}{2E_1E_3(1-\cos\theta_{13})}$$

collinear limit, $\theta\to 0$: $\simeq \frac{1}{2E_1E_3\theta^2} \Rightarrow |\mathcal{M}|^2 \sim \frac{\theta^2}{\theta^4}$ ← from numerators

soft limit, $E_3 \to 0$: interference term in $\Big|$  $\Big|^2$

gives $|\mathcal{M}|^2 \sim \frac{p_1\cdot p_2}{p_1\cdot p_3\; p_2\cdot p_3} \approx \frac{1}{E_3^2}$

in phase space integral, $\frac{d^3p_3}{2E_3} = \frac{1}{2E_3}E_3^2 dE_3 \sin\theta\, d\theta\, d\varphi \sim E_3\, dE_3\; \theta\, d\theta$

$\Rightarrow$ logarithmic singularities in both limits.