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INTEGRATION BY PARTS: THE ALGORITHM TO CALCULATE β-FUNCTIONS IN 4 LOOPS

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The following statement is proved: the counterterm for an arbitrary 4-loop Feynman diagram in an arbitrary model is calculable within the minimal subtraction scheme in terms of rational numbers and the Riemann ζ-function in a finite number of steps via a systematic "algebraic" procedure involving neither integration of elementary, special, or any other functions, nor expansions in and summation of infinite series of any kind. The number of steps is a rapidly increasing function of the complexity of the diagram. To demonstrate further possibilities offered by the technique we compute the 5-loop diagram contributing to the anomalous field dimension $\gamma_2(g)$ in the φ^4 model, that defied, heretofore, analytical calculation by other methods.

1. Introduction

The efforts directed at the refinement of calculational methods within the perturbative approach in quantum field theory need not be justified. Suffice it to say that, firstly, a wide range of important problems for many a theoretician to exercise their computational abilities is provided by QCD [1]. Secondly, a better understanding of the underlying physics and mathematics will, sooner or later, reveal the lacking information for incorporating low-order calculations with higher-order asymptotic estimates [2] in a (Borel-like) summation technique (for a review see [3]). Such a technique will be capable, hopefully, of increasing the quantitative predictive power of the perturbation method, which leaves much to be desired yet in QCD.

The present paper, being a continuation of the previous research [4, 5] and an extended exposition of the results outlined in [30], treats of a certain class of perturbative calculations. Namely, our aim here is to discuss how and why one can analytically calculate the renormalization group (RG) quantities, i.e., β-functions and all sorts of anomalous dimensions which enter the RG equations [6] as coefficients, at the 4-loop level in any model within the minimal subtraction (MS) scheme [7]. By now it is a well-known fact that the RG functions are relevant to the description of the high-energy behaviour of both inclusive and exclusive processes in QCD (see e.g., [8]).

Since the volume of algebraic manipulations involved is enormous, we have not attempted to present any numerical results for gauge theories. This is to be postponed until computer programs are written and tested. But the unexpected simplicity of our starting idea is in such contrast with the power of the method built upon it, that a separate discussion seems justified.

The paper has the following structure. In sect. 2 we present the preliminary information for the reader to understand why we concentrate on the evaluation of propagator-type massless integrals in the subsequent sections. In sect. 3 the principal idea of this work – integration by parts of dimensionally regularized integrals – is explained in simple terms, and in sects. 4–6 the algorithm of calculations is systematically developed and studied in a formal manner. In sect. 7, to exhibit the possibilities of the ideas behind the algorithm we calculate the contribution of the most intricate 5-loop diagram to the field anomalous dimension $\gamma_2(g)$ of the φ^4 model. In sect. 8 various aspects of the method are commented upon, and its relation to ref. [24] is clarified. Sect. 9 summarizes our results. Appendices A and B describe effective methods to calculate primitive integrals defined in sect. 2. Appendix C (to which we draw the reader's attention) is devoted to a subject which is not connected directly with the main body of the paper but throws new light upon it.

2. Formulation of the problem

It is a well-known fact that the RG functions are connected with counterterms of the lagrangian, i.e., with divergent parts of diagrams, e.g.,

$$\gamma = \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\bigg|_{g_\mathrm{B},\, m_\mathrm{B}} \ln Z. \tag{2.1}$$

Here Z is a renormalization constant, γ the corresponding anomalous dimension, μ the parameter of the subtraction scheme chosen, and g_B and m_B are, correspondingly, the bare coupling constant and mass of the model.

It is clear, therefore, that an astute choice of regularization and renormalization methods can facilitate calculations. At present the method of dimensional regularization [9] is, undoubtedly, the most convenient one to deal with infinities in Feynman integrals. As to the renormalization procedure, the MS scheme [7] and its various versions allow considerable simplifications in the calculation of counterterms of diagrams contributing to the corresponding renormalization constants, as is explained below. Moreover, contrary to the common belief that the MS scheme is only applicable to the study of high-energy asymptotics because of the independence of the corresponding running coupling constants on masses, it has been shown recently [10] that the MS scheme can be modified without any loss in its calculational convenience so as to become applicable to such problems as the broken symmetry restoration at energies near the mass of the intermediate bosons in grand unified theories, where the mass dependence is essential.

The MS scheme prescribes to subtract pure poles in $\varepsilon = \frac{1}{2}(4-D)$, D being the space-time dimension, through which divergences of Feynman integrals manifest themselves. The advantage of this recipe lies in that the dependence of counterterms on dimensional parameters of the model becomes polynomial [11] and, in fact, known beforehand. As was shown in [12], this allows one to perform the IR rearrangement, i.e., to simplify the infrared structure of a diagram without affecting its counterterm which is, essentially, its overall ultraviolet singularity. Various versions of the IR rearrangement differing from the original one are given in [4, 5].

Referring the reader to those papers for details and examples, we only formulate the main result: the problem of calculating the counterterm of an arbitrary l-loop diagram with an arbitrary number of masses and external momenta within the MS scheme can be reduced through the IR rearrangement to the problem of calculating to $O(\varepsilon^0)$ some $(l-1)$-loop massless integrals with only one external momentum. We denote such massless propagator-type integrals as the p-integrals. A p-integrand can have a complicated numerator, which without loss of generality can be taken to be a polynomial in scalar products of one external and some internal momenta. The corresponding denominator consists of factors like $(p-k)^{2n}$, $n = 1,2,3$, etc., cf. eq. (3.9). Such factors, and not just $(p-k)^2$ as one would expect in the case of usual diagrams, can result from differentiation with respect to the masses and external momenta at the stage of IR rearrangement [12, 5, 13], and have to be considered in the analysis.

In the earlier paper [5] we have shown how an arbitrary 2-loop and many 3-loop p-integrals can be calculated analytically via the Gegenbauer polynomial x-space technique (GPXT). In the present paper we combine GPXT with integration by parts and prove that an arbitrary 3-loop p-integral of the above described type can be calculated analytically to $O(\varepsilon^0)$. In fact, we shall prove in sects. 4–6 that a p-integral $I(\mathrm{k}, \varepsilon)$, k being the external momentum, can be exactly expressed as,

$$I(k,\varepsilon) = \sum_i C_i(\varepsilon)\mathrm{Pr}_i(k,\varepsilon) + C_L(\varepsilon)L_0(k,\varepsilon) + C_N(\varepsilon)N_0(k,\varepsilon). \tag{2.2}$$

Here Pr_i are "primitive" integrals, i.e., calculable in terms of the Euler Γ-functions by few applications of the one-loop integration formulae of appendix A (see also subsect. 4.1), and the sum goes over some finite set of primitive integrals. $L_0(k,\varepsilon)$ and $N_0(k,\varepsilon)$ are the two diagrams of the φ^3 model shown in fig. 1. Note also that various powers of k^2 multiplying each term on the r.h.s. of eq. (2.2) in order to preserve the right overall dimension, are omitted. $C_i(\varepsilon)$, $C_L(\varepsilon)$ and $C_N(\varepsilon)$ are simple rational functions of ε with the following properties. If $I(k,\varepsilon)$ is planar, then $C_N = 0$ and C_L is regular near $\varepsilon = 0$. If $I(k,\varepsilon)$ is non-planar then C_L is $O(\varepsilon)$ and C_N regular (see, however, remark (iii) of sect. 5). C_i can have a singularity at $\varepsilon = 0$ not stronger than ε^{-2}.

Fig. 1. The two diagrams of the scalar φ^3 model, which cannot be evaluated through integration by parts only.

So, in order to guarantee the calculability of any 3-loop p-integral to $O(\varepsilon^0)$, one only has to calculate L_0 and N_0 also to $O(\varepsilon^0)$. This can be easily done by means of GPXT [5], once and for all:

$$L_0(k,\varepsilon) = N_0(k,\varepsilon) + O(\varepsilon) = \frac{20\zeta(5)}{(4\pi)^6 k^4} + O(\varepsilon). \tag{2.3}$$

There exists an instructive explanation of the equality of L_0 and N_0 to $O(\varepsilon^0)$, which is presented in appendix C.

So, we conclude that once the algorithm of obtaining the representation (2.2) for any $I(k,\varepsilon)$ is given, the problem of calculating 4-loop RG functions no longer exists, at least, theoretically.

We would like to point out that the method we are going to describe also considerably simplifies 3-loop RG-calculations. This is because with the result (2.2) one can attack 3-loop diagrams in a straightforward manner, without performing the full IR rearrangement, though one would still have to put all masses and some of the external momenta to zero.

3. Simple example

The main idea of our method is very simple and can be best explained with a simple example. Consider the 2-loop scalar diagram of fig. 2. Its expression in momentum (p) space is*

$$F(k) = \int \frac{d^Dq\, d^Dp}{(2\pi)^{2D}} \frac{1}{p^2(p-k)^2 q^2(q-k)^2(p-q)^2}. \tag{3.1a}$$

The same integral in position (x) space is

$$F(x_3) \sim \int d^Dx_1\, d^Dx_2 \frac{1}{(x_3-x_1)^{2\lambda}(x_3-x_2)^{2\lambda} x_1^{2\lambda} x_2^{2\lambda} (x_2-x_1)^{2\lambda}}, \tag{3.1b}$$

where D is the space-time dimension, and $\lambda = \frac{1}{2}(D-2)$. Factors in the denominators of eqs. (3.1a) and (3.1b) are in a one-to-one correspondence.

* All integrals are defined over euclidean space.

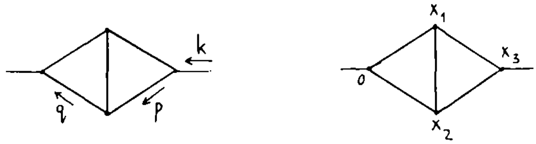

Fig. 2. A simple scalar diagram, which can be evaluated through integration by parts. (a) and (b) correspond to eqs. (3.1a) and (3.1b), respectively.

Now we write the following identity in p-space:

$$0=\int\frac{\mathrm{d}^Dq\mathrm{d}^Dp}{(2\pi)^{2D}}\frac{\partial}{\partial p^\mu}\left\{\frac{1}{p^2(p-k)^2q^2(q-k)^2}\frac{(p-q)^\mu}{(p-q)^2}\right\}. \tag{3.2a}$$

This identity becomes obvious in x-space,

$$0=\int\mathrm{d}^Dx_1\mathrm{d}^Dx_2\frac{\{(x_1-x_3)^\mu+(x_3-x_2)^\mu+(x_2-x_1)^\mu\}}{(x_1-x_3)^{2\lambda}(x_3-x_2)^{2\lambda}x_1^{2\lambda}x_2^{2\lambda}}\frac{(x_2-x_1)^\mu}{(x_2-x_1)^{2(\lambda+1)}}. \tag{3.2b}$$

Differentiating with respect to p in eq. (3.2a) and making use of identities like

$$2(p-q)^\mu(p-k)^\mu=(p-q)^2+(p-k)^2-(q-k)^2, \tag{3.3}$$

we obtain [here $\varepsilon=\frac{1}{2}(4-D)$]

$$\begin{aligned}&\varepsilon\int\frac{\mathrm{d}^Dp\mathrm{d}_q^D}{(2\pi)^{2D}}\frac{1}{p^2(p-k)^2q^2(q-k)^2(p-q)^2}\\&=\int\frac{\mathrm{d}_p^D\mathrm{d}_q^D}{(2\pi)^{2D}}\left\{\frac{1}{q^2(p-q)^2p^2(p-k)^4}-\frac{1}{p^2(p-k)^4q^2(q-k)^2}\right\}.\end{aligned} \tag{3.4}$$

Eq. (3.4) is pictured in fig. 3. It is a well-known fact that the integrals on the r.h.s. of eq. (3.4) can be expressed in terms of Γ-functions; all the necessary formulae are given in appendix A.

In short, we have managed to express a non-trivial 2-loop diagram through diagrams of much simpler structure, exactly to all orders in ε. The result is amazing*, if one considers the simplicity of the trick by which it has been obtained.

* The authors of sophisticated methods of evaluating Feynman diagrams [5, 14, 15, 21] should agree that it really is.

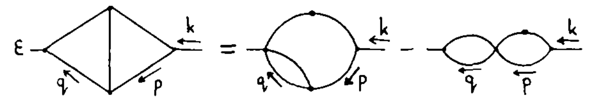

Fig. 3. A pictorial form of eq. (3.4). A dot on the line means an additional power of p^2 in the denominator, or, equivalently, the insertion of a φ^2 vertex.

But this is only a first example: in subsequent sections the algorithm will be developed systematically.

In deriving eq. (3.4) we have used:

(i) An identity of the following form*:

$$0=\int \mathrm{d}^D p \frac{\partial}{\partial p^{\mu}} f(p, \ldots), \tag{3.5a}$$

or, in x-space,

$$0=(x_1-x_3)^{\mu}+(x_3-x_2)^{\mu}+(x_2-x_1)^{\mu}. \tag{3.5b}$$

This identity can be considered as a consequence of translational invariance of dimensionally regularized integrals in p-space,

$$\int \mathrm{d}^D p f(p)=\int \mathrm{d}^D p f(p+q). \tag{3.6}$$

It can also be viewed as integration by parts in p-space.

(ii) Momentum conservation, i.e., translational invariance in x-space expressed in eq. (3.3) which can be viewed as integration by parts in x-space.

(iii) Cancellation of squared combinations of momenta in the numerator and denominator of the diagram, which is equivalent to shrinking the corresponding line to a point:

$$1=p^2 \frac{1}{p^2} \rightarrow \square \frac{1}{(x-y)^{2\lambda}} \sim \delta^{(D)}(x-y). \tag{3.7}$$

(iv) The fact that all divergent expressions we are dealing with at the intermediate stages are well-defined, and formal manipulations otherwise incorrect, justified within dimensional regularization.

* To our knowledge, such identities within dimensional regularization were considered with practical purposes in ref. [16] in a different context. Similar identities in x-space were used in ref. [21].

It is also worth stressing that the integrals on the r.h.s. of eq. (3.4) are calculable owing to the following simple facts:

(i) A simple power goes over into a simple power under Fourier transformation within dimensional regularization,

$$\frac{1}{q^{2\alpha}}=\frac{\Gamma(\frac{1}{2}D-\alpha)}{(2\pi)^{D/2}\Gamma(\alpha)}\int \mathrm{d}^D x \frac{e^{iqx}}{(x^2)^{D/2-\alpha}} \tag{3.8}$$

(note that eq. (3.8) is incorrect for $\alpha=\frac{1}{2}D+m$ or $\alpha=-m$, m non-negative integer, but we shall not encounter such cases in the present work).

