- summing up all four diagrams (see pg. 31, 33)

$$= ig^2\,\delta^{a_1a_2}\left(g^{\mu_1\mu_2}q^2 - q^{\mu_1}q^{\mu_2}\right)\int_0^1 dx\;*$$

$$*\left\{-4x(1-x)\sum_f I_2^0\left(m_f^2 - x(1-x)q^2\right) + N_c\left[\left(1-\tfrac{d}{2}\right)(1-2x)^2+2\right] I_2^0\left(-x(1-x)q^2\right)\right\}$$

now, use (pg.31) $I_2^0(\Delta) = \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\frac{1}{\Delta^{2-\frac{d}{2}}} = \frac{1}{2-\frac{d}{2}}\frac{\Gamma(3-\frac{d}{2})}{(4\pi)^{d/2}}\frac{1}{\Delta^{2-d/2}}$

$$\overset{d=4-2\varepsilon}{\approx} \frac{1}{\varepsilon}\frac{1}{(4\pi)^2} + O(\varepsilon^0)$$

$$\Rightarrow \{\dots\} \approx \frac{1}{\varepsilon}\frac{1}{(4\pi)^2}\left\{-4x(1-x)N_f + N_c\left[-(1-2x)^2+2\right]\right\} + O(\varepsilon^0)$$

$$\Rightarrow \int_0^1 dx\,\{\dots\} \approx \frac{1}{\varepsilon}\frac{1}{(4\pi)^2}\left\{-\frac{2}{3}N_f + \frac{5}{3}N_c\right\} + O(\varepsilon^0)$$

$$\approx i\frac{g^2}{16\pi^2}\delta^{a_1a_2}\left(g^{\mu_1\mu_2}q^2 - q^{\mu_1}q^{\mu_2}\right)\frac{1}{\varepsilon}\left\{\boxed{\frac{5}{3}}N_c - \frac{2}{3}N_f + O(\varepsilon)\right\}$$

$\hookrightarrow \left(\frac{13}{6}-\frac{\xi}{2}\right)$ for general value of gauge parameter ξ

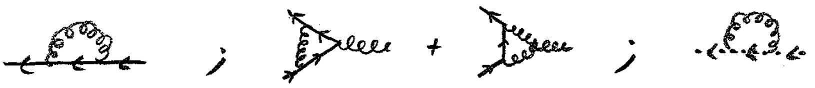
$\rightsquigarrow$ cannot (yet) take limit $\varepsilon\to 0$

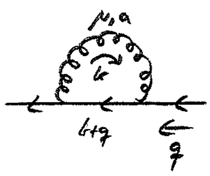
need to remove the $\frac{1}{\varepsilon}$ divergence by a counterterm (see pg 29; § 3.3 below)

$\rightsquigarrow$ want to first compute other 1-loop divergences

3.2 more 1-loop divergences in QCD

$\to$ goal: evaluate 1-loop diagrams (again, in dim. reg. $d=4-2\varepsilon$ and Feynman gauge $\xi=1$)



- 

$$= \int\frac{d^dk}{(2\pi)^d}\left(ig\gamma^\mu T^a\right)\frac{i(\not k+\not q+m)}{(k+q)^2-m^2+i\varepsilon}\left(ig\gamma_\mu T^a\right)\left(\frac{-i}{k^2}\right)$$

$$\gamma^\mu\gamma_\mu = \gamma^\mu\gamma^\nu g_{\mu\nu} = \tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}g_{\mu\nu} = \tfrac{1}{2}\,2g^{\mu\nu}\mathbb{1}\,g_{\mu\nu} = d\,\mathbb{1}$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = \{\gamma^\mu,\gamma^\nu\}\gamma_\mu - \gamma^\nu\gamma^\mu\gamma_\mu = 2\gamma^\nu - \gamma^\nu d = (2-d)\gamma^\nu$$

$$T^a_{ij}T^a_{jk} = \tfrac{1}{2}\left(\delta_{ik}\delta_{jj} - \tfrac{1}{N_c}\delta_{ij}\delta_{jk}\right) = \frac{N_c^2-1}{2N_c}\delta_{ik}$$

$$= -g^2\frac{N_c^2-1}{2N_c}\mathbb{1}_{N_c\times N_c}\int\frac{d^dk}{(2\pi)^d}\frac{(2-d)(\not k+\not q)+md\,\mathbb{1}_{\text{Dirac}}}{(k^2)\left((k+q)^2-m^2\right)}$$

6

Fy par, $\frac{1}{()()} = \int_0^1 dx \frac{1}{[(1-x)k^2 + x(k^2+2kq+q^2-m^2)]^2} = \int_0^1 dx \frac{1}{[(k+xq)^2 + \underbrace{x(1-x)q^2 - xm^2}_{\equiv -\Delta}]^2}$

