LHC ⇒ need cross sections σ
+ scatt amplitudes $\mathcal{A}$

• compute $\mathcal{A}$

have QCD Fy rules for ~ 40 years

⇒ every significant SM scattering process
should have been calc'd by now
to exp'lly needed accuracy !?

$\sigma_{LO} \sim |\mathcal{A}_{tree}|^2$

NLO $q\bar{q} \to gg$ (Jets), $q\bar{q} \to Wg$ etc

NNLO $e^+e^- \to$ Hadrons etc

bottlenecks: many ext legs

needed, e.g. QCD bg to $t\bar{t}$ production in $p\bar{p}$ coll
hadronic t-decay $t \to Wb \to q\bar{q}b$
⇒ bg starts at 6 jets!

⇝ efficient techniques for tree + loop amplitudes needed,
as input for LO, NLO, .. σ calc's

• in principle: draw all Fy diags
evaluate (reduction + masters)

• in practice: extremely hard when #legs ↗
many diags
many terms per diag (& complicated)
many Grassmann var's

⇒ intermediate expressions huge
but answer simple

⇝ lots of recent development
hot topic, see e.g. [KITP, The Harmony of Scatt Ampl, Apr-Jul 2011,
many online-talks, -tutorials]

- many new techniques inspired by calculations in string theory
  → but don't require its detailed knowledge!

basics, [L. Dixon, TASI 95, Calculating Scatt Ampl efficiently, hep-ph/9601359]

QCD amplitudes, tree + 1 loop
color + helicity decomposition
analytic properties, cuts + poles : "unitarity methods"
recursion rel's, SUSY rearrangements

developments [J.M. Drummond, CERN winter school 2010, Hidden Simplicity of Gauge Theory Amplitudes, arXiv: 1010.2418]

testing ground: max. susy. YM theory (N=4 SYM)
structure of A similar to QCD, but simpler
Yangian symmetry { non-free (but solvable?!) 4d QFT
{ gauge theory, QCD cousin
= grav in AdS space
recursion → complete sol'n for A_{tree}
beyond tree → superconf. sy?

recent short intro [L. Dixon, Scattering Ampl: the most perfect microscopic structures in the universe, arXiv: 1105.0771]

QCD – N=4 SYM – SUGRA

• Some motivation [M. Spradlin, talk @ KITP 7. Apr. 11, A practical guide to the art of N=4 Amplitudeology]

→ paradigm shift:

(1) enumerate principles (locality, analyticity ...) ⟺ (1) write down soln

(2) solve ⟺ (2) deduce principles from understanding its structure

ex massless ϕ^3, tree-level 4-particle amplitude

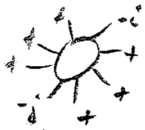
$$A = \frac{10s^2}{tu} + \frac{20s}{t} + \ldots 12 \text{ terms} - \frac{11}{s} \overset{s+t+u=0}{\overset{\downarrow}{=}} -\frac{1}{s} - \frac{1}{t} - \frac{1}{u}$$

⇒ need "good" variables!

particles w/ spin: spinor helicity var's, λ_i^a; $\langle ij\rangle$, $[ij]$

ex spin-1: helicity ±1, 2 phys states

MHV amplitude (planar, color-ordered, partial, tree level)



$$= \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle}$$

[Parke/Taylor 86, Berends/Giele 88]

⇒ BCFW recursion rels

→ simplicity gets discovered:

perturbative data + numerical experimentation

⇒ twistor space etc: N^k MHV [Witten, topological string theory]

ex found symm's among A's (dual conformal sy)

⇒ understand at level of $\mathcal{L}$?

Yangian symmetry; Grassmannian Integral; ..

$$A = \int d^4p_1 \ldots d^4p_L \ \Sigma \ \text{Fy diags}$$

↳ rel. simple integrand (exciting new technology)

↳ still very hard, BUT ↓

ex DDS remainder fct (N=4 SYM, 2loop, 6legs, MHV)

[Del Duca/Duhr/Smirnov arXiv: 1003.1702, pp 99-116] $R^{(2)}_{6,WL}(u_1,u_2,u_3)$

⇒ [Goncharov/Spradlin/Vergu/Volovich 1006.5703, 1 line] "hope where there was once

Preprint typeset in JHEP style - PAPER VERSION

IPPP/10/21, DCPT/10/42
CERN-PH-TH/2010-059

The Two-Loop Hexagon Wilson Loop in $\mathcal{N} = 4$ SYM

Vittorio Del Duca
PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland
INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy
E-mail: vittorio.del.duca@cern.ch

Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
E-mail: claude.duhr@durham.ac.uk