(ii) Fourier transform of a convolution is a product of Fourier transforms.

(iii) The group property of simple powers: $x^{2\alpha}\cdot x^{2\beta}=x^{2(\alpha+\beta)}$ and $p^{2\alpha}\cdot p^{2\beta}=p^{2(\alpha+\beta)}$.

Summarizing, one may say that the whole procedure consists in using algebraic identities like eqs. (3.5b) and (3.3) and Fourier transforming between x- and p-space with the help of eq. (3.8) and its parametric derivatives.

It should be clear, that repeating the same trick several times, one can calculate the following integral:

$$\int \frac{\mathrm{d}^D p\,\mathrm{d}^D q}{(2\pi)^{2D}}\frac{1}{p^{2\alpha}q^{2m}(p-q)^{2l}(p-k)^{2\beta}(q-k)^{2n}}, \tag{3.9}$$

where l, m and n are integers >0, and α and β are arbitrary. Further details are given in sect. 4.

4. The algorithm

Now we proceed to construct a set of recursive relations which will enable one to evaluate to $O(\varepsilon^0)$ an arbitrary 2- and 3-loop p-integral with an arbitrary scalar numerator and propagators of the form $(p^2)^{-m}$, m integer.

4.1. ONE-LOOP INTEGRALS, PRIMITIVE INTEGRALS, NOTATIONS

Consider the expression

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{\mathcal{P}(p,k)}{p^{2\alpha}(p-k)^{2\beta}}, \tag{4.1}$$

where $\mathcal{P}(p,k)$ is a scalar polynomial constructed of p, k, and, perhaps, some other vectors. It is convenient to rewrite eq. (4.1) as

$$\left\{\int \frac{\mathrm{d}^D p_1\,\mathrm{d}^D p_2}{(2\pi)^{2D}}(2\pi)^D\delta(p_1+p_2-k)\right\}\frac{\mathcal{P}(p_1,p_2)}{p_1^{2\alpha_1}p_2^{2\alpha_2}}. \tag{4.2}$$

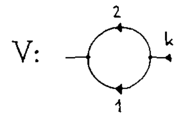

Fig. 4. The generic graph of one-loop primitive integrals.

Now we can introduce the condensed notation which will be indispensable in what follows,

$$\text{eq. (4.2)} = \mathcal{P}(p_1, p_2)V(\alpha_1, \alpha_2). \tag{4.3}$$

All information about momentum conservation, topology of the diagram and numeration of lines can be visualized via a "generic" graph, as in fig. 4. It will be our rule to assign a function name (V, in the present example) to every generic graph, so that one will not need to write explicitly expressions like eqs. (4.1) and (4.2), which become rather cumbersome at the 3-loop level. Also, we shall often call a line integer, positive, etc., if such is the corresponding α_i.

Returning to the one-loop case, we observe that, in general, both p_1 and p_2 are present in $\mathcal{P}$, and first of all one should substitute, say, p_2 by $k - p_1$. If the factor p_1^2 appears after such a substitution, it should be absorbed into V:

$$p_1^{2\sigma}V(\alpha_1, \alpha_2) = V(\alpha_1 - \sigma, \alpha_2). \tag{4.4}$$

So we come to the problem of calculating the following integral:

$$\mathcal{P}_n(p_1)V(\alpha_1, \alpha_2), \tag{4.5}$$

where $\mathcal{P}_n(p_1)$ does not contain p_1^2 and

$$\mathcal{P}_n(\lambda p_1) = \lambda^n \mathcal{P}_n(p_1). \tag{4.6}$$

This problem is extensively treated in appendix A, where it is explained how eq. (4.5) can be calculated most conveniently in terms of the so-called G-functions, which are themselves combinations of the Euler Γ-functions. Here we only note that eq. (4.5) can be calculated for arbitrary complex α_i and the result is a sum of the terms like

$$\frac{\mathcal{P}_i(k)}{k^{2\beta_i}} \tag{4.7}$$

(recall that k is the external momentum; β_i depend on α_i and ε).

Therefore, we can define an important class of "primitive" integrals*, or recursively one-loop integrals, as they were called in [5]. This class consists of the diagrams amenable to exact evaluation by repeated use of the one-loop integration

* Each primitive integral is a p-integral but not vice versa.

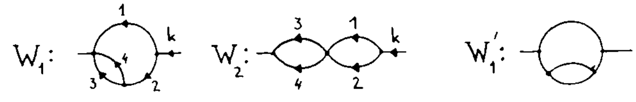

Fig. 5. The generic graphs of 2-loop primitive integrals. Note that the graph W_1 is equivalent to W_1'.

formulae of appendix A. The 2-loop generic primitive graphs are shown in fig. 5. Integration of primitive integrals proceeds as a simple sequence of substitutions. Numerators adding no essential complications, we illustrate the idea with a simple example without numerator:

$$\begin{aligned} W_1(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= G(\alpha_3,\alpha_4)\,p_2^{2(2-\varepsilon-\alpha_3-\alpha_4)}V(\alpha_1,\alpha_2) \\ &= G(\alpha_3,\alpha_4)V(\alpha_1,\alpha_2+\alpha_3+\alpha_4-2+\varepsilon) \\ &= k^{2(4-2\varepsilon-\alpha_1-\alpha_2-\alpha_3-\alpha_4)}G(\alpha_3,\alpha_4)G(\alpha_1,\alpha_2+\alpha_3+\alpha_4-2+\varepsilon), \end{aligned} \tag{4.8}$$

where $G(\alpha,\beta)=G(\alpha,\beta,0,0)$ is the function defined in eq. (A.2). Basically different generic 3-loop primitive graphs are presented in fig. 6.

As soon as a complete set of rules analogous to eq. (4.8) is formulated, one is left with algebraic substitutions which can be implemented, e.g., as a SCHOONSCHIP [17] program in a straight-forward manner.

4.2. TWO-LOOP INTEGRALS, PRELIMINARIES

There exists one topological class of non-primitive 2-loop p-diagrams, fig. 7a. For our purposes we only have to consider numerators which depend on 6 vectors: $p_1,\ldots,p_5$ and k. Then, recalling that we are always dealing with scalar numerators, we expand scalar products of different vectors as

$$2p_1\cdot p_2 = p_1^2+p_2^2-p_5^2. \tag{4.9}$$

Such relations follow from momentum conservation and reflect the topology of the

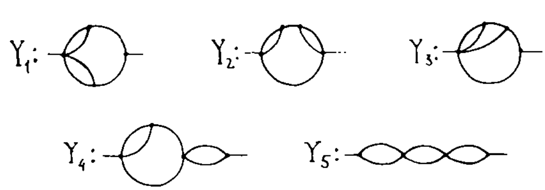

Fig. 6. All different generic graphs of the primitive 3-loop integrals.

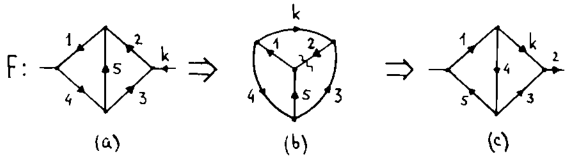

Fig. 7. (a) The only non-primitive generic 2-loop graph. (b) and (c) correspond to the same set of momentum conservation relations as (a).

diagram. For example (see fig. 7a),

$$p_2 + p_5 - p_4 = k, \tag{4.10}$$

whence

$$p_2^2 + p_4^2 + p_5^2 - 2p_4 \cdot p_5 - 2p_4 \cdot p_2 + 2p_2 \cdot p_5 = k^2, \tag{4.11}$$

and, finally,

$$2p_2 \cdot p_4 = p_1^2 + p_3^2 - p_5^2 - k^2. \tag{4.12}$$

Our first observation is that a numerator of the type under consideration can always be disposed of via such relations as eqs. (4.12) and (4.4). In general, however, scalar invariants $p_1^2, \dots, p_n^2$ from the denominator of the diagram and the squared external momentum k^2 do not form a full basis in which to expand scalar products. A simple example is provided by the W_1 graph of fig. 5: the product $k \cdot p_4$ can not be got rid of. Therefore, in general, one should add some "extra" invariants to the basis. The choice of the extra invariants is usually not unique and can be made from the viewpoint of calculational convenience. Thus, the invariant $2k \cdot p_4$ in the case of the W_1 graph of fig. 5 is more convenient than, say, $p_1 \cdot p_4$, as can be understood from appendix A. We shall always indicate a convenient choice of the extra invariants in complicated cases.

Another observation concerns the intrinsic symmetry of the totality of momentum conservation relations. The relations corresponding to fig. 7a can be equivalently expressed via figs. 7b and 7c. Using this symmetry, one can, e.g., immediately expand $2p_5 \cdot k$ by a simple relabelling of vectors in eq. (4.12). In subsequent sections, however, the mappings analogous to the mapping of fig. 7a onto fig. 7c will be shown to play a more important heuristic role in the study of 3-loop p-integrals.

Returning to the 2-loop case, consider the integral associated with the graph of fig. 7a [recall our conventions, eqs. (4.1)–(4.3)],

$$F(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5). \tag{4.13}$$

If at least one α_i is integer $\leqslant 0$, then F reduces to the types W_1 or W_2 of fig. 5. If, say, both α_1 and α_2 are integers $\leqslant 0$, then $F=0$. For general α_i, to our knowledge, eq. (4.13) has not been calculated in a closed form, though a 2-fold infinite series can be obtained if one uses the Slater theorem on integration of generalized hypergeometric functions [18] in the framework of GPXT [5]*. However, for our purposes it is sufficient to consider three types of parameters α_i:

(1) all α_i integer – the 2-loop case proper – the type F_1;

(2) $\alpha_1 = n+\varepsilon$, n and $\alpha_2,\ldots,\alpha_5$ integer – the type F_2;

(3) $\alpha_5 = n+\varepsilon$, n and $\alpha_1,\ldots,\alpha_4$ integer – the type F_3.

The last two cases are needed for 3-loop calculations in sect. 5 (cf. the graphs $R_1 - R_4$ of fig. 9), but it is convenient to discuss them here.

4.3. TYPES F_1 AND F_2: THE RULE OF TRIANGLE

The main tool of our analysis are the recursion relations based on the trick discussed in sect. 3. Let us now examine them in a more formal manner.

Choose an oriented closed circuit C, e.g., consisting of lines 1, 5 and 4 of fig. 7a. This will be written as

$$C = \{+p_1, +p_5, +p_4\}. \tag{4.14}$$

Write,

$$\partial_C = +\frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_5} + \frac{\partial}{\partial p_4}. \tag{4.15}$$

The sign before $\partial/\partial p_i$ is plus, if p_i flows in the direction of C, and minus in the opposite case.

Choose a vector P,

$$P = \sum_i \beta_i p_i + \beta k, \tag{4.16}$$

where β_i and β are arbitrary scalars; e.g., $P = p_5$.

The following identity holds (consult sect. 3, if necessary):

$$0 = (\partial_C \cdot P) F(\{\alpha\}). \tag{4.17}$$

It is implied that differentiation goes *before* integration; it will be always clear from the context what is meant in each case.

* By way of argument, this observation invalidates the assertions of [15], that p-space has advantages over x-space in this class of calculations. Apart from the simplicity which is characteristic of the RG calculations in x-space, there are the 5-loop φ^4 model calculations of ref. [19] and the integral N_0 of fig. 1 of the present paper, that *cannot* be performed without reference to x-space. Anyway, after [19, 5] and the present work the discussion seems unnecessary.

Perform differentiation and expand scalar products in the basis $\{p_i^2, k^2\}$, as in eqs. (4.9) and (4.12), and absorb p_i^2 into F as in eq. (4.4). In this way we get 6 relations, each one being an identity connecting functions F depending on different sets of α_i [see e.g., eq. (4.22)]. These identities are independent in the sense that the corresponding differential operators are independent. Six is the number of independent momenta after momentum conservation is taken into account, e.g., p_3, p_4, and k, multiplied by the number of independent circuits, e.g., $C_1 = \{p_1, p_5, p_4)$ and $C_2 = \{-p_5, p_2, p_3\}$, cf. [20]. The number of independent circuits is equal to the number of independent integration momenta.

Recall now that one knows the dimensionality d_F of F in terms of α_i and ε beforehand, whence three more relations:

$$d_F F = \left(k\frac{\mathrm{d}}{\mathrm{d}k}\right)F = k\left(\frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}\right)F, \tag{4.18}$$

$$(D + d_F)\frac{p_1 \cdot k}{k^2}F = \left(\frac{\mathrm{d}}{\mathrm{d}k}\cdot p_1\right)F = \left(\frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}\right)p_1 F, \tag{4.19}$$

$$(D + d_F)\frac{p_2 \cdot k}{k^2}F = \left(\frac{\mathrm{d}}{\mathrm{d}k}\cdot p_2\right)F = \left(\frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}\right)p_2 F, \tag{4.20}$$

where

$$d_F = 2\left(D - \sum_{i=1}^{5} \alpha_i\right).$$

Eqs. (4.18)–(4.20) can be incorporated in the scheme of eq. (4.17) if one chooses $C = \{k, p_1, p_2\}$.

To summarize, for an l-loop p-integral one can write down $(l+1)^2$ independent identities. Note that the identity

$$0 = \Box_C F \equiv (\partial_C \cdot \partial_C)F = \partial_C \cdot \sum P_j F_j \tag{4.21}$$

follows from eq. (4.17). Note also that the mapping of one generic graph onto another, analogous to fig. 7, induces a one-to-one correspondence between the two sets of vectors P and the two sets of circuits C, which leads to a one-to-one correspondence between the two sets of identities.

At present we do not possess a general method of studying recursions of the described type. However, the rule of triangle, which we are now going to explain, makes life much easier in many cases. Let us write down eq. (4.17) explicitly for

$P=p_5$ and $C=\{p_1, p_5, p_4\}$:

$$F(\alpha_1,\dots,\alpha_5)=(-2\alpha_5-\alpha_1-\alpha_4+D)^{-1}$$
$$\times\{\alpha_1[F(\alpha_1+1,\alpha_2,\alpha_3,\alpha_4,\alpha_5-1)-F(\alpha_1+1,\alpha_2-1,\alpha_3,\alpha_4,\alpha_5)]$$
$$+\alpha_4[F(\alpha_1,\alpha_2,\alpha_3,\alpha_4+1,\alpha_5-1)-F(\alpha_1,\alpha_2,\alpha_3-1,\alpha_4+1,\alpha_5)]\}, \tag{4.22}$$

or in a still more condensed notation*,

$$F(\{\alpha\})=(-2\alpha_5-\alpha_1-\alpha_4+4-2\varepsilon)^{-1}$$
$$\times\{\alpha_1\cdot\mathbf{1}^+[\mathbf{5}^- - \mathbf{2}^-]+\alpha_4\cdot\mathbf{4}^+[\mathbf{5}^- - \mathbf{3}^-]\}F. \tag{4.23}$$

Let now α_2, α_3 and α_5 be positive integer and α_1 and α_4 arbitrary complex. It is clear that applying the substitution (4.23) as many times as necessary, one can express $F(\{\alpha\})$ in terms of integrals with the 5th, 2nd, or 3rd line shrunk to a point, i.e., in terms of primitive integrals of the classes W_1 and W_2 (see fig. 5), which can immediately be turned into products of the G-functions according to eq. (4.8). In this way any diagram of types F_1 and F_2 can be calculated exactly to all orders in ε.