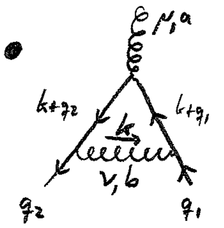
shift $k \to k - xq$

$$= -g^2 \frac{N^2-1}{2N}\int \frac{d^dk}{(2\pi)^d}\int_0^1 dx \frac{(2-d)(\overset{\to 0\ (\text{odd})}{\not k} + (1-x)\not q) + md}{(k^2-\Delta)^2}$$

Wick rotate, use basic tadpole from pg 31

$$= -ig^2 \frac{N^2-1}{2N}\int_0^1 dx\, \underbrace{I_2^0(xm^2 - x(1-x)q^2)}_{\approx \frac{1}{\varepsilon}\frac{1}{(4\pi)^2}\text{, see pg. 34}} \left\{(2-d)(1-x)\not q + md\right\}$$

$$\approx i\frac{g^2}{16\pi^2}\frac{N_c^2-1}{2N_c}\frac{1}{\varepsilon}\left\{\underset{\hookrightarrow \xi}{\boxed{1}}\not q - \underset{\hookrightarrow (3+\xi)}{\boxed{4}}m + O(\varepsilon)\right\}$$ for general value of ξ

- 

$$= \int \frac{d^dk}{(2\pi)^d}(ig)^3 T^b T^a T^b \frac{\gamma^\nu i(\not k+\not q_2+m)\gamma^\mu i(\not k+\not q_1+m)\gamma_\nu (-i)}{((k+q_2)^2-m^2)\quad((k+q_1)^2-m^2)\quad k^2}$$

$$\gamma^{\mu\nu\rho}{}_\mu = (2g^{\mu\nu}-\gamma^\nu\gamma^\mu)\gamma^\rho{}_\mu = 2\underbrace{\gamma^{\rho\nu}}_{=2g^{\rho\nu}-\gamma^{\nu\rho}} - \gamma^\nu\underbrace{\gamma^{\mu\rho}{}_\mu}_{=(2-d)\gamma^\rho} = 4g^{\nu\rho} - (4-d)\gamma^{\nu\rho}$$

$$\gamma^{\mu\nu\rho\sigma}{}_\mu = (2g^{\mu\nu}-\gamma^\nu\gamma^\mu)\gamma^{\rho\sigma}{}_\mu = 2\underbrace{\gamma^{\rho\sigma}\gamma^\nu}_{\cancel{=2g^{\rho\sigma}}-\gamma^{\sigma\rho}} - \gamma^\nu(\cancel{4g^{\rho\sigma}} - (4-d)\gamma^{\rho\sigma}) = -2\gamma^{\sigma\rho\nu} + (4-d)\gamma^{\nu\rho\sigma}$$

$$T^b_{ij}T^a_{jk}T^b_{kl} = T^a_{jk}\,\frac12\left(\delta_{il}\delta_{jk} - \frac1N\delta_{ij}\delta_{kl}\right) = \frac12\delta_{il}\overset{0}{\cancel{T^a_{jj}}} - \frac{1}{2N}T^a_{il} = -\frac{1}{2N}T^a$$

$$= g^3\left(-\frac{1}{2N}T^a\right)\int\frac{d^dk}{(2\pi)^d}\frac{N^\mu}{((k+q_2)^2-m^2)((k+q_1)^2-m^2)k^2}$$

$$N^\mu = m^2(2-d)\gamma^\mu + m(k+q_2)_\nu\left(4g^{\nu\mu}-(4-d)\gamma^{\nu\mu}\right) + m(k+q_1)_\nu\left(4g^{\mu\nu}-(4-d)\gamma^{\mu\nu}\right)$$
$$+ (k+q_2)_\nu(k+q_1)_\sigma\left[-2\gamma^{\sigma\mu\nu}+(4-d)\gamma^{\nu\mu\sigma}\right]$$

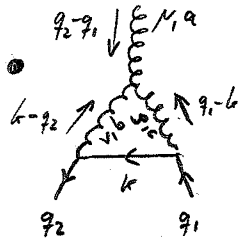
superficial degree of divergence of this integral: $\frac{k^6}{k^6} \to \log$!