Vladimir A. Smirnov
Nuclear Physics Institute of Moscow State University
Moscow 119992, Russia
E-mail: smirnov@theory.sinp.msu.ru

ABSTRACT: In the planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, the conformal symmetry constrains multi-loop n-edged Wilson loops to be given in terms of the one-loop n-edged Wilson loop, augmented, for $n \geq 6$, by a function of conformally invariant cross ratios. That function is termed the remainder function. In a recent paper, we have displayed the first analytic computation of the two-loop six-edged Wilson loop, and thus of the corresponding remainder function. Although the calculation was performed in the quasi-multi-Regge kinematics of a pair along the ladder, the Regge exactness of the six-edged Wilson loop in those kinematics entails that the result is the same as in general kinematics. We show in detail how the most difficult of the integrals is computed, which contribute to the six-edged Wilson loop. Finally, the remainder function is given as a function of uniform transcendental weight four in terms of Goncharov polylogarithms. We consider also some asymptotic values of the remainder function, and the value when all the cross ratios are equal.

KEYWORDS: QCD, MSYM, small x.

$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231};1\right)H(0;u_1)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)H(0;u_1)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},0;1\right)H(0;u_1)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{u_3-1}{u_1+u_3-1};1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0;1\right)H(0;u_1)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H(0;u_1)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0;1\right)H(0;u_1)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1;1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1;1\right)H(0;u_1)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1;1\right)H(0;u_1)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H(0;u_1)-\frac{3}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H(0;u_1)+$$
$$\frac{3}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H(0;u_1)+\frac{3}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H(0;u_1)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H(0;u_1)-$$
$$\frac{1}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_2)-\frac{3}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_2)-$$
$$\frac{3}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_2)+\frac{1}{4}G\left(0,\frac{u_2-1}{u_2+u_3-1},\frac{1}{1-u_3};1\right)H(0;u_2)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0;1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_2-1}{u_2+u_3-1},\frac{1}{1-u_3};1\right)H(0;u_2)+\frac{1}{2}G\left(\frac{1}{u_1},0,\frac{1}{u_2};1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)H(0;u_2)-\frac{3}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)H(0;u_2)+$$
$$\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_2)+\frac{1}{2}G\left(\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_2)+$$
$$\frac{1}{2}G\left(\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},1,\frac{1}{u_3};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_2-1}{u_2+u_3-1},1;1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H(0;u_2)+\frac{1}{2}G\left(\frac{1}{u_3},0,\frac{1}{u_2};1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)+$$

$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231};1\right)H(0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321};1\right)H(0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{u_3};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3};1\right)H(0;u_2)+$$
$$\frac{1}{2}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1};1\right)H(0;u_2)+\frac{1}{2}\mathcal{G}\left(0,v_{213},\frac{1}{1-u_2};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2};1\right)H(0;u_2)-\frac{1}{2}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132};1\right)H(0;u_2)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)H(0;u_2)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)H(0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},0;1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0;1\right)H(0;u_2)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0;1\right)H(0;u_2)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213};1\right)H(0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231};1\right)H(0;u_2)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)H(0;u_2)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},1;1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},1;1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321};1\right)H(0;u_2)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)H(0;u_2)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},1;1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{u_3};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H(0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0;1\right)H(0;u_2)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0;1\right)H(0;u_2)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)H(0;u_2)+$$
$$\frac{3}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1};1\right)H(0;u_2)+\frac{3}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1;1\right)H(0;u_2)-$$

$$\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1;1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1;1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H(0;u_2)-$$
$$\frac{3}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H(0;u_2)-\frac{3}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H(0;u_2)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_1)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_1)H(0;u_2)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)H(0;u_1)H(0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right)H(0;u_1)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H(0;u_1)H(0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H(0;u_1)H(0;u_2)+\frac{5}{24}\pi^2H(0;u_1)H(0;u_2)-$$
$$\frac{1}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(0;u_3)-\frac{1}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_3)+$$
$$\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H(0;u_3)-\frac{3}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_3)-$$
$$\frac{3}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_3)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H(0;u_3)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},1,\frac{1}{u_3};1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},1;1\right)H(0;u_3)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H(0;u_3)+\frac{1}{2}G\left(\frac{1}{u_1},0,\frac{1}{u_3};1\right)H(0;u_3)-$$
$$\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_3)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0;1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H(0;u_3)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1};1\right)H(0;u_3)+\frac{1}{2}G\left(\frac{1}{u_2},0,\frac{1}{u_3};1\right)H(0;u_3)-$$
$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_3)-$$
$$\frac{3}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)H(0;u_3)-\frac{3}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)H(0;u_3)+$$
$$\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_3)+$$
$$\frac{1}{2}G\left(\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_3)+\frac{1}{2}G\left(\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_3)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132};1\right)H(0;u_3)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231};1\right)H(0;u_3)+$$