The lesson is that as soon as the diagram contains a subgraph as shown in fig. 8, one can use the analogue of eq. (4.23) to simplify the integral. This recipe will be often used below.

Now some remarks are in order.

(a) Applying eq. (4.23) once, we get,

$$F(\alpha,1,1,\beta,1)=\frac{G(1,1)}{(2-2\varepsilon-\alpha-\beta)}\{\alpha[G(\alpha+1,\beta)-G(\alpha+1,\beta+\varepsilon)]+(\alpha\leftrightarrow\beta)\}. \tag{4.24}$$

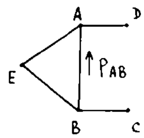

Fig. 8. The subgraph which makes possible the application of the rule of triangle. The lines AD, AB and BC should be integer $\geqslant 0$, each of the vertices A and B should have 3 incidental lines, the vertices D and C may coincide. C and P in eq. (4.17) should be chosen as ABE and P_{AB}, correspondingly.

* The reader is advised to invent some sort of pictorial notation to make the understanding of such equations easier. This has not been done in the text in order to save space and avoid ambiguities due to numerators in subsequent sections.

Historically, this was the first exact result in dimensional regularization for a non-primitive diagram obtained in [4] by GPXT. This formula was rederived in [21]. The latter authors managed to obtain eq. (4.24) via an algebraic procedure, which is more sophisticated than ours. They used the results of [22] and the conformal invariance properties of massless integrals. It is not clear to us, however, whether their method can be applied to 3-loop p-integrals with the same efficiency as integration by parts combined with GPXT.

(b) Combining eq. (4.18) with the following equation obtained from dimensional considerations,

$$d_F(d_F+D-2)F=\frac{\mathrm{d}^2}{\mathrm{d}k_\mu \mathrm{d}k_\mu}F=\left(\frac{\partial}{\partial p_1}+\frac{\partial}{\partial p_2}\right)^2 F, \tag{4.25}$$

one can express $F(\alpha,\gamma,1,1,1)$ through primitive integrals:

$$F(\alpha,\gamma,1,1,1)=\frac{(1-\varepsilon-\alpha-\gamma)}{\varepsilon(1-2\varepsilon)}\{\alpha\gamma(1-\varepsilon-\alpha-\gamma)^{-1}G(1,\alpha+1)G(1,\gamma+1)$$
$$+\left[\alpha G(1,\alpha+1)G(1,\alpha+\gamma+\varepsilon)+(\alpha\leftrightarrow\gamma)\right]\}. \tag{4.26}$$

As far as we know this result is new; the best that, say, GPXT can offer is to represent the l.h.s. of eq. (4.26) as a twofold series [4, 5].

(c) There is considerable ambiguity in the form of representing formulae like eq. (4.24). Thus, in [4, 5] and [21] different formulae were given though both equivalent to eq. (4.24). Even the representation in terms of the G-functions is not unique. Indeed, consider the integral $F(5,1,1,6,2)$. It can be reduced to the G-functions by eq. (4.23) in two steps. But if one rewrites it as $F(1,5,6,1,2)$ (recall the left-right symmetry of fig. 7a), one then has to apply eq. (4.23) to it $(5-1)+(6-1)+(2-1)+1=11$ times. In the case when all α_i are integers the choice can be easily made, and if, say, $\alpha_4=n+\varepsilon$, n and the rest of α_i being integer, then one simply has no choice but to exploit eq. (4.23). But at the 3-loop level, we are not sure we have found the shortest way of reducing diagrams to the G-functions, although the recipe for obtaining the final results in the most compact form would be of great value at the stage of computer calculations.

(d) If $\alpha+\beta=2+\mathrm{O}(\varepsilon)$, then ε^{-1} appears as a factor on the r.h.s. of eq. (4.24). One can easily see that the following general relation expressed in a symbolic form holds*:

$$F=\mathrm{O}(\varepsilon^{-1})G\times G. \tag{4.27}$$

* The absence of higher poles in eq. (4.27) can be seen if one uses the recursions similar to eqs. (4.28) and (4.29) instead of eq. (4.23).

As the analysis of subsect. 5.4 shows, the appearance of such pole factors is a regular feature of integration by parts. This forces one to expand the G-functions in ε to higher order than would be needed otherwise. This fact adds to the problems with time and memory at the stage of computer realization of the algorithm, if one recalls the volume of calculations.

4.4. TYPE F_3

The rule of triangle is not applicable to the 2-loop integrals of the type F_3, i.e., the integrals of the form

$$F(\alpha_1, \alpha_2, \alpha_3, \alpha_4, n+\varepsilon),$$

$\alpha_1, \dots, \alpha_4$ being integers >0, and n an arbitrary integer. In fact, we have not succeeded in expressing such diagrams in terms of the G-functions only, though any such diagram can be reduced to a sum of products of the G-functions and $F(1,1,1,1,1+\varepsilon) \equiv F'(1+\varepsilon)$. The latter function can be calculated with some labour by GPXT to order ε^2, which is sufficient for our purposes, but we prefer, for the sake of symmetry with the non-planar case (sect. 6), to express it in terms of the integral $L_0(\varepsilon)$ of fig. 1. This last step is postponed until subsect 5.4. The reduction to the G-functions and $F'(1+\varepsilon)$ proceeds in two steps. At first a general F_3 type integral is reduced to the G-functions and $F'(n+\varepsilon)$. Then $F'(n+\varepsilon)$ is expressed via $F'(1+\varepsilon)$.

The first step can be done easily* with the recursion obtained from eq. (4.17) by choosing $C = \{p_1, p_4, p_5\}$ and $P = p_1$, and then shifting $\alpha_4 \to \alpha_4 - 1$:

$$(\alpha_4 - 1)F(\{\alpha\}) = \{(2\alpha_1 + \alpha_4 + \alpha_5 - 5 + 2\varepsilon)\mathbf{4}^- + (\alpha_4 - 1)\mathbf{1}^{-1} + \alpha_5[\mathbf{1}^-\mathbf{4}^-\mathbf{5}^+ - \mathbf{2}^-\mathbf{4}^-\mathbf{5}^+]\}F. \tag{4.28}$$

One should apply it until no more non-primitive terms with $\alpha_4 > 1$ are present, then use the left-right and up-down symmetries of fig. 7a and apply eq. (4.28) again until no more non-primitive terms with α_1, α_2, or $\alpha_3 > 1$ are present.

The second step consists in combining eq. (4.18) applied to $F'(\alpha_5)$ and eq. (4.23) applied to $F'(\alpha_5 + 1)$, whence the following relation:

$$0 = (\alpha_5 + \varepsilon)F'(\alpha_5 + 1) + (\alpha_5 - 1 + 2\varepsilon)F'(\alpha_5) + 2(1 - 2\alpha_5 - 3\varepsilon)G(1, 1+\alpha_5)G(1, 1+\alpha_5+\varepsilon). \tag{4.29}$$

A peculiar feature of the derivation is that one always arrives at eq. (4.29) no matter how eqs. (4.17)–(4.20) are combined. In other words, one cannot get a relation

* We are grateful to S.A. Larin for pointing out to us this recursion, which is shorter than the original procedure.

connecting $F'(\alpha_5+1)$ and $F'(\alpha_5)$, which is linearly independent of eq. (4.29), and express $F'(\alpha_5)$ through primitive integrals.

Another important property can be easily observed in eqs. (4.28) and (4.29). Whatever F_3 type diagram different from $F'(1+\varepsilon)$ one takes as an input for the above described procedure, the output always contains $F'(1+\varepsilon)$ multiplied by ε and, perhaps, a rational function of ε regular at $\varepsilon=0$. It is this property, somewhat mysterious (which is not unusual with dimensionally regularized Feynman integrals), that will enable us to prove that only $L_0(\varepsilon=0)$ is needed in calculating any 3-loop p-integral and, consequently, any 4-loop RG function.

5. Three-loop p-integrals: the planar case

Before we plunge into the detail, it is worthwhile to have an overview of what we are to accomplish. In fig. 9 are shown 9 non-primitive, basically different generic graphs for the integrals that can and do occur in calculating arbitrary 4-loop counterterms. Any graph of fig. 9 can be considered as a particular case of the ladder, Mercedez, and non-planar graphs of level I after some lines are shrunk to a point. Recall that the generic graphs show only the topology of integrals. In general, the ith line of a generic graph corresponds to $(p_i^2)^{-\alpha_i}$, α_i being integer and p_i the momentum flowing through this line. In addition, the integral can have a scalar

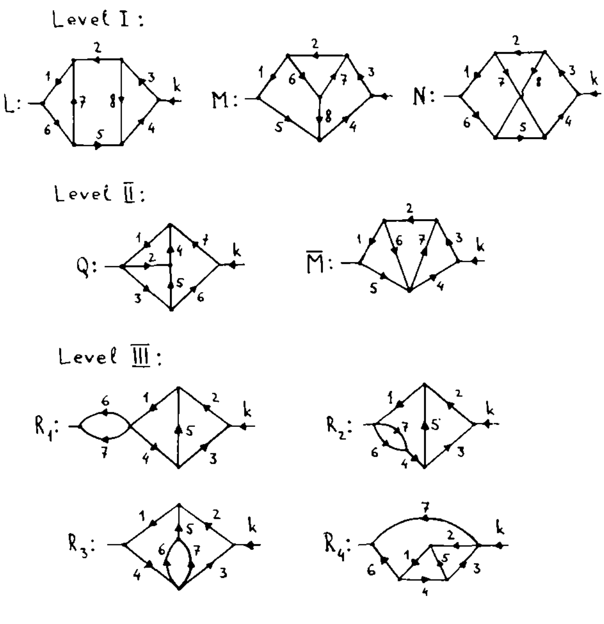

Fig. 9. The generic graphs of the non-primitive 3-loop p-integrals. On performing the one-loop subintegration in the integrals of the classes R_1-R_3 and R_4, one obtains sums of integrals of classes F_1-F_3 and F_1 correspondingly, multiplied by the G-functions (see subsect. 5.3 and appendix B).

polynomial of p_i and of the external momentum k in the numerator. As in subsect. 3.1, such numerators can be expanded in a full set of invariants which consists of p_i^2, k^2, and the extra invariants specified below for each class of diagrams. After this it may happen that either all $\alpha_i > 0$ or, for some i, $\alpha_i \leq 0$. For example, on expanding the numerator of some integral of the class N it may happen that in some term the 6th line becomes non-positive, e.g., $\alpha_6 = 0$. In this case the 6th line shrinks to a point and this term goes over into the class Q with a suitable reordering of the indices (cf., fig. 10),

$$N(\alpha_1,\ldots,\alpha_5,0,\alpha_7,\alpha_8) = Q(\alpha_8,\alpha_1,\alpha_5,\alpha_2,\alpha_7,\alpha_3,\alpha_4). \tag{5.1}$$

If $\alpha_6 < 0$ and/or there is a power of the extra invariant in the numerator, then $(p_6^2)^{|\alpha_6|}$ and the extra invariant should be re-expanded in a new set of invariants corresponding to the class Q.

If, on the other hand, $\alpha_i > 0$, for all i, then one should apply suitable recursion relations as specified in the subsequent sections in order to express this N-class integral through integrals of lower levels. In this way all integrals should be reduced to a sum of primitive ones, those of the classes R_1–R_4, and the two integrals of fig. 1. Table 1 shows which class goes over into which when some line shrinks to a point.

This is only a general scheme. We have yet to specify how to choose the extra invariants in each case, what recursion relations to use when all $\alpha_i > 0$, and what additional information besides the G-functions one needs in order to arrive at the final analytical result. We also should comment upon how to handle integrals of the classes R_i. All these issues will be covered step by step in what follows.

5.1. CLASS L

It is convenient to begin the description of the algorithm with planar diagrams and postpone the discussion of the non-planar case till the very end, because the reduction of the N-class integrals is much more difficult but adds little to general understanding.

Consider an integral of the class L, fig. 9. A convenient choice of the extra invariant is $2p_7 \cdot p_8$. So we write the following general expression (all α_i integer):

$$(2p_7\cdot p_8)^{\alpha_9} L(\alpha_1,\ldots,\alpha_8) \equiv L'(\alpha_1,\ldots,\alpha_8,\alpha_9), \tag{5.2}$$

$$\alpha_1,\ldots,\alpha_8 > 0, \qquad \alpha_9 \geq 0.$$

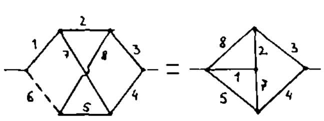

Fig. 10. Shrinking the 6th line of the graph N to a point, one obtains the graph Q with a non-standard numeration of lines.

TABLE 1

Class	Which line shrinks	New class
L	1,6,3,4	R_2
	2,5	$\bar{M}$
	7,8	R_1
N	2,5,7,8	$\bar{M}$
	1,3,4,6	Q
M	1,3	Q
	2	R_3
	4,5	R_4
	6,7	R_2
	8	$\bar{M}$
Q	1,2,3	primitive
	4,5	R_2
	6,7	R_4
$\bar{M}$	1,3	R_2
	2	R_3
	4,5,6,7	primitive

Recalling the rule of triangle of subsect. 4.3, we write the following identity:

$$0=(\partial_C\cdot P)L'(\{\alpha\}),\qquad C=\{p_8,p_4,p_3\},\qquad P=p_8. \tag{5.3}$$

In the explicit form it reads

$$L'(\{\alpha\})=(4-2\varepsilon+\alpha_9-\alpha_3-\alpha_4-2\alpha_8)^{-1}$$
$$\times\{\alpha_4\mathbf{4}^+(\mathbf{8}^--\mathbf{5}^-)+\alpha_3\mathbf{3}^+(\mathbf{8}^--\mathbf{2}^-)\}L'. \tag{5.4}$$

Using eq. (5.4) several times one can obtain that either α_8, α_5, or $\alpha_2=0$, which means that the L-class integral is expressed in terms of the classes R_1 and $\bar{M}$.

