2.2 Generalization : Yang-Mills Lagrangian

geometric construction can be generalized :

invariance under local phase rotations

→ invariance under any (continuous) symmetry group

here, use 3d rotation group (O(3) or SU(2)) for brevity

→ in the end, simple generalization to arbitrary local symmetry.

- consider $\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$ (doublet of Dirac fields)

$\equiv V(x)$

demand invariance under local 3d rotations : $\boxed{\psi(x) \to \boxed{e^{i\frac{\sigma^i}{2}\alpha^i(x)}}\,\psi(x)}$

(Matrix = ? $e^{\text{Matrix}} = \sum_{n=0}^{\infty} \frac{(\text{Matrix})^n}{n!}$)

((where σ^i = Pauli matrices $= \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$; $\sum_{i=1}^{3}$ suppressed))

Q: construct invariant Lagrangian ?

→ need again a covariant derivative !

→ now, compensating phase factor has to be a matrix,

with transformation $U(y,x) \to V(y)\,U(y,x)\,V^\dagger(x)$

→ again, $U(x,x)=1$ and $U^\dagger U = U U^\dagger = \mathbb{1}$ unitary

⇒ can expand in terms of (hermitean : $\sigma^\dagger = \sigma$) SU(2)- generators :

$$U(x+\varepsilon n, x) \approx \mathbb{1} + \underbrace{ig}_{\text{convention}}\,\varepsilon n^\mu \underbrace{A^i_\mu \frac{\sigma^i}{2}}_{\uparrow\text{ new (matrix-valued) vector field}} + O(\varepsilon^2)$$

(sign different from QED, since we take $e<0$, $g>0$)

⇒ covariant derivative : (($n^\mu D_\mu \psi = \lim_{\varepsilon\to 0} \frac{\psi(x+\varepsilon n) - U(x+\varepsilon n, x)\psi(x)}{\varepsilon}$))

$$D_\mu = \partial_\mu - ig\, A^i_\mu \frac{\sigma^i}{2}$$

where $A^i_\mu(x)\frac{\sigma^i}{2} \longrightarrow V(x)\left(A^i_\mu(x)\frac{\sigma^i}{2} + \frac{i}{g}\partial_\mu\right)V^\dagger(x)$ (consistency w/ U-trafo)

now, infinitesimally, $\psi \to (1 + i\alpha^i\frac{\sigma^i}{2} + \ldots)\psi$

$$D_\mu\psi \to \underset{\text{-----}}{(\text{check!})} = (1 + i\alpha^i\frac{\sigma^i}{2})\,D_\mu\psi$$

again, transforms the same way as field $\psi(x)$ ✓

((also valid for finite transformations (check!)))

⇒ again, $\bar\psi(x)\left(i\gamma^\mu D_\mu - m\right)\psi(x)$ is locally invariant.

- gauge-invariant terms containing A_μ^i only?

  → here, use construction via D_μ:

  from above, we have $[D_\mu, D_\nu]\psi(x) \to V(x)\,[D_\mu, D_\nu]\psi(x)$ (**)

now, note that

$$[D_\mu, D_\nu]\psi = \underbrace{[\partial_\mu, \partial_\nu]}_{0}\psi - ig\left(\left[\partial_\mu, A_\nu^i \tfrac{\sigma^i}{2}\right] + \left[A_\mu^i \tfrac{\sigma^i}{2}, \partial_\nu\right]\right)\psi - g^2\left[A_\mu^i \tfrac{\sigma^i}{2}, A_\nu^j \tfrac{\sigma^j}{2}\right]\psi$$

does not vanish, as in QED ⇒ $= A_\mu^i A_\nu^j \left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = A_\mu^i A_\nu^j\, i\varepsilon^{ijk}\frac{\sigma^k}{2}$

$(\sigma^i\sigma^j = \delta^{ij}\mathbb{1} + i\varepsilon^{ijk}\sigma^k)$

$$= -ig\underbrace{\left(\partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g\,\varepsilon^{ijk}A_\mu^j A_\nu^k\right)}_{\equiv F_{\mu\nu}^i(x)}\frac{\sigma^i}{2}\psi$$

$\equiv F_{\mu\nu}^i(x)$ ≡ (non-Abelian) field strength tensor

→ as before, $[D_\mu, D_\nu]$ is not a derivative, but a constant (matrix)!

