- important integrals

$$\Gamma(n) = (n-1)! = \int_0^\infty dt\, t^{n-1} e^{-t}$$

$$\int_0^\infty dx\, \frac{x^{2a-1}}{(x^2+1)^b} = \frac{\Gamma(a)\,\Gamma(b-a)}{2\,\Gamma(b)} \qquad (\text{if } \mathrm{Re}[a]>0,\ \mathrm{Re}[b-a]>0)$$

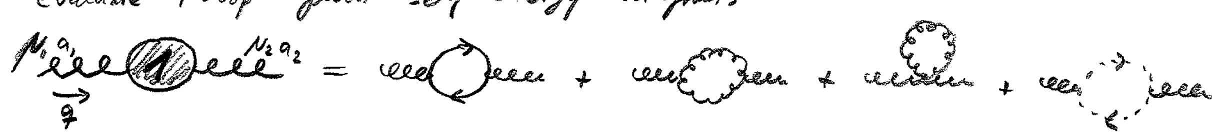
$(A_i = \hat{A}_i + i\varepsilon)$

$$\frac{1}{A_1^{a_1} A_2^{a_2}\cdots A_n^{a_n}} = \frac{\Gamma(a_1+\dots+a_n)}{\Gamma(a_1)\cdot\ldots\cdot\Gamma(a_n)} \int_0^1 dx_1 \dots dx_n\, \frac{x_1^{a_1-1}\cdot\ldots\cdot x_n^{a_n-1}\,\delta(1-\sum_{i=1}^n x_i)}{[x_1 A_1 + \dots + x_n A_n]^{a_1+\dots+a_n}}$$

Feynman parametrization

3.1 one-loop divergences in QCD

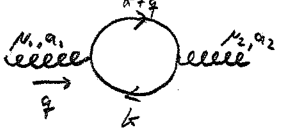
→ goal: evaluate 1-loop gluon self-energy diagrams



use dimensional regularization, $d^4k \to d^dk$, $d = 4-2\varepsilon$

use Feynman gauge $\xi=1$ for gluon propagator (for simplicity)

- 1st diagram: consider one quark flavor, do $\sum_f$ in the end



$$= -\mathrm{Tr} \int \frac{d^dk}{(2\pi)^d} \left(ig\gamma^{\mu_1}T^{a_1}\right) \frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon} \left(ig\gamma^{\mu_2}T^{a_2}\right) \frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2+i\varepsilon}$$

↑ one closed fermion loop

↑ trace over γ's and T's

remember: Dirac matrices

def $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$; $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$

$tr(\mathbb{1}) \equiv N$ (we take $N=4$; also $N=2^{d/2}$, $N=d$ seen)

$tr(\gamma^\mu) = 0$; $tr(\gamma^\mu\gamma^\nu\gamma^\rho) = 0$; ...

$tr(\gamma^\mu\gamma^\nu) = N g^{\mu\nu}$; $tr(\gamma^{1234}) = N(g^{12}g^{34} - g^{13}g^{24} + g^{14}g^{23})$; ...

$$= -g^2 (tr\mathbb{1}) \overbrace{(tr\, T^{a_1}T^{a_2})}^{\frac{1}{2}\delta^{12}\text{, see p. 9}} \int \frac{d^dk}{(2\pi)^d} \frac{N^{\mu_1\mu_2}(k,k+q)}{(k^2-m^2+i\varepsilon)((k+q)^2-m^2+i\varepsilon)}$$

$$N^{13}(k,p) \overset{tr(\gamma's)}{=} m^2 g^{13} + m\cdot 0 + (g^{12}g^{34} - g^{13}g^{24} + g^{14}g^{23})\,k_2 p_4$$

$$= (m^2 - k\cdot p)\,g^{13} + k^1p^3 + p^1k^3$$

for denominator, use Feynman parameters, see p. 30

$$\frac{1}{(k^2-m^2+i\varepsilon)((k+q)^2-m^2+i\varepsilon)} = \int_0^1 dx \frac{1}{[(1-x)(k^2-m^2+i\varepsilon)+x(k^2+2kq+q^2-m^2+i\varepsilon)]^2}$$

$$= \int_0^1 dx \frac{1}{[(k+xq)^2+\underbrace{x(1-x)q^2-m^2}_{\equiv -\Delta}+i\varepsilon]^2}$$

now, shift $k \to k - xq$ under integral $\int d^dk$

$$= -g^2\cdot 4\cdot \tfrac{1}{2}\delta^{a_1a_2}\int\frac{d^dk}{(2\pi)^d}\int_0^1 dx \frac{N^{\mu_1\mu_2}(k-xq,\, k+(1-x)q)}{[k^2-\Delta+i\varepsilon]^2}$$