However, exploiting magic triangles leads to the accumulation of the factors ε^{-1}, as shows the following realistic example:

$$L'(1,2,1,1,1,1,2,2,2). \tag{5.5}$$

In reducing eq. (5.5) with the help of eq. (5.4) one gets ε^{-2} as a coefficient of some lower level terms. The reason is that the quantity

$$\Delta = 4 + \alpha_9 - \alpha_3 - \alpha_4 - 2\alpha_8 \tag{5.6}$$

oscillates around $\Delta = 0$ as a recursion (5.4) is being repeatedly applied. Obviously, if one could force Δ to vary monotonously, then the dangerous point would be passed over only once.

This can be achieved in the following way. Consider the identity:

$$0 = (\partial_C \cdot P) L'(\{\alpha\}),$$

$$C = \{p_2, -p_8, p_5, -p_7\}, \qquad P = p_7. \tag{5.7}$$

On shifting $\alpha_8 \to \alpha_8 - 1$ and $\alpha_9 \to \alpha_9 - 1$, it can be written explicitly as,

$$(\alpha_8 - 1) L'(\{\alpha\}) = \{2(\alpha_9 - 1)\mathbf{7}^-\mathbf{9}^- - (2\alpha_7 + \alpha_2 + \alpha_5 - \alpha_9 - 3 + 2\varepsilon)$$

$$+ \alpha_2 \mathbf{2}^+ (\mathbf{1}^- - \mathbf{7}^-) + \alpha_5 \mathbf{5}^+ (\mathbf{6}^- - \mathbf{7}^-)\} \mathbf{8}^- \mathbf{9}^- L'. \tag{5.8}$$

Applying eq. (5.8) to eq. (5.2) one arrives at one of the two possibilities: either $\alpha_8 = 1$, or $\alpha_9 = 0$. Now, if one applies eq. (5.4) *after* the recursion (5.8) has been worked out, one will have at worst simple poles in ε in the coefficients of lower level terms. Indeed, if $\alpha_8 = 1$, then Δ decreases by 1 with every application of eq. (5.4), while the terms with $\mathbf{8}^-$ in eq. (5.4), which would otherwise cause Δ to increase, are automatically out of play: α_8 becomes zero in those terms, and they go over into the class R_1. If, on the other hand, $\alpha_9 = 0$, then $\Delta < 0$ with the exception of the case when $\Delta = 0$ with positive α_i, i.e., $\alpha_8 = \alpha_3 = \alpha_4 = 1$. But now Δ increases by 1 each time eq. (5.4) is applied, except in the terms in which the 8th line shrinks to a point.

On the other hand, one cannot get rid of ε^{-1} completely as the example $L'(1,\dots,1,0)$ and eq. (5.5) show.

The result of applying eqs. (5.8) and (5.4) to a general integral of the class L can be symbolically summarized as,

$$L = \mathrm{O}(\varepsilon^{-1})\overline{M} + \mathrm{O}(\varepsilon^{-1})R_1. \tag{5.9}$$

5.2. CLASSES M AND $\overline{M}$; THE RULE OF MAPPING

We choose $2p_2 \cdot k$ to be the extra invariant for the classes M and $\overline{M}$. It is convenient to consider the classes M and $\overline{M}$ simultaneously by allowing α_8 of the class $\overline{M}$ to be non-positive, which is equivalent to choosing the second extra invariant of $\overline{M}$ as $(p_6 - p_7)^2$.

Consider the integral

$$(2p_2 \cdot k)^{\alpha_9} M(\alpha_1,\dots,\alpha_8) \equiv M'(\alpha_1,\dots,\alpha_8,\alpha_9),$$

$$\alpha_1,\dots,\alpha_7 > 0, \qquad \alpha_9 \geqslant 0. \tag{5.10}$$

In order to find suitable recursions we avail ourselves of the intrinsic symmetry introduced in subsect. 4.2. The underlying observation is that both graphs L and M of fig. 9 can be obtained by cutting different lines of the graph of fig. 11b. So, there exits a mapping of the momentum structure of the graph L onto that of the graph M, such that the extra invariant of the class L corresponds to that of M. (It is better to say that the two invariants are *chosen* to be in such a correspondence.) As mentioned in subsect. 4.3 after eq. (4.21), there also exists a one-to-one correspondence between the recursions of the classes L and M.

The heuristic "rule of mapping" prescribes to employ for M the recursions which correspond to those employed for L. Of course, the rule implies that the absence of higher order poles in the coefficients of eq. (5.9) will be carried over to the class M. This can be easily understood if one notes that the recursion of the rule of triangle affects the subgraph shown in fig. 8 in the same way in M and L, i.e., that the number of generated poles in ε depends only on the inner structure of this subgraph.

So, the mapping of fig. 11 transforms the circuit C and the vector P of eq. (5.7) into $C = \{p_1, p_2, p_3, k\}$ and $P = k$. The corresponding identity is

$$d_M M'(\{\alpha\}) = \left(k \cdot \frac{\mathrm{d}}{\mathrm{d}k}\right) M'(\{\alpha\})$$

$$= k \cdot \left(\frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2} + \frac{\partial}{\partial p_3} + \frac{\partial}{\partial k}\right) M'(\{\alpha\}),$$

$$d_M = -2\left(\sum_{i=1}^{8} \alpha_i - \alpha_9 - 6 + 3\varepsilon\right), \tag{5.11}$$

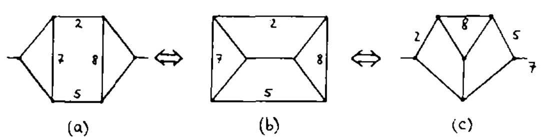

Fig. 11. The mapping of the momentum structure of the class L(a) onto that of the class M(c).

or, explicitly, after shifting $\alpha_2 \to \alpha_2 - 1$ and $\alpha_9 \to \alpha_9 - 1$,

$$(\alpha_2 - 1)M'(\{\alpha\}) = \left\{ \left(-13 + \alpha_1 + \alpha_3 - \alpha_9 + 2\alpha_2 + 2\sum_{i=4}^{8} \alpha_i + 6\varepsilon \right) \right.$$
$$\left. + 2(\alpha_9 - 1)\mathbf{9}^- + \alpha_1 \mathbf{1}^+ (\mathbf{5}^- - 1) + \alpha_3 \mathbf{3}^+ (\mathbf{4}^- - 1) \right\} \mathbf{2}^- \mathbf{9}^- M'. \tag{5.12}$$

Eq. (5.4) goes over into the following:

$$0 = (\partial_C \cdot P)M'(\{\alpha\}),$$
$$C = \{p_2, p_6, p_7\}, \qquad P = p_2, \tag{5.13}$$

or, explicitly,

$$M'(\{\alpha\}) = (4 + \alpha_9 - 2\alpha_2 - \alpha_6 - \alpha_7 - 2\varepsilon)^{-1}$$
$$\times \{\alpha_6 \mathbf{6}^+ (\mathbf{2}^- - \mathbf{1}^-) + \alpha_7 \mathbf{7}^+ (\mathbf{2}^- - \mathbf{3}^-)\} M'. \tag{5.14}$$

Eq. (5.12) should be applied as many times as possible in order to prepare the ground for eq. (5.14) to operate without producing excessive poles in ε. The result can be expressed as

$$M = \mathrm{O}(\varepsilon^{-1})R_3 + \mathrm{O}(\varepsilon^{-1})Q + \mathrm{O}(\varepsilon^{-1})R_2, \tag{5.15a}$$

$$\overline{M} = \mathrm{O}(\varepsilon^{-1})R_3 + \mathrm{O}(\varepsilon^{-1})R_2. \tag{5.15b}$$

5.3. CLASSES Q AND $R_1 - R_4$

Consider now the classes R_1, R_2, and R_3 (fig. 9). Their common feature is that they have a one-loop subintegration (lines 6 and 7 in each case) that can be performed immediately. The two extra invariants in each case can be chosen in the following way: $2p_6 \cdot p_1$ and $2p_6 \cdot p_2$ for R_1 and R_2, and $2p_6 \cdot p_1$ and $2p_6 \cdot k$ for R_3. The subintegration can be performed with the formula (A.1). Having done this, we arrive, correspondingly, at the integrals of the types F_1, F_2, and F_3 which have been analyzed in subsects. 4.3 and 4.4. The final result can be expressed as

$$R_{1,2} = \mathrm{O}(\varepsilon^0)G \times F_{1,2} = \mathrm{O}(\varepsilon^{-1})G \times G \times G, \tag{5.16}$$

and

$$R_3 = \mathrm{O}(\varepsilon^0)G \times F_3. \tag{5.17}$$

As to the class R_4, it is less obvious that such integrals with a numerator consisting of powers of two extra invariants, say, $2p_1 \cdot k$ and $2p_2 \cdot k$, can be reduced to the F_1 type integrals. This fact being demonstrated in appendix C, we write here only the final result:

$$R_4 = O(\varepsilon^0) G \times F_1. \tag{5.18}$$

It should be understood that, in principle, the R_4 integrals can be reduced to the primitives via the rule of triangle despite the presence of a numerator, but then the control over powers of ε will be lost, so that eq. (5.18) can not be established.

The previous remark holds true also for the integrals of the class Q (see fig. 9). This time the handle on the poles in ε again may be found in the rule of mapping, subsect. 5.2. Indeed, the graph Q is connected with the graph $\overline{M}$ via the mapping shown in fig. 12. According to the rule of mapping, one should choose the two extra invariants of the class Q in such a way that they correspond to those of the class $\overline{M}$, then construct two recursions corresponding to eqs. (5.12) and (5.14), and prove by a direct analysis that the result is

$$Q = O(\varepsilon^{-1}) R_2 + O(\varepsilon^{-1}) Y_1 + O(\varepsilon^{-1}) R_4. \tag{5.19}$$

All the details are left to the reader.

5.4. SUMMING UP THE PLANAR CASE

Let us summarize our achievements. Combining the results of sects. 4 and 5, we can write the following symbolic expressions for the integrals of the classes Q, M, $\overline{M}$ and L:

$$Q = O(\varepsilon^{-2}) G \times G \times G, \tag{5.20}$$

$$M = O(\varepsilon^{-3}) G^3 + O(\varepsilon^{-1}) G \times F_3, \tag{5.21a}$$

$$\overline{M} = O(\varepsilon^{-2}) G^3 + O(\varepsilon^{-1}) G \times F_3, \tag{5.21b}$$

$$L = O(\varepsilon^{-3}) G^3 + K \times G \times F_3, \tag{5.22}$$

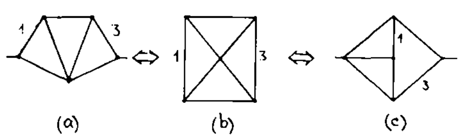

Fig. 12. The mapping of the momentum structure of the class $\overline{M}$(a) onto that of the class Q(c).

where

$$K = O(\varepsilon^{-2}).$$

The first remark concerns the properties of the G-functions defined in appendix A. For the arguments that can occur in practice, the Laurent expansion of a G-function around $\varepsilon = 0$ reads

$$G = \sum_{n=-1}^{\infty} \varepsilon^n C_{G,n}, \tag{5.23}$$

where $C_{G,n}$ contains the Riemann function $\zeta(k)$ with $2 \leqslant k \leqslant n$, the Euler constant γ, and $\ln 4\pi$, within the canonical dimensional regularization [9]. By now it is a well-known fact that γ, $\ln 4\pi$, and $\zeta(2)$ cancel from the RG-functions calculated in the MS scheme and its versions (see appendix A). So, ignoring the F_3 terms in eqs. (5.21) and (5.22) for the moment, we conclude that only rational numbers, $\zeta(3)$, $\zeta(4)$, and $\zeta(5)$ can appear in the 4-loop RG functions in the MS scheme.

In subsect. 4.4 it has been shown that

$$F_3 = C \times F'(1+\varepsilon) + \text{primitives}, \tag{5.24}$$

where $C = O(\varepsilon)$ unless $F_3 = F'(1+\varepsilon)$. On the other hand, using the algorithm described in the preceding sections, one obtains the identity shown in fig. 13 (the inessential powers of the external momentum are suppressed whence the varying dimensionality of different terms). Cast into analytical form, it reads

$$\frac{1}{(4\pi)^2 \varepsilon} G(1,1,0,0) F'(1+\varepsilon) = \frac{1}{3(1-2\varepsilon)} L_0(\varepsilon) + \text{primitives}. \tag{5.25}$$

Noticing that the G-function multiplying F_3 in eqs. (5.21) and (5.22) differs from $G(1,1,0,0)$ by a rational function of ε regular at $\varepsilon = 0$ and that $G(1,1,0,0) = O(\varepsilon^{-1})$, we can immediately combine eqs. (5.25) and (5.26) with eq. (5.21) to obtain,

$$M \text{ or } \overline{M} = O(\varepsilon^{-3}) G^3 + O(\varepsilon^0) L_0(\varepsilon). \tag{5.26}$$

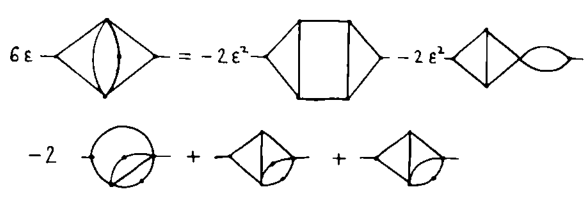

Fig. 13. An identity resulting from integration by parts.

To complete the proof we must show that a similar equation, namely,

$$L = \mathrm{O}(\varepsilon^{-3})G^3 + \mathrm{O}(\varepsilon^0)L_0(\varepsilon), \tag{5.27}$$

holds for the class L. At first sight, the coefficient $K = \mathrm{O}(\varepsilon^{-2})$ in eq. (5.22) as compared with $\mathrm{O}(\varepsilon^{-1})$ in eq. (5.21) plus the possibility of $F_3 = F'(1+\varepsilon)$ in eq. (5.25), do not allow this.

However, the fact is that K can acquire the strongest singularity ε^{-2} only when $F_3 \neq F'(1+\varepsilon)$ and, therefore, C in eq. (5.25) becomes $\mathrm{O}(\varepsilon)$, cancelling the superfluous singularity and leading to eq. (5.27). The full proof is remarkably unilluminative and consists in a lengthy analysis of various inequalities for α_i, the parameters of the integrals involved. But a precise idea of how the mechanism works can be formed through the examination of the way this same cancellation occurs in eq. (5.25). With this remark the proof of our theorem of sect. 2 for the case of planar diagrams is completed.