$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321};1\right)H(0;u_3)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1};1\right)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{u_3};1\right)H(0;u_3)+$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2};1\right)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{u_3-1}{u_2+u_3-1};1\right)H(0;u_3)-$$
$$\frac{1}{2}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1};1\right)H(0;u_3)+\frac{1}{2}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2};1\right)H(0;u_3)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(0,v_{321},\frac{1}{1-u_3};1\right)H(0;u_3)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132};1\right)H(0;u_3)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)H(0;u_3)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)H(0;u_3)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1;1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1};1\right)H(0;u_3)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{u_3};1\right)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0;1\right)H(0;u_3)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0;1\right)H(0;u_3)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213};1\right)H(0;u_3)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231};1\right)H(0;u_3)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)H(0;u_3)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},0;1\right)H(0;u_3)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{u_3-1}{u_2+u_3-1};1\right)H(0;u_3)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0;1\right)H(0;u_3)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)H(0;u_3)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0;1\right)H(0;u_3)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)H(0;u_3)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321};1\right)H(0;u_3)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)H(0;u_3)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)H(0;u_3)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},1;1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H(0;u_3)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},1;1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)H(0;u_3)-$$
$$\frac{3}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1};1\right)H(0;u_3)-\frac{3}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1;1\right)H(0;u_3)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1;1\right)H(0;u_3)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2};1\right)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1;1\right)H(0;u_3)+$$

$$\frac{3}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H(0;u_3)+\frac{3}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H(0;u_3)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H(0;u_3)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H(0;u_3)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_2+u_3};1\right)H(0;u_1)H(0;u_3)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)H(0;u_1)H(0;u_3)+$$
$$\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_1)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)H(0;u_1)H(0;u_3)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H(0;u_1)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H(0;u_1)H(0;u_3)+$$
$$\frac{5}{24}\pi^2H(0;u_1)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)H(0;u_3)+$$
$$\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_2)H(0;u_3)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)H(0;u_2)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H(0;u_2)H(0;u_3)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H(0;u_2)H(0;u_3)+\frac{5}{24}\pi^2H(0;u_2)H(0;u_3)+$$
$$3H(0;u_2)H(0,0;u_1)H(0;u_3)+3H(0;u_1)H(0,0;u_2)H(0;u_3)+$$
$$\frac{1}{4}H(0;u_2)H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)H(0;u_3)+\frac{1}{2}H(0;u_1)H(0,1;(u_1+u_2))H(0;u_3)+$$
$$\frac{1}{4}H(0;u_1)H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right)H(0;u_3)+\frac{1}{2}H(0;u_2)H(0,1;(u_2+u_3))H(0;u_3)+$$
$$\frac{3}{4}H(0;u_2)H(1,0;u_1)H(0;u_3)+\frac{3}{4}H(0;u_1)H(1,0;u_2)H(0;u_3)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H(0,0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H(0,0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H(0,0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H(0,0;u_1)-\frac{23}{24}\pi^2H(0,0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H(0,0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H(0,0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H(0,0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H(0,0;u_2)-$$
$$\frac{25}{4}H(0,0;u_1)H(0,0;u_2)-\frac{23}{24}\pi^2H(0,0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H(0,0;u_3)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H(0,0;u_3)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H(0,0;u_3)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H(0,0;u_3)+3H(0;u_1)H(0;u_2)H(0,0;u_3)-$$
$$\frac{25}{4}H(0,0;u_1)H(0,0;u_3)-\frac{25}{4}H(0,0;u_2)H(0,0;u_3)-\frac{23}{24}\pi^2H(0,0;u_3)+\frac{1}{12}\pi^2H(0,1;u_1)+$$
$$\frac{1}{12}\pi^2H(0,1;u_2)-\frac{1}{24}\pi^2H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{2}H(0;u_1)H(0;u_2)H(0,1;(u_1+u_2))+$$