$\Rightarrow$ look at $k \gg \{q_1, q_2, m\}$ only, to extract this leading log div

$$\sim g^3\left(-\frac{1}{2N}T^a\right)\int\frac{d^dk}{(2\pi)^d}\frac{[-2\gamma^{\sigma\mu\nu}+(4-d)\gamma^{\nu\mu\sigma}]k_\nu k_\sigma}{(k^2)^3} + \text{const.}$$

numerator $\to [\ldots]\frac{g_{\nu\sigma}k^2}{d} = \frac{k^2}{d}\left[-2\gamma^{\sigma\mu}{}_\sigma + (4-d)\gamma^{\nu\mu}{}_\nu\right] \overset{(\text{pg. }34)}{=} \frac{k^2}{d}(2-d)^2\gamma^\mu$

$$= g^3\left(-\frac{1}{2N}T^a\right)\frac{(2-d)^2}{d}\gamma^\mu \underbrace{\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^2}} = i I_2^0(0) \quad \text{(after Wick rot.)}$$

- 

$$= \int \frac{d^dk}{(2\pi)^d} (ig)^2 T^b T^c \; \frac{\gamma_\nu \, i(\not{k}+m)\, \gamma_\rho}{(k^2-m^2)} \; \frac{(-i)}{(k-q_2)^2} \; \frac{(-i)}{(k-q_1)^2} \; g f^{abc} \, *$$

$$* \left[(2q_2-q_1-k)^\rho g^{\mu\nu} + (2k-q_1-q_2)^\mu g^{\nu\rho} + (2q_1-q_2-k)^\nu g^{\rho\mu} \right]$$

$$\Big\lceil\; T^b T^c f^{abc} = \tfrac{1}{2}[T^b, T^c] f^{abc} = \tfrac{i}{2} \underbrace{f^{bcd} T^d f^{abc}}_{= N_c \delta^{ad} \text{ (see pg. 33)}} = \tfrac{i}{2} N_c T^a$$

again, extract only leading log div, $k \gg \{q_1, q_2, m\}$

$$\sim -g^3 \frac{N_c}{2} T^a \int \frac{d^dk}{(2\pi)^d} \frac{\gamma_\nu \not{k} \gamma_\rho \left[-k^\rho g^{\mu\nu} + 2k^\mu g^{\nu\rho} - k^\nu g^{\rho\mu} \right]}{(k^2)^3} + \text{const.}$$

replace $k^\mu k^\nu \to \frac{g^{\mu\nu} k^2}{d}$

numerator $\sim \gamma_\nu \gamma_\sigma \gamma_\rho \left[-g^{\sigma\rho} g^{\mu\nu} + 2 g^{\sigma\mu} g^{\nu\rho} - g^{\sigma\nu} g^{\rho\mu} \right]$

$$= -\gamma^{\mu\rho}{}_{\rho} + 2\gamma^{\rho\mu}{}_{\rho} - \gamma_\nu{}^{\nu\mu} \overset{(\text{pg. }34)}{=} -d\gamma^\mu + 2(2-d)\gamma^\mu - d\gamma^\mu = 4(1-d)\gamma^\mu$$

$$= -g^3\, 2N_c T^a \frac{1-d}{d} \gamma^\mu \underbrace{\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^2}}_{= i I_2^0(0) \text{ (after Wick rot.)}}$$

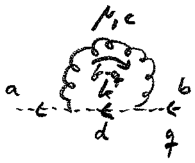
- sum of last two diagrams

 + 

$$= -ig^3 T^a \gamma^\mu I_2^0(0) \left\{ \frac{1}{2N} \frac{(2-d)^2}{d} + 2N \frac{1-d}{d} \right\} + \text{const.}$$

$$\overset{d=4-2\varepsilon}{\approx} -ig T^a \gamma^\mu \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} \left\{ \underbrace{\boxed{1}}_{\to\, \xi} \frac{1}{2N} - \frac{3}{4} \underbrace{\boxed{2}}_{\to\, (\xi+1)} N + \mathcal{O}(\varepsilon) \right\}$$

for general value of ξ

- 

$$= \int \frac{d^dk}{(2\pi)^d} \left(-g f^{acd} q^\mu \right) \left(-g f^{deb} k_\mu \right) \frac{i}{k^2} \frac{-i}{(k-q)^2}$$

$$= -g^2 \underbrace{f^{acd} f^{cdb}}_{= N_c \delta^{ab} \text{ (pg. 33)}} \int \frac{d^dk}{(2\pi)^d} \frac{q \cdot k}{k^2 (k-q)^2}$$

denominator invariant under $k \to q-k$

write numerator $k = \frac{1}{2}k + \frac{1}{2}k \to \frac{1}{2}k + \frac{1}{2}(q-k) = \frac{1}{2}q$

$$= -g^2 N_c \delta^{ab} \frac{q^2}{2} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 (k-q)^2}$$

extract leading log div, $k \gg q$; Wick rot ; use tadpole

$$\sim -ig^2 N_c \delta^{ab} \frac{q^2}{2} I_2^0(0) + \text{const.}$$

$$\overset{d=4-2\varepsilon}{\approx} -i \frac{g^2}{16\pi^2} \delta^{ab} q^2 \frac{1}{\varepsilon} \left\{ \underbrace{\boxed{2}}_{\to\, (3-\xi)} \frac{N_c}{4} + \mathcal{O}(\varepsilon) \right\}$$

for general value of ξ