6. The non-planar case

Let us now consider the non-planar case (fig. 9, the generic graph N). The non-planar 3-loop generic graph is characterized by two features: the inapplicability of the rule of triangle and a very high symmetry. Indeed, apart from the trivial reflexions, N is invariant with respect to the transformation of fig. 14.

Moreover, if one applies the transformations introduced in subsect. 4.2 to the graph N, one always obtains the same graph with a different labelling of the lines. This property demonstrates a high intrinsic symmetry of the totality of momentum conservation relations, and an immediate guess is that the ideas of the rule of mapping (subsect. 5.2) will be particularly useful in this case. We shall see that the guess is right. Another practical consequence is that one is given much freedom in choosing the extra invariant for the graph N: the dot product of any pair of momenta corresponding to non-adjoining lines, e.g., $2p_2 \cdot p_5$ or $2p_2 \cdot k$, can be chosen. We shall use the choice $2p_2 \cdot p_5$.

The strategy will be the following. An arbitrary integral of the form

$$(2p_2 \cdot p_5)^{\alpha_9} N(\alpha_1, \ldots, \alpha_8) \equiv N'(\alpha_1, \ldots, \alpha_8, \alpha_9), \tag{6.1}$$

with $\alpha_1, \ldots, \alpha_9$ integer > 0, will be expressed in terms of the integral with $\alpha_9 = 0$ and

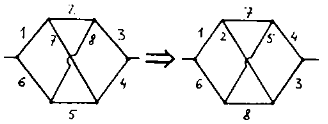

Fig. 14. The additional symmetry of the graph N.

$\alpha_1 = \cdots = \alpha_8 = 1$, which has been denoted as $N_0(\varepsilon)$, plus integrals with one of the internal lines shrunk to a point. These latter have already been taken care of in the preceding sections, and we shall never mention them in what follows in order to avoid repetition of the phrase "plus integrals...to a point". Also, we shall not indicate that $N_0(\varepsilon)$ can be produced by some recursion, since this diagram is fundamental in our approach in the sense that it is not subjected to further treatment by integration by parts. In fact, none of the recursions described below are applicable to it.

We shall achieve our goal in two steps. First, suitable recursions will be applied to eq. (6.1) in order to reduce it to a sum of terms with either

$$(\mathrm{a})\ \alpha_9 = 0, \text{ or } (\mathrm{b})\ \alpha_1 = \cdots = \alpha_8 = 1, \alpha_9 > 0.$$

Second, the two cases will be treated separately until the desired result is obtained.

6.1. THE FIRST STEP

In principle, all that one needs for the first step is eq. (6.5) plus the recursions obtained from it by the rule of mapping of sect. 5.2. In this way, however, one would have to choose a new extra invariant for each new recursion. This would have forced one to re-expand the numerator, which in general is a power $\geqslant 1$ of the extra invariant, in a new set of invariants when going over from one recursion to another. Looking at the example of such a re-expansion provided by eq. (6.6), the reader familiar with how SCHOONSCHIP works will agree readily that this is something to be avoided. Another inconvenience encountered on this way is the generation of the terms with some α_i negative, when a power > 1 of the extra invariant is re-expanded according to eqs. (6.3) or (6.6). A negative α_i, as opposed to $\alpha_i = 0$, means a more complicated numerator with all the after effects caused hereby, e.g., the need for the machinery of appendix A to be applied to some such terms.

However, it is possible to construct a slightly more involved procedure based on precisely the same idea, but which has none of these drawbacks. Let us now give the list of the recursions needed at the first step, in the order in which they should be applied, with brief explanations of their effect.

Consider the equation

$$0 = (\partial_C \cdot P) N'(\alpha_1, \ldots, \alpha_9), \tag{6.2}$$

with $P = p_2$ and $C = \{p_2, p_1, p_6, -p_8\}$. Recalling that

$$2 p_2 \cdot p_6 = 2 p_2 \cdot p_5 + p_2^2 + p_8^2 - p_3^2, \tag{6.3}$$

and shifting $\alpha_6 \to \alpha_6 - 1$, $\alpha_9 \to \alpha_9 - 1$, we get:

$$(\alpha_6 - 1) N'(\{\alpha\}) = \big[(\alpha_6 - 1)(\mathbf{3}^- - \mathbf{2}^- - \mathbf{8}^-) + (3 - 2\varepsilon + \alpha_9 - \alpha_1 - \alpha_8 - 2\alpha_2)\mathbf{6}^- \\ + \alpha_1 \mathbf{1}^+ \mathbf{6}^- (\mathbf{7}^- - \mathbf{2}^-) + \alpha_8 \mathbf{8}^+ \mathbf{6}^- (\mathbf{3}^- - \mathbf{2}^-)\big] \mathbf{9}^- N'. \tag{6.4}$$

Eq. (6.4) is to be applied until $\alpha_9 = 0$ or $\alpha_6 = 1$. Similar recursions are then used to obtain $\alpha_9 = 0$, or $\alpha_1 = \alpha_3 = \alpha_4 = \alpha_6 = 1$ if $\alpha_9 \neq 0$.

Consider now eq. (6.2) with $P = p_2$ and $C = \{p_2, p_7, -p_5, -p_8\}$. After shifting $\alpha_5 \to \alpha_5 - 1$ and $\alpha_9 \to \alpha_9 - 1$ we get,

$$(\alpha_5 - 1)N'(\{\alpha\}) = \big[(2\alpha_2 + \alpha_7 + \alpha_8 - \alpha_9 - 3 + 2\varepsilon) + 2(\alpha_9 - 1)\mathbf{2}^-\mathbf{9}^- + \alpha_7\mathbf{7}^+(\mathbf{2}^- - \mathbf{1}^-) + \alpha_8\mathbf{8}^+(\mathbf{2}^- - \mathbf{3}^-)\big]\mathbf{5}^-\mathbf{9}^- N'. \qquad (6.5)$$

This equation should be applied until $\alpha_9 = 0$ or $\alpha_5 = 1$. Then the symmetric transform of eq. (6.5) with the same C but with $P = p_5$ should be used until $\alpha_9 = 0$ or $\alpha_2 = 1$.

At this point all diagrams with $\alpha_9 \neq 0$ have $\alpha_1 = \cdots = \alpha_6 = 1$.

Consider now eq. (6.2) with $P = p_7$ and $C = \{p_2, p_7, -p_5, -p_8\}$. Noticing that

$$2p_7\cdot p_8 = -2p_2\cdot p_5 + p_1^2 + p_3^2 + p_4^2 + p_6^2 - p_2^2 - p_5^2 - p_7^2 - p_8^2 - k^2, \qquad (6.6)$$

we get our next recursion:

$$\begin{aligned}(\alpha_8 - 1)N'(\{\alpha\}) = \big[&(\alpha_8 - 1)(\mathbf{1}^- + \mathbf{3}^- + \mathbf{4}^- + \mathbf{6}^- - \mathbf{2}^- - \mathbf{5}^- - \mathbf{7}^- - 1)\\ &+ (\alpha_9 - 1)\mathbf{8}^-\mathbf{9}^-(\mathbf{1}^- + \mathbf{4}^- - \mathbf{2}^- - \mathbf{5}^- - 2\cdot\mathbf{7}^-)\\ &+ (5 - 2\varepsilon - \alpha_2 - \alpha_5 - \alpha_8 - 2\alpha_7)\mathbf{8}^-\\ &+ \alpha_2\mathbf{2}^+\mathbf{8}^-(\mathbf{1}^- - \mathbf{7}^-) + \alpha_5\mathbf{5}^+\mathbf{8}^-(\mathbf{4}^- - \mathbf{7}^-)\big]\mathbf{9}^- N'. \qquad (6.7)\end{aligned}$$

In fact, the r.h.s. of eq. (6.7) can be simplified if one notes that the integrals to which this recursion is applied are symmetric with respect to the simultaneous permutations in the following pairs of lines,

$$1 \leftrightarrow 4, 2 \leftrightarrow 5, 3 \leftrightarrow 6, \qquad \text{7 and 8 remain invariant.} \qquad (6.8)$$

Recalling that $\alpha_1 = \cdots = \alpha_6 = 1$ in these integrals, we rewrite eq. (6.7) as

$$\begin{aligned}(\alpha_8 - 1)N'(\{\alpha\}) = \big[&(\alpha_8 - 1)(2(\mathbf{1}^- + \mathbf{3}^- - \mathbf{2}^-) - \mathbf{7}^- - 1)\\ &+ 2(\alpha_9 - 1)\mathbf{9}^-\mathbf{8}^-(\mathbf{1}^- - \mathbf{2}^- - \mathbf{7}^-) + (3 - 2\varepsilon - 2\alpha_7 - \alpha_8)\mathbf{8}^-\\ &+ 2\cdot\mathbf{2}^+\mathbf{8}^-(\mathbf{1}^- - \mathbf{7}^-)\big]\mathbf{9}^- N'. \qquad (6.9)\end{aligned}$$

This recursion and its analogue with $P = p_8$ are to be applied until $\alpha_9 = 0$ or, if $\alpha_9 \neq 0$, then $\alpha_7 = \alpha_8 = 1$.

It is clear, however, that eq. (6.9) generates terms with α_2 or $\alpha_5 > 1$. To such terms eq. (6.5) or its symmetric transform should immediately be applied, and so on. To make sure that this shuttle-like procedure is not infinite we observe the following fact. The terms on the r.h.s. of eq. (6.9) which violate the equalities $\alpha_2 = \alpha_5 = 1$, either are such that α_1 or $\alpha_4 = 0$, which is welcome [recall that eq. (6.9) is only applied to the terms with $\alpha_1 = \alpha_3 = \alpha_4 = \alpha_6 = 1$], or that the quantity $\sigma = \alpha_2 + \alpha_5 + \alpha_7 + \alpha_8$ is reduced by 1 as compared with the l.h.s. of eq. (6.9). Obviously, sooner or later we obtain $\sigma = 4$, which means that either $\alpha_2 = \alpha_5 = \alpha_7 = \alpha_8 = 1$ or one of the lines contributing to σ is shrunk. With this the first step is completed.

6.2. CASE (a): $\alpha_9 \neq 0$ BUT $\alpha_1 = \cdots = \alpha_8 = 1$

Taking into account the symmetries of the integrals in question and the relation,

$$2p_2 \cdot p_5 = p_1^2 + p_3^2 - p_7^2 - p_8^2 - 2p_2 \cdot k, \tag{6.10}$$

we obtain

$$N'(\{\alpha\}) = \left[2(\mathbf{3}^- - \mathbf{8}^-) - 2(k \cdot p_2)\right]\mathbf{9}^- N'. \tag{6.11}$$

On dimensional grounds,

$$\begin{aligned} 2k^\mu\left[p_2^\mu \mathbf{9}^- N'\right] &\equiv 2k^\mu\left[k^\mu (k^2)^{-3-3\varepsilon+\alpha_9} Z\right] \\ &= k^2(\alpha_9 - 1 - 4\varepsilon)^{-1} \frac{\mathrm{d}}{\mathrm{d}k^\mu}\left[k^\mu (k^2)^{-3-3\varepsilon+\alpha_9} Z\right] \\ &= k^2(\alpha_9 - 1 - 4\varepsilon)^{-1} \frac{\mathrm{d}}{\mathrm{d}k^\mu}\left[p_2^\mu \mathbf{9}^- N'\right], \end{aligned} \tag{6.12}$$

where Z is a dimensionless auxiliary.

One can also perform the differentiation *before* integration:

$$\frac{\mathrm{d}}{\mathrm{d}k} = \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2} + \frac{\partial}{\partial p_3}. \tag{6.13}$$

Finally, we get

$$N'(\{\alpha\}) = \left[2(\mathbf{3}^- - \mathbf{8}^-) + (\alpha_9 - 1 - 4\varepsilon)^{-1}\left(2 \cdot \mathbf{3}^+ \mathbf{9}^- (\mathbf{2}^- - \mathbf{8}^-) + 1 + 2\varepsilon - \alpha_9\right)\right]\mathbf{9}^- N' \tag{6.14}$$

Note that if $\alpha_9 = 1$, then the terms on the r.h.s. of eq. (6.14) which could acquire the factor ε^{-1} cancel owing to the symmetry of the integral. With this the discussion of case (a) is completed.

6.3. CASE (b): $\alpha_9 = 0$

Now we turn to the case when $\alpha_9 = 0$ but $\alpha_i > 1$ for some or all $i = 1,\dots,8$. First we put $P = p_2$ and $C = \{p_2, p_7, -p_5, -p_8\}$ in

$$0 = (\partial_C \cdot P) N(\alpha_1,\dots,\alpha_8), \tag{6.15}$$

and find ourselves in a predicament. Indeed, the resulting identity contains a term with a numerator, namely,

$$\alpha_5 (2 p_2 \cdot p_5) \mathbf{5}^+ N(\{\alpha\}), \tag{6.16}$$

and this numerator can not be got rid of by expanding $2p_2 \cdot p_5$ in p_i^2. On the other hand, the initial diagram has no numerator. Such a situation is typical if one does not or can not follow the rule of triangle.

As mentioned above, the absence of triangular subgraphs can partly be compensated for by the high symmetry of the graph N. To see this, we employ eq. (6.15) once more, setting $P = p_1$ and $C = \{p_1, p_6, p_5, -p_7\}$. The r.h.s. of the new identity contains the term

$$\alpha_5 (2 p_1 \cdot p_5) \mathbf{5}^+ N(\{\alpha\}). \tag{6.17}$$

Since

$$2 p_1 \cdot p_5 = 2 p_2 \cdot p_5 - p_4^2 + p_5^2 + p_7^2, \tag{6.18}$$

we exclude the term with the numerator from the two identities to obtain the following result:

$$(\alpha_6 - 1) N(\{\alpha\}) = \big[(\alpha_5 + \alpha_6 + \alpha_8 + 2\alpha_1 + 2\alpha_2 + 2\alpha_7 - 9 + 4\varepsilon)\mathbf{6} \\ + (\alpha_6 - 1)\mathbf{1}^- + \alpha_8 \mathbf{8}^+ \mathbf{6}^- (\mathbf{2}^- - \mathbf{3}^-) + \alpha_5 \mathbf{5}^+ \mathbf{6}^- (\mathbf{7}^- - \mathbf{4}^-) \big] N'. \tag{6.19}$$

This recursion is applied until $\alpha_6 = 1$. Analogous recursions are then used to obtain $\alpha_1 = \alpha_3 = \alpha_4 = 1$.

An important observation consists in that every application of (6.19) reduces the quantity $\sigma = \alpha_1 + \cdots + \alpha_8$ by 1.