$$\frac{1}{12}\pi^2H(0,1;(u_1+u_2))+\frac{1}{12}\pi^2H(0,1;u_3)+\frac{1}{4}H(0;u_1)H(0;u_3)H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right)-$$
$$\frac{1}{24}\pi^2H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right)+\frac{1}{12}\pi^2H(0,1;(u_1+u_3))-\frac{1}{24}\pi^2H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right)+$$
$$\frac{1}{12}\pi^2H(0,1;(u_2+u_3))-\frac{1}{2}G\left(0,\frac{1}{u_1+u_2};1\right)H(1,0;u_1)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_1+u_3};1\right)H(1,0;u_1)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(1,0;u_1)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(1,0;u_1)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(1,0;u_1)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)H(1,0;u_1)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(1,0;u_1)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right)H(1,0;u_1)-\frac{3}{4}H(0,0;u_2)H(1,0;u_1)-\frac{3}{4}H(0,0;u_3)H(1,0;u_1)+$$
$$\frac{1}{4}H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right)H(1,0;u_1)-\frac{1}{3}\pi^2H(1,0;u_1)-\frac{1}{2}G\left(0,\frac{1}{u_1+u_2};1\right)H(1,0;u_2)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_2+u_3};1\right)H(1,0;u_2)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)H(1,0;u_2)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(1,0;u_2)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(1,0;u_2)+$$
$$\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(1,0;u_2)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(1,0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)H(1,0;u_2)-\frac{3}{4}H(0,0;u_1)H(1,0;u_2)-\frac{3}{4}H(0,0;u_3)H(1,0;u_2)+$$
$$\frac{1}{4}H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)H(1,0;u_2)-\frac{1}{4}H(1,0;u_1)H(1,0;u_2)-\frac{1}{3}\pi^2H(1,0;u_2)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_1+u_3};1\right)H(1,0;u_3)-\frac{1}{2}G\left(0,\frac{1}{u_2+u_3};1\right)H(1,0;u_3)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(1,0;u_3)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)H(1,0;u_3)+$$
$$\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(1,0;u_3)-\frac{1}{3}\pi^2H(1,0;u_3)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(1,0;u_3)+$$
$$\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(1,0;u_3)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)H(1,0;u_3)+$$
$$\frac{3}{4}H(0;u_1)H(0;u_2)H(1,0;u_3)-\frac{3}{4}H(0,0;u_1)H(1,0;u_3)-\frac{3}{4}H(0,0;u_2)H(1,0;u_3)+$$
$$\frac{1}{4}H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right)H(1,0;u_3)-\frac{1}{4}H(1,0;u_1)H(1,0;u_3)-\frac{1}{4}H(1,0;u_2)H(1,0;u_3)+$$
$$\frac{1}{24}\pi^2H(1,1;u_1)+\frac{1}{24}\pi^2H(1,1;u_2)+\frac{1}{24}\pi^2H(1,1;u_3)+\frac{1}{2}H(0;u_3)H(0,0,0;u_1)+$$
$$\frac{1}{2}H(0;u_3)H(0,0,0;u_2)+\frac{1}{2}H(0;u_1)H(0,0,0;u_3)-\frac{1}{2}H(0;u_2)H\left(0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)-$$
$$\frac{1}{2}H(0;u_3)H\left(0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)-H(0;u_1)H(0,0,1;(u_1+u_2))-$$
$$H(0;u_2)H(0,0,1;(u_1+u_2))-\frac{1}{2}H(0;u_1)H\left(0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)-$$

