1.4 Elements of group theory

- the color charge introduced above can be treated much more rigorously.
  - → symmetry at work.
  - → before (re-) learning the connection symmetry ⇔ charge from QED (cf. § 2.1), let us review some basic facts about the theory of continous symmetry groups
  - → (much) more detail eg. in [H. Georgi: Lie Algebras in Particle Physics] or at http://www.physik.uni-bielefeld.de/~laine/symmetrien/cover.html

- our provisional color assignment to gluons (cf. § 1.3) $g_1 \dots g_8$ can be rewritten in a different basis (just different linear combinations) of 3x3 - Matrices (label eg. rows by colors, columns by anticolors $\begin{array}{c|ccc} & \bar r & \bar g & \bar b \\ r & \cdot & \cdot & \cdot \\ g & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot \end{array}$)

("generalization of Pauli matrices")

$$T^1 = \frac{1}{2}\begin{pmatrix} 0&1&0\\1&0&1\\0&0&0\end{pmatrix} \quad T^2 = \frac{1}{2}\begin{pmatrix} 0&-i&0\\i&0&0\\0&0&0\end{pmatrix} \quad T^3 = \frac{1}{2}\begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0\end{pmatrix} \quad T^4 = \frac{1}{2}\begin{pmatrix} 0&0&1\\0&0&0\\1&0&0\end{pmatrix}$$

$$T^5 = \frac{1}{2}\begin{pmatrix} 0&0&-i\\0&0&0\\i&0&0\end{pmatrix} \quad T^6 = \frac{1}{2}\begin{pmatrix} 0&0&0\\0&0&1\\0&1&0\end{pmatrix} \quad T^7 = \frac{1}{2}\begin{pmatrix} 0&0&0\\0&0&-i\\0&i&0\end{pmatrix} \quad T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$$

→ actually, $T^a = \frac{1}{2}\lambda^a$, where λ^a are "Gell-Mann matrices", $a = 1 \dots 3^2-1 = 1 \dots 8$

→ they form a possible representation of the infinitesimal generators of the "special unitary group" $SU(3)$, the fundamental representation

→ some important properties (check ?!):

$$[T^a, T^b] = i f^{abc} T^c \qquad \text{Lie algebra}$$

↖ antisymmetric structure constants

$$\{T^a, T^b\} = \frac{1}{3}\delta^{ab}\mathbb{1}_{3\times3} + d^{abc}T^c$$

↖ symmetric structure constants

$$\Rightarrow T^a T^b = \frac{1}{2}\left(\frac{1}{3}\delta^{ab}\mathbb{1}_{3\times3} + (d^{abc} + i f^{abc})T^c\right)$$

$$T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{kl}\right) \qquad \text{Fierz identity}$$

→ often we will need traces

$$Tr(T^a) = 0$$

$$Tr(T^a T^b) = \frac{1}{2}\delta^{ab} \qquad \text{normalization}$$

etc.

→ could calculate f^{abc} by multiplying Lie algebra with T^d, then taking trace:

$$f^{abc} = \frac{2}{i}\left(Tr(T^aT^bT^c) - Tr(T^bT^aT^c)\right)$$

result (check ?!) : $f^{123} = 1$, $f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}$,

$f^{458} = f^{678} = \frac{\sqrt{3}}{2}$; rest by antisymmetry

- from a more general viewpoint, we have just seen one example of a broader mathematical concept: representations of Lie Groups

math: group contains abstract entities that obey certain algebraic rules

QFT: interested in groups of unitary operators acting on vector space of states

here: interested in continuously generated groups

contain elements arbitrarily close to identity.

can reach general group element by repeated action of infinitesimal ones

$$g(\alpha) = 1 + i\alpha^a T^a + O(\alpha^2)$$

T^a: Hermitean op's; "generators" of symm. group

α^a: group parameters

a group with this structure is called a "Lie group"

- the set T^a spans space of infinitesimal group transformations

⇒ commutator is a linear combination of generators

$$[T^a, T^b] = i f^{abc} T^c$$

vector space spanned by generators + commutator ≡ Lie Algebra

→ $$[T^a,[T^b,T^c]] + [T^b,[T^c,T^a]] + [T^c,[T^a,T^b]] = 0$$ Jacobi identity

$$\Rightarrow f^{abe}f^{ecd} + f^{ace}f^{edb} + f^{ade}f^{ebc} = 0$$

- for us, symmetry $\hat{=}$ unitary transformation of a set of fields

→ interested in Lie groups with finite # of generators: "compact"

- classification of Lie Algebras

→ if one T^a commutes with all others: Abelian subgroup, $\psi \to e^{i\alpha}\psi$, $U(1)$ (group of phase rotations)

→ if set of T^a's cannot be divided into two mutually commuting sets: "simple"

→ general Lie algebra ≡ direct sum of non-Abelian simple components + additional Abelian generators

(Killing, Cartan ca. 1894)

→ $SU(N)$ $\left((UU^\dagger = 1, \det U = 1)\right)$, $SO(N)$ $\left((RR^T = 1, \det R = 1)\right)$, $Sp(N)$; $G_2, F_4, E_{6,7,8}$ is complete set of compact simple Lie groups!

1.5 Notation and conventions

- natural units $\hbar = c = k_B = 1$

$$\Rightarrow [\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1} = \text{GeV}^{-1}$$

- vectors + tensors

indices $\mu = 0,1,2,3$ or t, x, y, z

metric tensor $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1,-1,-1,-1)$

four-vectors $x^\mu = (x^0, \vec{x})$; $\partial_\mu = \partial_{x^\mu} = (\partial_0, \vec{\nabla})$

totally antisymmetric tensor $\varepsilon^{0123} \equiv 1$ $\left(\Rightarrow \varepsilon_{0123} = -1,\ \varepsilon^{1230} = -1 \text{ etc} \right)$

- matrices

Pauli $\sigma^i \sigma^j = \delta^{ij} \mathbb{1}_{2\times 2} + i \varepsilon^{ijk} \sigma^k$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac $\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}$

standard basis: $\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix}$, $\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$

$$\gamma^5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

- Einstein summation convention

e.g. $p_\mu x^\mu \equiv \sum_{\mu=0}^{3} p_\mu x^\mu = p_0 x^0 + (-\vec{p}) \cdot \vec{x} = p^0 x^0 - \vec{p} \cdot \vec{x}$

e.g. $A^a T^a \equiv \sum_a A^a T^a$