Without loss of generality we may now consider the case when $\alpha_5 > 1$. From dimensional considerations one has

$$\left(k \frac{\mathrm{d}}{\mathrm{d}k} \right) N(\{\alpha\}) = 2 \left(6 - \sum_{i=1}^{8} \alpha_i - 3\varepsilon \right) N(\{\alpha\}). \tag{6.20}$$

Performing differentiation before integration,

$$\frac{\mathrm{d}}{\mathrm{d}k}=\frac{\partial}{\partial k}-\frac{\partial}{\partial p_4}-\frac{\partial}{\partial p_5}-\frac{\partial}{\partial p_6}, \tag{6.21}$$

one gets the following term on the l.h.s. of eq. (6.20):

$$\alpha_5(2k\cdot p_5)\mathbf{5}^+N(\{\alpha\})=\alpha_5\left(-7+\sum_{i=1}^{8}\alpha_i+4\varepsilon\right)^{-1}\left(-\frac{\mathrm{d}}{\mathrm{d}k}\cdot p_5\right)\mathbf{5}^+N(\{\alpha\}). \tag{6.22}$$

[This equation has been obtained through counting dimensions as in eq. (6.12).] Using eq. (6.21) once more, we arrive at the following relation, after shifting $\alpha_5\to\alpha_5-1$:

$$\begin{aligned}&(\alpha_5-1)(\alpha_4+\alpha_6+2\alpha_5-4+2\varepsilon)N(\{\alpha\})\\&\quad=\Big[d\big((2d+2-2\varepsilon-\alpha_4-\alpha_6)+\alpha_4\mathbf{4}^+\mathbf{3}^-+\alpha_6\mathbf{6}^+\mathbf{1}^-\big)\mathbf{5}^-\\&\qquad+(\alpha_5-1)(\alpha_4\mathbf{4}^+\mathbf{7}^-+\alpha_6\mathbf{6}^+\mathbf{8}^-)-(d+\alpha_5-1)(\alpha_4\mathbf{4}^++\alpha_6\mathbf{6}^+)\mathbf{5}^-\Big]N,\end{aligned} \tag{6.23}$$

where

$$d=\sum_{i=1}^{8}\alpha_i-8+4\varepsilon.$$

Recall that this relation is intended to be applied only to the terms with $\alpha_1=\alpha_3=\alpha_4=\alpha_6=1$.

There are two kinds of terms on the r.h.s. of eq. (6.23), apart from those with a line shrunk to a point, namely, the terms with $\sigma=\alpha_1+\cdots+\alpha_8$ reduced by 1 as compared with the l.h.s., and those with σ unchanged but with $\sigma'=\alpha_2+\alpha_5+\alpha_7+\alpha_8$ reduced by 1. To the latter terms of eq. (6.19) should immediately be applied, with the result of reducing σ by 1 in those terms, and so on. It should be clear that the process cannot last forever. Case (b) is settled.

To summarize, in this section we have constructed a set of relations which enables one to express an arbitrary non-planar diagram with a scalar numerator in terms of $N_0(\varepsilon)$ and simpler diagrams with some line shrunk to a point, whose evaluation has already been discussed in sects. 4 and 5. With this the proof of our main theorem formulated in sect. 2 is completed.

7. More complicated example

The possibilities of the integration by parts method are not exhausted by the algorithm constructed in the preceding sections. Thus, there exist rather complicated 5-loop integrals to which the above described ideas can be applied with great effect, though neither the rule of triangle of subsect. 4.4, nor the rule of mapping of subsect. 4.2, are applicable there.

Consider the calculation of the 5-loop correction to the field anomalous dimension γ_2 of the φ^4 model which has important applications to the calculation of critical exponents governing the behaviour of statistical systems like ferromagnets near the critical point (for a review see [3]). There are 11 different diagrams contributing to γ_2 at the 5-loop level. 10 of them were computed analytically in [19]. After application of the operator $\mathrm{d}^2/\mathrm{d}k^\mu \mathrm{d}k^\mu$ the 11th diagram shown in fig. 15a transforms into the sum of two diagrams (figs. 15b and c). The second one did not yield to analytical calculation [19].

We are interested in the pole part of this diagram, which contributes to γ_2. Since this diagram is logarithmically divergent and does not contain divergent subgraphs we may reattach the external momentum k as shown in fig. 16a without changing the divergent part. (This is a simple example of the IR rearrangement, see sect. 2.) So we have to find the pole part of the following expression*:

$$\begin{aligned} &4G(1,1+4\varepsilon)\{(2p_1\cdot p_2)S(2,2,1,1,1,1,1,1,\varepsilon)\} \\ &= \frac{4}{5\varepsilon}\{(2p_1\cdot p_2)S(2,2,1,1,1,1,1,1,\varepsilon=0)\} + \mathrm{O}(\varepsilon^0) \\ &= \frac{A}{\varepsilon} + \mathrm{O}(\varepsilon^0), \end{aligned} \tag{7.1}$$

where the symbol $S(\{\alpha\})$ stands for the value of the generic integral pictured in fig. 16b, with the factor $(2p_1\cdot p_2)$ multiplying the corresponding *integrand*. For the sake

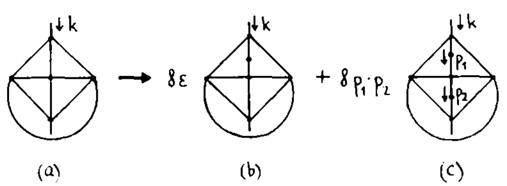

Fig. 15. (a) The most complicated diagram of the φ^4 model contributing to the anomalous dimension of the field $\gamma_2(g)$ at the 5-loop level; (b) and (c) are the diagrams resulting from double differentiation with respect to K.

* In this section all calculations are performed within the G-scheme (see appendix A and refs. [5, 13]).

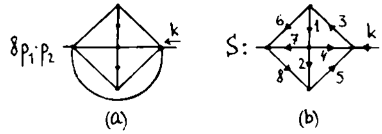

Fig. 16. (a) The result of application of the IR rearrangement to the diagram (c) of fig. 15. (b) The generic graph for the diagram of the class S.

of brevity we shall omit some last α_i which are equal to 1, e.g.,

$$S(2,2,1,1,1,1,1,1) \to S(2,2),$$

etc. and suppress, as usual, all inessential powers of the squared external momenta k^2.

A little meditation upon the integral $(2p_1 \cdot p_2)S(2,2)$ reveals that it can not be simplified by the rule of triangle, nor can the annoying numerator* be transformed away by means of the tricks used in sect. 6 in the case of non-planar diagrams. Nevertheless, the $O(\varepsilon^0)$ term of the diagram can be computed via an involved procedure using integration by parts.

To find A let us consider a simple identity resulting from momentum conservation:

$$(2p_1 \cdot p_2)S(2,2) = (2p_1 \cdot p_2)(2p_1 \cdot p_3)S(2,2,2) - (2p_1 \cdot p_2)S(1,2,2) + (2p_1 \cdot p_2)S(2,2,2,1,1,0). \quad (7.2)$$

Now the last term on the r.h.s. of eq. (7.2) is easily computed by the methods of the preceding sections with the following result:

$$(2p_1 \cdot p_2)S(2,2,2,1,1,0) = (4\pi)^{-8}\left(-\frac{6\zeta(3)}{\varepsilon} + 28\zeta(3) - 9\zeta(4) + O(\varepsilon) \right). \quad (7.3)$$

The application of the operator $d^2/dk^\mu dk^\mu$ to the integral $(2p_1 \cdot p_2)S(1,2)$ gives

$$-16\varepsilon(1-5\varepsilon)(2p_1 \cdot p_2)S(1,2) = 8(1+\varepsilon)(2p_1 \cdot p_2)S(1,2,2) - 4(2p_1 \cdot p_2)S(0,2,2,1,1,2). \quad (7.4)$$

The last integral on the r.h.s. of eq. (7.4) is computed by straightforward use of eq. (4.26):

$$-4(2p_1 \cdot p_2)S(0,2,2,1,1,2) = \frac{8}{(4\pi)^8}\left(\frac{6\zeta(3)}{\varepsilon} + 9\zeta(4) - 24\xi(3) + O(\varepsilon) \right). \quad (7.5)$$

* Annoying because the same integral without numerator can be dealt with the aid of GPXT of ref. [5].

On the other hand, a simple calculation based on the IR rearrangement and the algorithm of sects. 3–6 yields

$$(2p_1\cdot p_2)S(1,2)=(4\pi)^{-8}\left(\frac{2\zeta(3)}{\varepsilon}+O(\varepsilon^0)\right). \tag{7.6}$$

Combining eqs. (7.4)–(7.6) we get,

$$(2p_1\cdot p_2)S(1,2,2)=(4\pi)^{-8}\left(-\frac{\zeta(3)}{\varepsilon}+26\zeta(3)-9\zeta(4)+O(\varepsilon)\right). \tag{7.7}$$

It is much more difficult to compute $(2p_1\cdot p_2)(2p_1\cdot p_3)S(2,2,2)$ in order to complete the evaluation of A. First of all we use the identity $p_1=p_2+p_4+p_7$ to get

$$\begin{aligned}
&(2p_1\cdot p_2)(2p_1\cdot p_3)S(2,2,2)\\
&\quad=(2p_1\cdot p_3)\left[2S(2,1,2)+((2p_2\cdot p_4)+(2p_2\cdot p_7))S(2,2,2)\right]\\
&\quad=\frac{\partial^2}{\partial p^3_\mu}\left[S(1)+((p_2\cdot p_4)+(p_2\cdot p_7))S(1,2)\right].
\end{aligned} \tag{7.8}$$

To proceed, let us consider the following identity:

$$\left(\frac{\partial}{\partial p_1^\mu}+\frac{\partial}{\partial p_3^\mu}\right)^2 S=\left(\frac{\partial}{\partial p_4^\mu}\right)^2 S, \tag{7.9a}$$

or, equivalently,

$$2\frac{\partial^2}{\partial p_1^\mu\partial p_3^\mu}S=(\Box_4-\Box_1-\Box_3)S,$$

$$\Box_i=\partial^2/\partial p_i^\mu\partial p_i^\mu, \tag{7.9b}$$

which is true for *any* integral S (recall that the chain p_1, p_3, p_4 is closed, see fig. 16b).

The identity (7.9b) plus some algebra gives the following decomposition for the r.h.s. of eq. (7.8):

$$\begin{aligned}
&(2p_1\cdot p_2)(2p_1\cdot p_3)S(2,2,2)\\
&\quad=-\varepsilon(2p_1\cdot p_2)S(2,2)\\
&\qquad+\varepsilon(2p_1\cdot p_2)\left[S(1,2,1,2)-S(1,2,2)\right]-(2p_2\cdot p_4)S(1,2,1,2).
\end{aligned} \tag{7.10}$$

Since the first term on r.h.s. of eq. (7.10) vanishes at $\varepsilon \to 0$ one has to compute only the remaining three integrals.

To compute, say, $\varepsilon(2p_1 \cdot p_2)S(1,2,1,2)$ we observe that the corresponding integral can be written as

$$\int \mathrm{d}^D p_4 \frac{\varepsilon}{p_4^4} f(p_4, k).$$

Now, one can perform integration over p_4 immediately by means of the following formula:

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{p^4} = -\pi^2 \delta^{(D)}(p), \tag{7.11}$$

provided that the remaining integral $f(p_4 = 0, k)$ is finite at $\varepsilon \to 0$, which is true in our case. So, one gets

$$\varepsilon(2p_1 \cdot p_2)S(1,2,1,2) = -\pi^2 f(p_4 = 0, k, \varepsilon) + \mathrm{O}(\varepsilon). \tag{7.12}$$

Note that although we are interested in the value of $f(p_4 = 0, k, \varepsilon)$ at $\varepsilon = 0$ it is convenient to keep $\varepsilon \neq 0$ till the end of the calculation (cf. remark (iii) of sect. 8).

Now one has to compute the 3-loop p-integral $f(p_4 = 0, k)$ up to $\mathrm{O}(\varepsilon^0)$, which can be easily done with the algorithm of the preceding sections. The same treatment is applicable to the integral $\varepsilon(2p_1 \cdot p_2)S(1,2,2)$. The results are

$$\varepsilon(2p_1 \cdot p_2)S(1,2,1,2) = (4\pi)^{-8}(-20\zeta(5) + 4\zeta(3) + \mathrm{O}(\varepsilon)), \tag{7.13a}$$

$$\varepsilon(2p_1 \cdot p_2)S(1,2,2) = (4\pi)^{-8}(-6\zeta(3) + \mathrm{O}(\varepsilon)). \tag{7.13b}$$

Note that eq. (7.13b) is in agreement with eq. (7.7).

To compute the last integral on the r.h.s. of eq. (7.10) we first use the identity analogous to eq. (7.9):

$$\begin{aligned}(2p_2 \cdot p_4)S(1,2,1,2) &= \frac{1}{2} \frac{\partial^2}{\partial p_2^\mu \partial p_4^\mu} S(1) \\ &= -\varepsilon S(1,1,1,1,2) + \varepsilon S(1,2) + \varepsilon S(1,1,1,2). \end{aligned} \tag{7.14}$$

Now it is not difficult to see that eq. (7.11) can be applied to the r.h.s. of eq. (7.14) due to the fact that the IR divergences produced by such a treatment in the last two integrals, cancel each other. A simple calculation gives,

$$(2p_2 \cdot p_4)S(1,2,1,2) = (4\pi)^{-8}(-20\zeta(5) + \mathrm{O}(\varepsilon)). \tag{7.15}$$

Finally, combining eqs. (7.2), (7.3), (7.7), (7.10), (7.13) and (7.15) we come to

$$(2p_1 \cdot p_2)S(2,2) = (4\pi)^{-8}(12\zeta(3) + O(\varepsilon)),$$

whence the long-awaited result:

$$A = \tfrac{48}{5}\zeta(3) = 11.5392\ldots, \tag{7.16}$$

in agreement with the numerical computation of ref. [19].

We hope this example demonstrates clearly enough both the calculational possibilities hidden in space-time structures of Feynman diagrams and the need for regular methods to study them.

8. Discussion

There are various points we would like to discuss in connection with the algorithm constructed in sects. 4–6.

(i) This algorithm is not unique. It is quite possible that a shorter set of recursions will be found. It would also be extremely useful to solve these recursions explicitly in terms of Γ-functions. This, however, seems to be a difficult task, not only because the resulting series of products of Γ-functions are complicated and unwieldy, but also because a direct summation of such series by the formulae given, e.g., in [23] seems to be impossible. To support this statement we refer the reader to remark (c) after eq. (4.26) where it has been shown how two different representations for the same integral can be obtained. We do not know at present how to prove their equivalence by a direct resummation, without reference to the original integral.