$$\frac{1}{2}H(0;u_3)H\left(0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)-H(0;u_1)H(0,0,1;(u_1+u_3))-$$
$$H(0;u_3)H(0,0,1;(u_1+u_3))-\frac{1}{2}H(0;u_2)H\left(0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-$$
$$\frac{1}{2}H(0;u_3)H\left(0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-H(0;u_2)H(0,0,1;(u_2+u_3))-$$
$$H(0;u_3)H(0,0,1;(u_2+u_3))-\frac{1}{2}H(0;u_2)H(0,1,0;u_1)-\frac{1}{2}H(0;u_3)H(0,1,0;u_2)-$$
$$\frac{1}{2}H(0;u_1)H(0,1,0;u_3)+\frac{1}{4}H(0;u_2)H\left(0,1,1;\frac{u_1+u_2-1}{u_2-1}\right)-$$
$$\frac{1}{4}H(0;u_3)H\left(0,1,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{4}H(0;u_1)H\left(0,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-$$
$$\frac{1}{4}H(0;u_3)H\left(0,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-\frac{1}{4}H(0;u_1)H\left(0,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+$$
$$\frac{1}{4}H(0;u_3)H\left(0,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+\frac{1}{2}H(0;u_2)H(1,0,0;u_1)-\frac{1}{2}H(0;u_3)H(1,0,0;u_1)-$$
$$\frac{1}{2}H(0;u_1)H(1,0,0;u_2)+\frac{1}{2}H(0;u_3)H(1,0,0;u_2)+\frac{1}{2}H(0;u_1)H(1,0,0;u_3)-$$
$$\frac{1}{2}H(0;u_2)H(1,0,0;u_3)-\frac{1}{4}H(0;u_3)H\left(1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)-$$
$$\frac{1}{4}H(0;u_2)H\left(1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)-\frac{1}{4}H(0;u_1)H\left(1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-$$
$$7H(0,0,0,0;u_1)-7H(0,0,0,0;u_2)-7H(0,0,0,0;u_3)+\frac{3}{2}H\left(0,0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+$$
$$3H(0,0,0,1;(u_1+u_2))+\frac{3}{2}H\left(0,0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+3H(0,0,0,1;(u_1+u_3))+$$
$$\frac{3}{2}H\left(0,0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+3H(0,0,0,1;(u_2+u_3))+\frac{9}{4}H(0,0,1,0;u_1)+$$
$$\frac{9}{4}H(0,0,1,0;u_2)+\frac{9}{4}H(0,0,1,0;u_3)-\frac{1}{2}H(0,1,0,0;u_1)-\frac{1}{2}H(0,1,0,0;u_2)-$$
$$\frac{1}{2}H(0,1,0,0;u_3)+\frac{1}{2}H\left(0,1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{2}H\left(0,1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+$$
$$\frac{1}{2}H\left(0,1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+H(0,1,1,0;u_1)+H(0,1,1,0;u_2)+H(0,1,1,0;u_3)-$$
$$\frac{1}{4}H\left(0,1,1,1;\frac{u_1+u_2-1}{u_2-1}\right)-\frac{1}{4}H\left(0,1,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-$$
$$\frac{1}{4}H\left(0,1,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+H\left(1,0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+H\left(1,0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+$$
$$H\left(1,0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+2H(1,0,1,0;u_1)+2H(1,0,1,0;u_2)+2H(1,0,1,0;u_3)+$$
$$\frac{1}{4}H\left(1,1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{4}H\left(1,1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+$$
$$\frac{1}{4}H\left(1,1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+\frac{1}{2}H(1,1,1,0;u_1)+\frac{1}{2}H(1,1,1,0;u_2)+\frac{1}{2}H(1,1,1,0;u_3)-$$
$$\frac{1}{24}\pi^2H(0;u_3)\mathcal{H}\left(1;\frac{1}{u_{123}}\right)-\frac{1}{24}\pi^2H(0;u_1)\mathcal{H}\left(1;\frac{1}{u_{231}}\right)-\frac{1}{24}\pi^2H(0;u_2)\mathcal{H}\left(1;\frac{1}{u_{312}}\right)+$$
$$\frac{1}{8}\pi^2H(0;u_3)\mathcal{H}\left(1;\frac{1}{v_{123}}\right)-\frac{1}{8}\pi^2H(0;u_3)\mathcal{H}\left(1;\frac{1}{v_{123}}\right)+\frac{1}{24}\pi^2H(0;u_2)\mathcal{H}\left(1;\frac{1}{v_{132}}\right)-$$