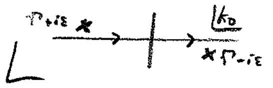
numerator $= \left(m^2-(k^2-x(1-x)q^2+\cancelto{0}{O(k)})\right)g^{\mu_1\mu_2} + 2\boxed{k^{\mu_1}k^{\mu_2}} - 2x(1-x)q^{\mu_1}q^{\mu_2} + \cancelto{0}{O(k)}$

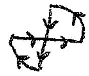
linear (in k) terms integrate to 0; $\;\to \frac{g^{\mu_1\mu_2}k^2}{d}$, see p. 29 bottom

$= \left(m^2+(\tfrac{2}{d}-1)k^2+x(1-x)q^2\right)g^{12} - 2x(1-x)q^1q^2$

to evaluate the $\int d^dk$ integral, perform a Wick rotation $k^0 = ik_E^0$

$k^2-m^2+i\varepsilon = k_0^2-\vec{k}^2-m^2+i\varepsilon = (k_0+\sqrt{\vec{k}^2+m^2}-i\varepsilon)(k_0-\sqrt{\ }+i\varepsilon)$



$0 =$  , $\to\; = -\downarrow\; = +\uparrow$: $k^0 = ik_E^0$

$$= -g^2\cdot 4\cdot\tfrac{1}{2}\delta^{a_1a_2}\int_0^1 dx\; i\left\{\left[(m^2+x(1-x)q^2)g^{12}-2x(1-x)q^1q^2\right]I_2^0(\Delta)+(1-\tfrac{2}{d})g^{12}I_2^1(\Delta)\right\}$$

where $I_n^a(\Delta) \equiv \int\frac{d^dk_E}{(2\pi)^d}\frac{(k_E^2)^a}{[k_E^2+\Delta]^n}$ (basic 1-loop "tadpole" integral)

$$= \frac{1}{(2\pi)^d}\frac{2\pi^{d/2}}{\Gamma(d/2)}\underbrace{\int_0^\infty dk_E\frac{k_E^{d-1+2a}}{(k_E^2+1)^n}}_{=\frac{\Gamma(a+d/2)\Gamma(n-a-d/2)}{2\Gamma(n)}}|\Delta|^{d/2+a-n}$$ see p. 30

$$= \frac{\Gamma(a+\frac{d}{2})\Gamma(n-a-\frac{d}{2})}{(4\pi)^{d/2}\Gamma(\frac{d}{2})\Gamma(n)|\Delta|^{n-a-d/2}}$$

note that $I_2^1(\Delta) = I_2^0(\Delta)\cdot|\Delta|\cdot\frac{d}{2}\frac{1}{1-\frac{d}{2}} = I_2^0(\Delta)\frac{|\Delta|}{\frac{2}{d}-1}$ $\quad((x\Gamma(x)=\Gamma(x+1)))$

$$= -g^2\cdot4\cdot\tfrac{1}{2}\delta^{12}\int_0^1 dx\; i\, I_2^0(\Delta)\left\{\big(\underbrace{m^2+x(1-x)q^2-|\Delta|}_{\Delta>0:\;=\,2x(1-x)q^2}\big)g^{12}-2x(1-x)q^1q^2\right\}$$

$$= -4ig^2\delta^{a_1a_2}(g^{\mu_1\mu_2}q^2-q^{\mu_1}q^{\mu_2})\int_0^1 dx\; I_2^0(m^2-x(1-x)q^2)\,x(1-x)$$

$$= \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\int_0^1 dx\frac{x(1-x)}{[m^2-x(1-x)q^2]^{2-d/2}}$$