It is also interesting to note that the coefficient C_L in eq. (2.2) has happened to be equal to zero in all examples of non-planar diagrams considered by us so far. The fact seems extremely amusing and appealing, but we have no proof of this property being always valid.

All these observations leave the impression of something important having been missed in our analysis, that could be very useful for both practical purposes and a better understanding of perturbative series, if these two things can be separated.

(ii) There is another problem which is not solved at present. Even if one is content with the algorithm in the form of recursion relations as opposed to a set of explicit closed formulae, there exists no recipe to construct recursions for evaluating the p-integrals, e.g., at the 4-loop level, which corresponds to the 5-loop RG functions. The method of "close examination" employed in this work is of little use in more complicated cases. Suffice it to say that it took some months to find the right way of computing the 5-loop diagram of fig. 15c (sect. 7) though the actual calculation had been performed in several hours. Also, as yet there is no criterion to decide whether a given 4-loop p-integral can be reduced to simpler ones via integration by parts, or whether it should be evaluated by means of independent methods, of which at present only one seems to be efficient enough – the Gegenbauer polynomial technique in x-space [5].

It is also not clear if there is any comprehensible connection between the lagrangian structure of diagrams in the input of the algorithm and the numbers in the output. It is quite possible that the use of this additional "lagrangian" information might be the clue to constructing less universal but more efficient algorithms.

(iii) Another observation concerns the method of dimensional regularization. There are two points where its role is crucial: the IR rearrangement and integration by parts. Indeed, no regularization, to our knowledge, except the dimensional one offers a natural definition of a subtraction scheme which gives so much flexibility in calculating divergent parts of diagrams as does the MS scheme (sect. 2). On the other hand, eq. (3.4) clearly shows how important is the dimensional regularization for the algorithm: at $\varepsilon = 0$ the l.h.s. of eq. (3.4) vanishes and its r.h.s. becomes meaningless due to the presence of both UV and IR divergences.

Such a situation is somewhat unusual. In fact, all methods of evaluating Feynman diagrams developed so far are but more or less straightforward extrapolations of methods designed to deal with convergent unregularized expressions, to the case of regularized integrals. Consequently, all orthodox methods, including GPXT, demonstrate nothing new when applied to convergent integrals with regularization removed.

Our algorithm is very different in this respect: it seems impossible to apply it when $D = 4$, yet the algorithm turns out to be remarkably efficient when D becomes a complex parameter, even in the case of convergent integrals, cf. eq. (3.4). In a sense the method of dimensional regularization plays the same role in the calculation of Feynman integrals as the complex analysis does in the calculation of ordinary integrals.

(iv) Now it is convenient to clarify the relation of the present paper with a work, in which integration by parts was used with the purposes similar to ours. We mean ref. [24].

In fact, the work of Bender et al. provided us with a hint of the idea lying in the foundation of the present paper. But before the idea took its final shape, the rule had been applied, which proved its fruitfulness in [4, 5]. The rule boils down to the following: take a method of calculating Feynman diagrams in momentum space and see if it looks more natural in position space, and, if so, enjoy the results. We believe the rule is not yet exhausted of its heuristic power, though its application may not be easy, as the present work shows.

The basic difference between our approach and that of [24], consists in that in [24] the momentum space relation for the massless propagator

$$\Box \frac{1}{p^2} = -4\pi^2 \delta^{(4)}(p), \tag{8.1}$$

which is only valid at $D=4$, plus the simple rule of integrating δ-functions, were taken as a starting point of the method, while in our algorithm the relation

$$\Box \frac{1}{(x^2)^{D/2-1}} = -2(D-2)\frac{\pi^{D/2}}{\Gamma(\frac{1}{2}D)}\delta^{(D)}(x) \tag{8.2}$$

plus the rules of integrating primitive diagrams (appendix A) are used. Eq. (8.1) corresponds to deleting a line of the Feynman graph, while eq. (8.2) corresponds to shrinking one.

As a result, our algorithm works naturally in the context of dimensional regularization and is therefore capable of dealing with divergent integrals, while the method of [24] is not, which imposes a severe restriction on its range of applicability.

From a technical point of view, the method of [24] leads to a system of linear equations (16 equations in the case of 3-loop counterterms), while our recursions provide, in a sense, a solution to it. Also, in our method irrationalities like $\zeta(3)$ can appear in the final results while the method of [24] always leads to rational numbers, being, therefore, inapplicable to computing diagrams with irrationalities in the result.

Finally, being much more powerful than the method of [24], the algorithm described here lacks one attractive feature of [24]. In that work, the hidden symmetry of two-loop integrals discovered in [25] (see also the discussion in appendix C) was put to use in order to considerably reduce the volume of calculations. We do not know yet, whether such a symmetry can be incorporated into our algorithm, though the idea seems promising.

(v) In [5] a modification of the MS-scheme was introduced – the G-scheme – which was found to be convenient in practical calculations. Referring the reader to appendix A for the definition and to refs. [5, 13] for further details, we only point out that since primitive integrals are products of the G-functions, eq. (2.2) [see also eqs. (5.26) and (5.27)] makes the G-scheme yet more useful and natural in calculations. Of course, its usefulness in phenomenology of QCD is another question.

(vi) Our algorithm is almost useless without fast computer programs. The first reason is that there are virtually thousands of different elementary integrals which must be calculated in order to obtain, say, the 4-loop QED β-function, not to mention QCD, plus the proliferation of terms at the intermediate steps. The second reason is that the algorithm consists of uniform primitive but rather numerous operations, which makes it unpleasant to a human's taste. But not to computer's and, to be sure, not to programmer's. (In fact, it was the modest desire to ease the programmer's lot as much as possible that was the initial concern in performing the present work – the results, however, surpassed all expectations). This is why we believe the algorithm provides a good basis for writing a set of programs in, say, the SCHOONSCHIP language [17], though the memory needed to store intermediate results and the time to process them are considerable by standards of the CDC-6500 machine. This contrasts strikingly with the fact that the final result even for the most complicated diagram never contains more than a dozen rational numbers.

(vii) There exist different ways to generalize our algorithm; e.g., one could try to carry out a similar analysis for the 5-loop RG functions. But as we have indicated above, this is a difficult task. Besides, already at the 4-loop level the number of different Feynman diagrams contributing to the RG functions is such that the evaluation of, say, the 4-loop QCD β-function is hardly possible at present. That is to say, the problem of handling the profusion of diagrams becomes decisive there.

As to the diagrams with masses or more than one external momentum, nothing definite can be said at present except that identities like eq. (3.2a) remain valid, and *some* simplification is also possible in those cases. But that is another story.

9. Conclusion

In this work the algorithm is presented, of analytical evaluation of arbitrary dimensionally regularized massless Feynman integrals with one external momentum and no more than 3 internal integrations, up to and including the $O(\varepsilon^0)$ terms, $\varepsilon = \frac{1}{2}(4-D)$, D being the space-time dimension. The algorithm is based on recursion relations obtained through integration by parts. The effect of the algorithm consists in substituting a given integral by a sum of simpler ones, each being calculable by the one-loop integration formulae of appendix A, plus the two integrals of fig. 1, which have been calculated by the Gegenbauer polynomial x-space technique to order ε^0. The algorithm was conceived to form a convenient basis to write programs for computer systems of algebraic manipulations.

Together with the method of the IR rearrangement (sect. 2) the algorithm guarantees that the counterterm of an arbitrary 4-loop diagram can be calculated within the MS-scheme, in terms of rational numbers, $\zeta(3)$, $\zeta(4)$, and $\zeta(5)$. This implies the analytical calculability of the β-functions and anomalous dimensions of field and composite operators in an arbitrary model at the 4-loop level.

We are grateful to Profs. V.A. Matveev and A.N. Tavkhelidze for continuing support and A.N. Vassiliev for informing us of ref. [21] prior to publication. We also wish to express our gratitude to S.G. Gorishny, A.L. Kataev and S.A. Larin for innumerable fruitful discussions of the computer implementation of the algorithm.

Appendix A

Here we are going to discuss at some length the evaluation of one-loop integrals with one external momentum and with arbitrary numerators. Such calculations have been performed in the simplest cases by many authors. In our method, however, one-loop integrals acquire major importance since *any* integral is finally expressed through them. Besides, the volume of calculations is such that one is not likely to fight one's way through vector algebra and Γ-functions without being properly equipped: SCHOONSCHIP alone does not suffice, as the preliminary tests have shown.

The tactics that has already been advocated [5, 13] consist in expressing primitive integrals (defined in subsect. 4.1) in terms of the G-functions which themselves are aggregates of the familiar Γ-functions [eqs. (A.1) and (A.2)]. Each loop gives one G-function. Then one transforms every G-function into a rational function of ε multiplied by a G-function of the standard arguments [eq. (A.3)] according to simple rules. At the very last stage of calculations these standard G-functions are expanded to a sufficiently high order in ε with the formulae that can be prepared in advance. Referring the reader to [5, 13] for further details, we concentrate here on what is the novel feature of multiloop calculations – the need for efficient programmable methods of handling complicated numerators.

We begin with a well-known (at least for lowest values of n) formula written in a new form:

$$\int \frac{\mathcal{P}_n(p)\,\mathrm{d}^D p}{p^{2\alpha}(p-k)^{2\beta}(2\pi)^D} = \frac{(k^2)^{2-\varepsilon-\alpha-\beta}}{(4\pi)^2} \times \sum_{\sigma \geqslant 0} G(\alpha,\beta,n,\sigma) k^{2\sigma} \left\{ \frac{1}{\sigma!} \left(\frac{\Box_p}{4} \right)^\sigma \mathcal{P}_n(p) \right\} \Bigg|_{p=k}, \tag{A.1}$$

where $\Box_p = \partial^2/\partial p_\mu \partial p_\mu$, and

$$G(\alpha,\beta,n,\sigma) = (4\pi)^\varepsilon \frac{\Gamma(\alpha+\beta-\sigma-2+\varepsilon)}{\Gamma(\alpha)\Gamma(\beta)} B(2-\varepsilon-\alpha+n-\sigma, 2-\varepsilon-\beta+\sigma). \tag{A.2}$$

$\mathcal{P}_n(p)$ is an arbitrary polynomial in p such that $\mathcal{P}_n(\lambda p) = \lambda^n \mathcal{P}_n(p)$. It can also depend on other vectors, e.g., $\mathcal{P}_5(p) = p^2(p\cdot k)(p\cdot q)p^\mu$.

The "standard" form of the G-functions is

$$G(1+m\varepsilon, 1+n\varepsilon, 0, 0) = (4\pi)^\varepsilon \frac{\Gamma((m+n+1)\varepsilon)}{\Gamma(1+m\varepsilon)\Gamma(1+n\varepsilon)} B(1-(m+1)\varepsilon, 1-(n+1)\varepsilon). \tag{A.3}$$

α and β in eq. (A.1) usually have the form $n + m\varepsilon$, n and m being integer. A very important property of the G-functions is that no G-function can have a singularity in

ε stronger than ε^{-1}. The definition of the G-scheme consists in normalizing momentum integrations in such a way that $G(1,1,0,0)=\varepsilon^{-1}$, which immediately excludes γ, $\ln 4\pi$, and $\zeta(2)$ from all expansions near $\varepsilon=0$ [5].

Eq. (A.1) can be obtained through Feynman parameters if one takes $\mathcal{P}_n(p)=(p\cdot q)^n$ and then notes that eq. (A.1) is independent of the actual choice of $\mathcal{P}_n$.

But if one has to perform more than 2 successive one-loop integrations it may be often more expedient to expand $\mathcal{P}_n(p)$ in harmonic polynomials:

$$\mathcal{P}_n(p)=\mathcal{P}^{(n)}(p)+d_1p^2\mathcal{P}^{(n-2)}(p)+\cdots, \tag{A.4}$$

where

$$\Box\mathcal{P}^{(i)}(p)=0, \qquad \mathcal{P}^{(i)}(\lambda p)=\lambda^i\mathcal{P}^{(i)}(p). \tag{A.5}$$

$\mathcal{P}^{(n)}(p)$ is called the harmonic projection of $\mathcal{P}_n(p)$ [26]. Note that if $\mathcal{P}_n(p)=p^{\mu_1}\cdots p^{\mu_n}$ then its harmonic projection,

$$\mathcal{P}^{(n)}(p)\equiv p^{(\mu_1\cdots\mu_n)}\equiv p^{\mu_1}\cdots p^{\mu_n}+c_1p^2g^{\mu_1\mu_2}p^{\mu_3}\cdots p^{\mu_n}+\cdots, \tag{A.6}$$

equals zero if any pair of indices is contracted:

$$p^{(\nu\nu\mu_3\cdots\mu_n)}=0. \tag{A.7}$$

This will be clear from the representation we now proceed to derive.

As in deriving eq. (A.1) we take

$$\mathcal{P}_n=(2p\cdot q)^n \tag{A.8}$$

without loss of generality. (We prefer to work with $2p\cdot q$ and $\frac{1}{4}\Box$ as single entities in order to avoid a tiresome counting of powers of 2, and an unnecessary arithmetic during computer calculations). The harmonic projection of $\mathcal{P}_n(p)$ is

$$\mathcal{P}^{(n)}(p)\equiv(2p\cdot q)^{(n)}=(2p\cdot q)^n+c_1p^2q^2(2p\cdot q)^{n-2}+\cdots. \tag{A.9}$$

Imposing the condition (A.5) one finds c_i,

$$\begin{aligned}(2p\cdot q)^{(n)}&=\sum_{\sigma\geqslant 0}(n+1-\varepsilon)_{-\sigma}(-)^\sigma\frac{(p^2q^2)^\sigma n!}{\sigma!(n-2\sigma)!}(2p\cdot q)^{n-2\sigma}\\&=\left\{\sum_{\sigma\geqslant 0}(n+1-\varepsilon)_{-\sigma}\frac{(-)^\sigma p^{2\sigma}}{\sigma!}\left(\frac{\Box_p}{4}\right)^\sigma\right\}(2p\cdot q)^n,\end{aligned} \tag{A.10}$$

where $(a)_b=\Gamma(a+b)/\Gamma(a)$ is the Pochhammer symbol [23].