$$\frac{1}{24}\pi^2H(0;u_3)\mathcal{H}\left(1;\frac{1}{v_{132}}\right)-\frac{1}{24}\pi^2H(0;u_1)\mathcal{H}\left(1;\frac{1}{v_{213}}\right)+\frac{1}{24}\pi^2H(0;u_3)\mathcal{H}\left(1;\frac{1}{v_{213}}\right)-$$
$$\frac{1}{8}\pi^2H(0;u_1)\mathcal{H}\left(1;\frac{1}{v_{231}}\right)+\frac{1}{8}\pi^2H(0;u_3)\mathcal{H}\left(1;\frac{1}{v_{231}}\right)+\frac{1}{8}\pi^2H(0;u_1)\mathcal{H}\left(1;\frac{1}{v_{312}}\right)-$$
$$\frac{1}{8}\pi^2H(0;u_2)\mathcal{H}\left(1;\frac{1}{v_{312}}\right)+\frac{1}{24}\pi^2H(0;u_1)\mathcal{H}\left(1;\frac{1}{v_{321}}\right)-\frac{1}{24}\pi^2H(0;u_2)\mathcal{H}\left(1;\frac{1}{v_{321}}\right)-$$
$$\frac{1}{4}H(0;u_2)H(0;u_3)\mathcal{H}\left(0,1;\frac{1}{u_{123}}\right)-\frac{1}{4}H(1,0;u_2)\mathcal{H}\left(0,1;\frac{1}{u_{123}}\right)+\frac{1}{24}\pi^2\mathcal{H}\left(0,1;\frac{1}{u_{123}}\right)+$$
$$\frac{1}{24}\pi^2\mathcal{H}\left(0,1;\frac{1}{u_{231}}\right)-\frac{1}{4}H(0;u_1)H(0;u_3)\mathcal{H}\left(0,1;\frac{1}{u_{231}}\right)-\frac{1}{4}H(1,0;u_3)\mathcal{H}\left(0,1;\frac{1}{u_{231}}\right)-$$
$$\frac{1}{4}H(0;u_1)H(0;u_2)\mathcal{H}\left(0,1;\frac{1}{u_{312}}\right)-\frac{1}{4}H(1,0;u_1)\mathcal{H}\left(0,1;\frac{1}{u_{312}}\right)+\frac{1}{24}\pi^2\mathcal{H}\left(0,1;\frac{1}{u_{312}}\right)-$$
$$\frac{1}{4}H(0;u_2)H(0;u_3)\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)+\frac{1}{4}H(0,0;u_2)\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)+$$
$$\frac{1}{4}H(0,0;u_3)\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)-\frac{1}{4}H(0;u_2)H(0;u_3)\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)+$$
$$\frac{1}{4}H(0,0;u_2)\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0,0;u_3)\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)-$$
$$\frac{1}{4}H(0;u_1)H(0;u_3)\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)+\frac{1}{4}H(0,0;u_1)\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)+$$
$$\frac{1}{4}H(0,0;u_3)\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)-\frac{1}{4}H(0;u_1)H(0;u_3)\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)+$$
$$\frac{1}{4}H(0,0;u_1)\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H(0,0;u_3)\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)-$$
$$\frac{1}{4}H(0;u_1)H(0;u_2)\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)+\frac{1}{4}H(0,0;u_1)\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)+$$
$$\frac{1}{4}H(0,0;u_2)\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_1)H(0;u_2)\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)+$$
$$\frac{1}{4}H(0,0;u_1)\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)+\frac{1}{4}H(0,0;u_2)\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)-$$
$$\frac{1}{2}H(0;u_2)H(0;u_3)\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)+\frac{1}{2}H(0,0;u_2)\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)+$$
$$\frac{1}{2}H(0,0;u_3)\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)+\frac{11}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)-\frac{1}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{132}}\right)-$$
$$\frac{1}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{213}}\right)-\frac{1}{2}H(0;u_1)H(0;u_3)\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)+\frac{1}{2}H(0,0;u_1)\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)+$$
$$\frac{1}{2}H(0,0;u_3)\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)+\frac{11}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)-\frac{1}{2}H(0;u_1)H(0;u_2)\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)+$$
$$\frac{1}{2}H(0,0;u_1)\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)+\frac{1}{2}H(0,0;u_2)\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)+\frac{11}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)-$$
$$\frac{1}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{321}}\right)+\frac{1}{2}H(0;u_2)\mathcal{H}\left(0,0,1;\frac{1}{u_{123}}\right)+\frac{1}{2}H(0;u_3)\mathcal{H}\left(0,0,1;\frac{1}{u_{123}}\right)+$$
$$\frac{1}{2}H(0;u_1)\mathcal{H}\left(0,0,1;\frac{1}{u_{231}}\right)+\frac{1}{2}H(0;u_3)\mathcal{H}\left(0,0,1;\frac{1}{u_{231}}\right)+\frac{1}{2}H(0;u_1)\mathcal{H}\left(0,0,1;\frac{1}{u_{312}}\right)+$$
$$\frac{1}{2}H(0;u_2)\mathcal{H}\left(0,0,1;\frac{1}{u_{312}}\right)+\frac{1}{4}H(0;u_3)\mathcal{H}\left(0,1,1;\frac{1}{u_{123}}\right)+\frac{1}{4}H(0;u_1)\mathcal{H}\left(0,1,1;\frac{1}{u_{231}}\right)+$$

