Reduction of Feynman integrals with integration by parts relations and the Laporta algorithm

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MLWG

Goal

- Reduce many Feynman integrals to a small set of so-called master integrals:
- Use integration by parts relations (IBP) [K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B **192** (1981) 159]
- and a systematic way to combine them:

The Laporta algorithm [S. Laporta, hep-ph/0102033]

Reduce Feynman integrals

- Why do we want to reduce Feynman integrals?
- Usually we are faced with many Feynman integrals in a state of the art perturbative computation
    - number of integrals $\neq$ number of diagrams
    - 1 Feynman diagram can result in thousands of Feynman integrals.
- Typically millions of Fintegrals to compute $\rightarrow$ impossible without computer algebra

Reminder: Perturbative Expansion

- Any perturbative calculation consist of a combinatoric, algebraic and analytic part.
  - combinatoric: Wick contractions
  - algebraic: Feynman rules, gamma algebra, expansions, projections, tensor decompositions, etc. + reduction
- The former ones are well suited problems for automatization.
- BUT: in order to calculate master integrals human intervention unavoidable.
  - Ideal starting point: small set of master integrals.

Integration by parts relations I

- n-loop Feynman integral with m external momenta:

$$F(p_1, \ldots, p_m) \equiv \int_{k_1,\ldots,k_n} d^d k_1 \ldots d^d k_n \prod_i \frac{(p \cdot k)_i^{b_i}}{(q_i^2 + m_i^2)^{a_i}}$$

- Integration by parts relations are generated by

$$\int_{k_1,\ldots,k_n} d^d k_1 \ldots d^d k_n \frac{\partial}{\partial k_j^\mu} \left(k_l^\mu \prod_i \frac{(p \cdot k)_i^{b_i}}{(q_i^2 + m_i^2)^{a_i}} \right) = 0$$

Zero-temperature massiv tadpole

- Example 1: $I(a) \equiv \int d^d k \frac{1}{(k^2+m^2)^a}$, 0pt 1-loop fct. $\rightarrow$ 1 IBP
- IBP relation $\partial_k k$:

$$
\begin{aligned}
0 &= \int d^d k \partial_k k \frac{1}{(k^2+m^2)^a} \\
&= \int d^d k \left\{ D \frac{1}{(k^2+m^2)^a} - 2a \frac{k^2+m^2-m^2}{(k^2+m^2)^{a+1}} \right\} \\
&= I(a)(D-2a) + 2am^2 I(a+1)
\end{aligned}
$$

- $\Rightarrow I(a+1) = -\frac{D-2a}{2am^2} I(a)$, for $a \geq 1$.

Finite-temperature tadpole

- Example 2: $I(a,b) \equiv \sum_{n=-\infty}^{\infty} \int d^{D-1}k \frac{K_0^b}{(K^2)^a}$, $K^2 = \omega_n^2 + k^2$
- IBP relation $\partial_k k$:

$$
\begin{aligned}
0 &= \sum\!\!\!\!\!\!\int_K \partial_k k \frac{K_0^b}{(K^2)^a} \\
&= \sum\!\!\!\!\!\!\int_K \left\{ (D-1)\frac{K_0^b}{(K^2)^a} - 2a \frac{\left(K^2 - K_0^2\right) K_0^b}{(K^2)^{a+1}} \right\} \\
&= I(a,b)(D-1-2a) + 2a\, I(a+1,b+2)
\end{aligned}
$$

- $\Rightarrow I(a+1,b+2) = -\frac{D-1-2a}{2a} I(a,b)$, for $a \geq 1$.

Zero-temperature sunset propagator

- Example 3: $I_{a,b,c,d,e} \equiv \int_{p,q} \frac{1}{(p^2)^a((p-k)^2)^b(q^2)^c((q-k)^2)^d((p-q)^2)^e}$
- 6 possible IBP relations: $\partial_p p, \partial_p q, \partial_p k, \partial_q p, \partial_q q, \partial_q k$
- Question: How do we combine the IBP relations to get a suitable reduction? In general not known → introduce unique ordering (Laporta algorithm)
- In this case we just take $\partial_p p - \partial_p q$:

$$0 + 0 = \int_{p,q} \partial_p \, (p - q) \, \frac{1}{p^2(p-k)^2 q^2 (q-k)^2 (p-q)^2}$$

- ... after two pages of algebra we find:

$$\frac{1}{2}(4 - D) I_{1,1,1,1,1} = I_{1,1,2,1,0} - I_{1,1,2,0,1}$$

Laporta algorithm

- As mentioned before, in general it can be quite involved to find the correct combinations of IBP relations: brute force algorithm (1981 - 2000)
- "Solution" Laporta algorithm [S. Laporta, hep-ph/0102033]
  - Key idea: Introduce lexicographic ordering: prescription in order to decide what is the most complicated integral out of a set of integrals.
- Example: Let us consider once again the sunset propagator:

$$I_{a,b,c,d,e} \equiv \int_{p,q} \frac{1}{(p^2)^a((p-k)^2)^b(q^2)^c((q-k)^2)^d((p-q)^2)^e}$$

- Obviously, $I_{1,1,1,1,1}$ is more difficult than $I_{1,1,2,0,1}$ or $I_{1,1,2,1,0}$.
- So, the first "rule" could be:
  - 1. Count positiv power of propagators $\sum_{a-e}\theta(\#)$, choose highest, if equal go to 2.

Lexicographic ordering

- 2. Compute abs. sum of powers of propagators $\sum_{a-e} |\#|$, choose highest, if equal go to 3.
- 3. Count zero propagators $\sum_{a-e} \delta(\#)$, choose lowest, if equal go to 4.
- 4. Choose integral with highest power on propagator e,d,c,b,a
- Example:

Rule	$I_{1,1,1,1,1}$	$I_{1,1,2,0,1}$	$I_{1,1,1,-1,1}$	$I_{1,1,1,2,1}$	$I_{2,1,1,1,1}$
1	5	4	4	5	5
2	5	5	5	6	6
3	0	1	0	0	0
4	-	-	-	d $\surd$	a

Schematic of simple Laporta implementation