It should be clear from eq. (A.10) that in order to get the harmonic projection $\mathcal{P}^{(n)}(p)$ of an arbitrary polynomial $\mathcal{P}_n(p)$, one only has to apply to it the following operator:

$$H_p^n = \sum_{\sigma \geqslant 0} (n+1-\varepsilon)_{\sigma} \frac{(-)^{\sigma} p^{2\sigma}}{\sigma!} \left(\frac{\Box}{4}\right)^{\sigma}. \tag{A.11}$$

Now eq. (A.7) follows from the identity:

$$H_p^n p^2 \mathcal{P}_{n-2}(p) = 0. \tag{A.12}$$

Since $(2p\cdot q)^{(n)}$ can be seen to be proportional to the Gegenbauer polynomial $C_n^{1-\varepsilon}(p\cdot q/(p^2q^2)^{1/2})$, one gets a useful equation from the well-known properties of $C_n^{1-\varepsilon}$:

$$(2p\cdot p)^{(n)} = p^{2n} \frac{(2-2\varepsilon)_n}{(1-\varepsilon)_n}. \tag{A.13}$$

To obtain the unknown coefficients in eq. (A.4) we follow the same pattern. Start with the equation,

$$(2p\cdot q)^n = (2p\cdot q)^{(n)} + d_1 p^2 q^2 (2p\cdot q)^{(n-2)} + \cdots, \tag{A.14}$$

apply $\Box_p$ to its both sides, then expand its l.h.s. in harmonic polynomials, and come to the result:

$$(2p\cdot q)^n = \sum_{\sigma \geqslant 0} \frac{1}{(n+2-\varepsilon)_{-\sigma}} \frac{(p^2q^2)^{\sigma} n!}{\sigma!(n-2\sigma)!} (2p\cdot q)^{(n-2\sigma)}. \tag{A.15}$$

The generalization of eq. (A.15) to arbitrary $\mathcal{P}_n(p)$ reads:

$$\mathcal{P}_n(p) = \sum_{\sigma \geqslant 0} \frac{1}{(n+2-\varepsilon)_{-\sigma}} p^{2\sigma} H_p^{(n-2\sigma)} \left\{ \frac{1}{\sigma!} \left(\frac{\Box_p}{4}\right)^{\sigma} \mathcal{P}_n(p) \right\}. \tag{A.16}$$

The usefulness of eq. (A.16) results from the simplicity of eq. (A.1) when the numerator of its integrand is harmonic. In that case the sum of the r.h.s. of eq. (A.1) reduces to a single term. In many cases this property together with eq. (A.13) can prevent the proliferation of terms due to successive use of (A.1) in its crude form, as can be seen from the following example,

$$(2p_5\cdot k)^n Y_3(\alpha_1,\ldots,\alpha_5), \tag{A.17}$$

where Y_3 corresponds to the generic graph in fig. 6. Another non-trivial application of the irreducible polynomials is given in appendix B.

It is interesting to note that even if Feynman parameters did not exist, one would still be able to perform one-loop integrations. Indeed, eq. (A.1) with $\mathcal{P}_n$ harmonic can be derived very easily [owing to eq. (A.7)] by Fourier transforming to x-space and then back to p-space [5]. Therefore, the calculability of the 4-loop counterterms in a closed analytical form stems entirely from fundamental space-time properties of massless diagrams.

As a by-product of the above derivation we get the following expansion from eq. (A.15):

$$\frac{(2x)^n}{n!}=\sum_{\sigma\geqslant 0}\frac{1}{(2-\varepsilon)_{n-\sigma}\sigma!}\,\frac{(n-2\sigma+1-\varepsilon)}{(1-\varepsilon)}C^{1-\varepsilon}_{n-2\sigma}(x). \tag{A.18}$$

An equivalent to eq. (A.18) can be found, to our knowledge, in [27], formula (7.311.2), in a disguised form. On the other hand, eq. (A.18) is extremely helpful in obtaining a plethora of expansions in the Gegenbauer polynomials, like those derived with much pain in [14a].

Appendix B

We are now going to show how irreducible polynomials of appendix A can be used in order to reduce an integral of class R_4 (see fig. 9) with an arbitrary numerator to a sum of G-functions multiplied by integrals of class F (fig. 7), by performing a one-loop integration.

Obviously, if the numerator is absent, then the knowledge of the dimension

$$d=2\left(4-2\varepsilon-\sum_{i=1}^{5}\alpha_i\right) \tag{B.1}$$

of the 2-loop p-subgraph of the R_4 graph is sufficient for writing the following relation (all irrelevant powers of momenta here and in the following are suppressed):

$$R_4(\{\alpha\})=G(\alpha_7,\alpha_6-\tfrac{1}{2}d)F(\alpha_1,\dots,\alpha_5). \tag{B.2}$$

The case with an arbitrary numerator can be dealt with in the same manner if one notes that the irreducible tensorial structure of an integrand survives all integrations over momenta, cf. eq. (A.1) with $\mathcal{P}_n(p)$ irreducible.

Consider now a general R_4 integral:

$$(2p_1\cdot k)^{n_1}(2p_2\cdot k)^{n_2}R_4(\alpha_1,\dots,\alpha_7). \tag{B.3}$$

Applying the expansion (A.16) with $p=k$ to the numerator of eq. (C.3) and absorbing into R_4 the factors p_1^2, p_2^2, and $2p_1\cdot p_2 = p_1^2+p_2^2-p_5^2$, which result from $(\frac{1}{4}\Box_k)^\sigma$ in eq. (A.16), we come to a linear combination of the terms like,

$$\mathcal{P}^{(n_1+n_2-2\sigma)}(k)R_4(\alpha_1-\sigma_1,\alpha_2-\sigma_2,\alpha_3,\alpha_4,\alpha_5-\sigma_5,\alpha_6,\alpha_7), \tag{B.4}$$

with $\sigma_1+\sigma_2+\sigma_5=\sigma$. $\mathcal{P}^{(\cdots)}(k)$ is irreducible with respect to k and depends also on p_1 and p_2. Identifying the 2-loop subgraph of R_4 with an F-integral which has p_6 as the external momentum, one can easily see the validity of the following chain of equations:

$$\begin{aligned}
\text{eq. (B.4)} &= \mathcal{P}^{(n_1+n_2-2\sigma)}(k)\frac{1}{p_7^{2\alpha_7}p_6^{2\alpha_6}}F(\cdots)\\
&= \mathcal{P}^{(\cdots)}(k)\big|_{p_1=p_2=p_6}\frac{1}{p_7^{2\alpha_7}p_6^{2\alpha_6}}X\\
&= \mathcal{P}^{(\cdots)}(k)\big|_{p_1=p_2=k}X\cdot G(\alpha_7,a,n_1+n_2-2\sigma,0)\\
&= \left\{\mathcal{P}^{(\cdots)}(k)\cdot F(\alpha_1+\sigma_1,\alpha_2+\sigma_2,\alpha_3,\alpha_4,\alpha_5+\sigma_5)\right\}G(\cdots),
\end{aligned} \tag{B.5}$$

where $a=-4+2\varepsilon+\sum_{i=1}^6\alpha_i-\sigma$, and the F-integral in the last line has the external momentum k. The result of the whole procedure can be summarized as

$$\begin{aligned}
\text{eq. (B.3)} = \sum_{\sigma\geq 0}&\frac{1}{(n_1+n_2+2-\varepsilon)_{-\sigma}}G\left(\alpha_7,-4+2\varepsilon+\sum_{i=1}^{6}\alpha_i-\sigma,n_1+n_2-2\sigma,0\right)\\
&\times H_k^{(n_1+n_2-2\sigma)}\left\{\frac{1}{\sigma!}\left(\frac{\Box_k}{4}\right)^\sigma(2p_1\cdot k)^{n_1}(2p_2\cdot k)^{n_2}\right\}F(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5).
\end{aligned} \tag{B.6}$$

Appendix C

The equality of the two diagrams of fig. 1 at $D=4$ is somewhat puzzling, since the two diagrams have very different topological structures. Nevertheless, there exists an instructive explanation of this fact, which is presented here.

Let us analyze the calculation of, say, the planar diagram through GPXT [5]. Though in p-space the diagram seems to be convergent, i.e., the result has the form

$$L_0(\varepsilon)(k^2)^{-4-6\varepsilon} \tag{C.1}$$

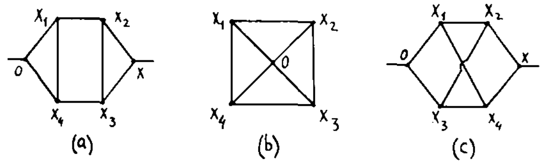

Fig. 17. (a) and (c) are the x-space diagrams corresponding to the p-space diagrams (a) and (b) of fig. 1; (b) is the diagram resulting from "gluing" external vertices of these diagrams.

with

$$L_0(\varepsilon) = [20\zeta(5) + \mathrm{O}(\varepsilon)](4\pi)^{-6},$$

one can easily see that if considered as a distribution in the sense of Schwartz [28], the expression (C.1) becomes singular. Indeed, if one smears eq. (C.1) with a suitable smooth function, say, $\exp(-k^2)$, the result will have a pole in ε due to the integration over the $k = 0$ region. In x-space this singularity shows itself as a pole in ε resulting from integration over large $\boldsymbol{x}_i$ (cf., fig. 17a):

$$\frac{1}{\varepsilon} L_0(\varepsilon)(x^2)^{4\varepsilon} \cdot \mathrm{const.} \tag{C.2}$$

If one performs Fourier transformation back to p-space, the factor of order ε emerges and kills the pole to produce a non-singular result (C.1). We conclude that to find $L_0(\varepsilon = 0)$, one has to extract the x-space IR singularity of the diagram.

In order to specify the origin of the singularity we employ the Weinberg theorem [29]. A simple power counting gives that the pole comes from the region where all $|\boldsymbol{x}_i| \to \infty$. So we introduce a cutoff $R > |\boldsymbol{x}|$ and split the region of integration into two parts: (i) the domain where for all $i\,|\boldsymbol{x}_i| > R$, and (ii) its complement. Taking $\varepsilon < 0$ so as to regularize the IR divergence, the integration over the first region gives,

$$R^{8\varepsilon}\left\{\frac{L_0}{\varepsilon} + \mathrm{O}\left(\frac{x^2}{R^2}, \varepsilon^0\right)\right\} \tag{C.3}$$

times a trivial factor, the same for both planar and non-planar diagrams. The second region produces an expression regular at $\varepsilon = 0$.

It is clear now that to evaluate the pole one can set $\boldsymbol{x} = 0$ after introducing the cutoff. The obtained integrand is easily seen to correspond to the diagram of fig. 17b which can be said to have resulted from fig. 17a by "gluing" its external vertices.

The wonderful thing here is that the same procedure applied to the non-planar diagram (fig. 17c) leads to exactly the same integral of fig. 17b, whence eq. (2.2) follows immediately!

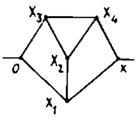

Fig. 18. The 3-loop scalar diagram of the φ^3 model which can not be calculated by straightforward gluing.

Now a few remarks are in order.

(i) The proof presented reveals explicitly all the conditions necessary for the gluing to be successful. Thus, e.g., it explains why the straightforward gluing fails to produce any result in case of the Mercedez diagram of fig. 18. This is because the corresponding integral has two IR dangerous regions: all $x_i \to 0$, and $x_2, x_3, x_4 \to \infty$ with x_1 fixed. The IR singular subgraph can also be observed through power counting in p-space.

(ii) Essentially, the gluing consists in simplifying the integral by rearranging its UV structure, i.e., introducing a lower cutoff and setting the irrelevant parameter x to 0, without affecting the IR pole. This is completely analogous to the IR rearrangement proposed in [4, 5] for evaluating the UV poles, and consisting in introducing an upper cutoff in x-space and setting to zero all irrelevant parameters such as masses and external momenta.

Eq. (2.3), however, can be derived in quite a different way. This second derivation, perhaps, less instructive and less rigorous than the previous one, will, nevertheless, enable us to incorporate the gluing and the symmetry first observed in [25] into a general scheme.

Consider the diagram of fig. 19a with zero external momenta in four dimensions. Introducing lower and upper cutoffs, respectively, λ and Λ, one gets the result $20\zeta(5)(4\pi)^{-8}\ln(\Lambda^2/\lambda^2)$. As is well-known, the coefficient of $\ln \Lambda^2$ is independent of the choice of the internal momenta because of the absence of divergent subgraphs (though the simplicity of the actual calculation depends crucially on this choice; but it is only the theoretical possibility to make different choices that counts in what follows). So, we choose the internal momenta as in fig. 19b after which the corresponding Feynman integral takes the following form

$$\int_{\lambda^2}^{\Lambda^2} \frac{p^2 \mathrm{d} p^2}{(4\pi)^2} I(p,\Lambda,\lambda) = \frac{20\zeta(5)}{(4\pi)^8} \ln\frac{\Lambda^2}{\lambda^2} + \mathrm{O}(1), \tag{C.4}$$

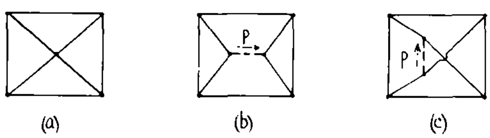

Fig. 19. (b) and (c) show the two choices of the internal momenta in the integral defined by diagram (a), which lead to eq. (2.3).

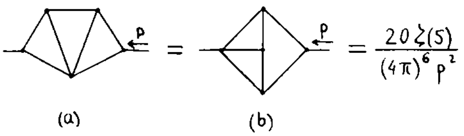

Fig. 20. The results in $D=4$ obtained via gluing without additional calculations.

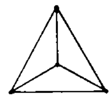

Fig. 21. The generic graph which produces all the symmetries of the 2-loop finite integrals discovered in ref. [25]. The symmetries are generated by cutting the internal lines of the graph.

where $I(p, \Lambda, \lambda)$ equals the diagram L_0 of fig. 1 evaluated at $D=4$ plus terms $O(p^{-2}\Lambda^{-2})$. A similar result for the diagram N_0 of fig. 1 can be obtained from fig. 19 whence follows eq. (2.3). Now, following the same pattern one obtains the result of fig. 20 at $D=4$ from fig. 19 without further calculations.

The general recipe can be formulated as follows. Choose a logarithmic divergent diagram without subdivergences, with a numerator, perhaps, and cut various lines or vertices. All the resulting p-diagrams have the same value at $D=4$, equal to the coefficient before the logarithm of the initial diagram times $(4\pi)^2$.

The symmetry discovered in [25] can be obtained in this way from the generic graph of fig. 21, if one adds a numerator and modifies the denominator so as to meet the conditions stated above. Note that the proof given in [25] is based on the internal properties of two-loop diagrams and is difficult to generalize. Correspondingly, our proof can be called external in the sense that it does not employ specific properties and is applicable to diagrams with an arbitrary loop number.

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