$$\frac{1}{4}H(0;u_2)\mathcal{H}\left(0,1,1;\frac{1}{u_{312}}\right)+\frac{1}{4}H(0;u_2)\mathcal{H}\left(0,1,1;\frac{1}{v_{123}}\right)-\frac{1}{4}H(0;u_3)\mathcal{H}\left(0,1,1;\frac{1}{v_{123}}\right)-$$
$$\frac{1}{4}H(0;u_2)\mathcal{H}\left(0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0;u_3)\mathcal{H}\left(0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0;u_1)\mathcal{H}\left(0,1,1;\frac{1}{v_{213}}\right)-$$
$$\frac{1}{4}H(0;u_3)\mathcal{H}\left(0,1,1;\frac{1}{v_{213}}\right)-\frac{1}{4}H(0;u_1)\mathcal{H}\left(0,1,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H(0;u_3)\mathcal{H}\left(0,1,1;\frac{1}{v_{231}}\right)+$$
$$\frac{1}{4}H(0;u_1)\mathcal{H}\left(0,1,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_2)\mathcal{H}\left(0,1,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_1)\mathcal{H}\left(0,1,1;\frac{1}{v_{321}}\right)+$$
$$\frac{1}{4}H(0;u_2)\mathcal{H}\left(0,1,1;\frac{1}{v_{321}}\right)+\frac{1}{4}H(0;u_3)\mathcal{H}\left(1,0,1;\frac{1}{u_{123}}\right)+\frac{1}{4}H(0;u_1)\mathcal{H}\left(1,0,1;\frac{1}{u_{231}}\right)+$$
$$\frac{1}{4}H(0;u_2)\mathcal{H}\left(1,0,1;\frac{1}{u_{312}}\right)+\frac{1}{4}H(0;u_2)\mathcal{H}\left(1,0,1;\frac{1}{v_{123}}\right)-\frac{1}{4}H(0;u_3)\mathcal{H}\left(1,0,1;\frac{1}{v_{123}}\right)-$$
$$\frac{1}{4}H(0;u_2)\mathcal{H}\left(1,0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0;u_3)\mathcal{H}\left(1,0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0;u_1)\mathcal{H}\left(1,0,1;\frac{1}{v_{213}}\right)-$$
$$\frac{1}{4}H(0;u_3)\mathcal{H}\left(1,0,1;\frac{1}{v_{213}}\right)-\frac{1}{4}H(0;u_1)\mathcal{H}\left(1,0,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H(0;u_3)\mathcal{H}\left(1,0,1;\frac{1}{v_{231}}\right)+$$
$$\frac{1}{4}H(0;u_1)\mathcal{H}\left(1,0,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_2)\mathcal{H}\left(1,0,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_1)\mathcal{H}\left(1,0,1;\frac{1}{v_{321}}\right)+$$
$$\frac{1}{4}H(0;u_2)\mathcal{H}\left(1,0,1;\frac{1}{v_{321}}\right)+H(0;u_2)\mathcal{H}\left(1,1,1;\frac{1}{v_{123}}\right)-H(0;u_3)\mathcal{H}\left(1,1,1;\frac{1}{v_{132}}\right)-$$
$$H(0;u_1)\mathcal{H}\left(1,1,1;\frac{1}{v_{231}}\right)+H(0;u_3)\mathcal{H}\left(1,1,1;\frac{1}{v_{231}}\right)+H(0;u_1)\mathcal{H}\left(1,1,1;\frac{1}{v_{312}}\right)-$$
$$H(0;u_2)\mathcal{H}\left(1,1,1;\frac{1}{v_{312}}\right)-\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{123}}\right)-\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{231}}\right)-$$
$$\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{312}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{123}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{213}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{321}}\right)-$$
$$\frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{312}}\right)-$$
$$\frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{312}}\right)+$$
$$\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{123}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{132}}\right)+\zeta_3H(0;u_1)+\zeta_3H(0;u_2)+\zeta_3H(0;u_3)+$$
$$\frac{5}{2}\zeta_3H(1;u_1)+\frac{5}{2}\zeta_3H(1;u_2)+\frac{5}{2}\zeta_3H(1;u_3)+\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{123}}\right)+\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{231}}\right)+$$
$$\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{312}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{312}}\right)+$$
$$\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{123}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{132}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{213}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{231}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{312}}\right)+$$
$$\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{321}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{213}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{231}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{312}}\right)+$$
$$\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{321}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{123}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{213}}\right)+$$
$$\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{231}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{312}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{321}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{123}}\right)+$$
$$\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{213}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{231}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{312}}\right)+$$

Classical Polylogarithms for Amplitudes and Wilson Loops

A. B. Goncharov,[1] M. Spradlin,[2] C. Vergu,[2] and A. Volovich[2]

[1]*Department of Mathematics, Brown University, Box 1917, Providence, Rhode Island 02912, USA*
[2]*Department of Physics, Brown University, Box 1843, Providence, Rhode Island 02912, USA*

We present a compact analytic formula for the two-loop six-particle maximally helicity violating remainder function (equivalently, the two-loop lightlike hexagon Wilson loop) in $\mathcal{N}=4$ supersymmetric Yang-Mills theory in terms of the classical polylogarithm functions Li_k with cross-ratios of momentum twistor invariants as their arguments. In deriving our formula we rely on results from the theory of motives.