⇒ from (**), the field strength is <u>not invariant</u> now, but transforms as $F_{\mu\nu}^i \frac{\sigma^i}{2} \to V(x)\, F_{\mu\nu}^i \frac{\sigma^i}{2} V^\dagger(x)$

⇒ can construct the locally invariant terms from <u>traces</u> (using cyclicity and $V^\dagger V = \mathbb{1}$)

e.g. $\mathrm{Tr}\left(F_{\mu\nu}^i \frac{\sigma^i}{2} F^{\mu\nu j}\frac{\sigma^j}{2}\right) = \frac{1}{2} F_{\mu\nu}^i F^{\mu\nu i} \equiv \frac{1}{2}\left(F_{\mu\nu}^i\right)^2$

- adding up: $\boxed{\mathcal{L} = \bar\psi\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{1}{4}\left(F_{\mu\nu}^i\right)^2}$ <u>Yang-Mills Lagrangian</u>
  - two parameters: m, g
  - variations → equations of motion: Dirac eqn + eqn for vector field
- generalize to other continuous symmetry groups:

  $V \to n \times n$ unitary matrices ; $\psi(x)$ is n-plet ; $\psi(x) \to V(x)\,\psi(x)$

  expand $V(x) \approx \mathbb{1} + i\,T^a \alpha^a(x) + O(\alpha^2)$

  ↑ $(T^a)^\dagger = T^a$ set of <u>generators</u> of symmetry group

  all as above, with $\frac{\sigma^i}{2} \to T^a$

  for def. of $F_{\mu\nu}^a$ use $[T^a, T^b] = i f^{abc}\, T^c$

  ↑ completely antisym. <u>structure const.</u>

- summary:

invariance of n-plet ψ under local "gauge" transformations $\psi(x) \to V(x)\psi(x)$

with $V(x) \equiv n\times n$ unitary matrices $= e^{iT^a\alpha^a(x)}$

where $(T^a)^\dagger = T^a$ are hermitean generators

with structure constants f^{abc} given by $[T^a, T^b] = if^{abc}T^c$

$\Rightarrow$ covariant derivative $D_\mu = \partial_\mu - igA_\mu^a T^a$

contains one vector field for each independent generator of local symmetry

$A_\mu^a T^a \to V(x)\left(A_\mu^a T^a + \frac{i}{g}\partial_\mu\right)V^\dagger(x)$ guarantees $D_\mu\psi(x) \to V(x)D_\mu\psi(x)$

$\Rightarrow$ field strength tensor $[D_\mu, D_\nu] = -ig F_{\mu\nu}^a T^a$

((or $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$))

transforms as $F_{\mu\nu}^a T^a \to V(x) F_{\mu\nu}^a T^a V^\dagger(x)$

($\rightsquigarrow$ for later reference: infinitesimal transformations

$\psi \to \psi + iT^a\alpha^a(x)\psi + O(\alpha^2)$

$A_\mu^a \to A_\mu^a + \left(f^{abc}A_\mu^b + \frac{1}{g}\delta^{ac}\partial_\mu\right)\alpha^c(x) + O(\alpha^2)$

$F_{\mu\nu}^a \to F_{\mu\nu}^a + f^{abc}F_{\mu\nu}^b\,\alpha^c(x) + O(\alpha^2)$)

$\Rightarrow$ most general gauge-invariant renormalizable Lagrangian

(conserving P, T): $\mathcal{L} = \bar\psi\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{1}{4}\left(F_{\mu\nu}^a\right)^2$

- jargon: Abelian symmetry group of QED

vs non-Abelian symmetry group of the more general theories above.

$\rightarrow$ non-Abelian gauge theory $\equiv$ QFT associated with a non-commuting local symmetry