INTRODUCTION

The past few years have witnessed revolutionary advances in our understanding of the structure of scattering amplitudes, especially in $\mathcal{N}=4$ supersymmetric Yang-Mills theory (SYM). It is easy to argue that the seeds of modern progress were sown already in the 1980s with the discovery of the Parke-Taylor formula for the simplest nontrivial amplitudes: tree-level maximally helicity violating (MHV) gluon scattering. The mere existence of such a simple formula for a quantity which otherwise would have been prohibitively difficult to calculate using traditional Feynman diagram methods signalled the tantalizing possibility that a great vista of unanticipated structure in scattering amplitudes awaited exploration.

In contrast to the situation at tree level, it is fair to say that recent progress at loop level has mostly been evolutionary rather than revolutionary, driven primarily by faster computers, improved algorithms (both analytic and numeric), and software for multiloop calculations which has been made publicly available. Yet we hope that a great new vista of unexplored structure awaits us also at loop level in SYM theory.

This paper is concerned with the planar two-loop six-particle MHV amplitude [1, 2], which in a sense is the simplest nontrivial SYM loop amplitude. The known infrared and collinear behavior of general amplitudes, conveniently encapsulated in the ABDK/BDS ansatz [3, 4], determines the n-particle MHV amplitude at each loop order $L \geq 2$ up to an additive finite function of kinematic invariants called the remainder function $R_n^{(L)}$. Given the presumption of dual conformal invariance [5, 6] for SYM amplitudes (not yet proven, but supported by all available evidence [1, 3, 4, 7, 8]), $R_n^{(L)}$ can depend on conformal cross-ratios only. Since there are no cross-ratios for $n = 4, 5$, the first nontrivial remainder function is $R_6^{(2)}$.

The same function $R_6^{(2)}$ is also believed [9–12] to arise as the expectation value of the two-loop lightlike hexagon Wilson loop in SYM theory [13, 14] (after appropriate subtraction of ultraviolet divergences, e.g. [15]). Numerical agreement between the two remainder functions was established in [1, 14]. In a heroic effort, Del Duca, Duhr, and Smirnov (DDS) explicitly evaluated the appropriate Wilson loop diagrams to obtain an analytic expression for $R_6^{(2)}$ as a 17-page linear combination of generalized polylogarithm functions [16, 17] (see also [18]).

The motivation for the present work is the belief that if SYM theory is really as beautiful and rich as recent developments indicate, then there must exist a more enlightening way of expressing the remainder function $R_6^{(2)}$. Ideally, like the Parke-Taylor formula at tree level, the expression should provide encouragement and guidance as we seek deeper understanding of SYM at loop level.

We present our new formula for $R_6^{(2)}$ in the next section and then describe the algorithm by which it was obtained.

THE REMAINDER FUNCTION $R_6^{(2)}$

The remainder function $R_6^{(2)}$ is usually presented as a function of the three dual conformal cross-ratios

$$u_1 = \frac{s_{12}s_{45}}{s_{123}s_{345}}, \quad u_2 = \frac{s_{23}s_{56}}{s_{234}s_{123}}, \quad u_3 = \frac{s_{34}s_{61}}{s_{345}s_{234}}, \quad (1)$$

of the momentum invariants $s_{i\cdots j} = (k_i + \cdots + k_j)^2$, though we will see shortly that cross-ratios of momentum twistor invariants are more natural variables. In terms of

$$x_i^\pm = u_i x^\pm, \qquad x^\pm = \frac{u_1 + u_2 + u_3 - 1 \pm \sqrt{\Delta}}{2u_1u_2u_3}, \quad (2)$$

where $\Delta = (u_1 + u_2 + u_3 - 1)^2 - 4u_1u_2u_3$, we find

$$R_6^{(2)}(u_1, u_2, u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+, x_i^-) - \frac{1}{2}\mathrm{Li}_4(1 - 1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1 - 1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}. \quad (3)$$

Here we use the functions

$$L_4(x^+, x^-) = \frac{1}{8!!}\log(x^+x^-)^4 + \sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+) + \ell_{4-m}(x^-)) \quad (4)$$

and

$$\ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n \mathrm{Li}_n(1/x)\right), \quad (5)$$

as well as the quantity

$$J = \sum_{i=1}^{3}(\ell_1(x_i^+) - \ell_1(x_i^-)). \quad (6)$$

Note that in the Euclidean region where all $u_i > 0$, the x_i^+ never enter the lower half-plane and the x_i^- never enter the upper half-plane. The expression (3) is valid in the Euclidean region with the understanding that the branch cuts of $\mathrm{Li}_n(x_i^+)$ and $\mathrm{Li}_n(1/x_i^-)$ are taken to lie below the real axis while the branch cuts of $\mathrm{Li}_n(x_i^-)$ and $\mathrm{Li}_n(1/x_i^+)$ are taken to lie above the real axis. (The quantities $x_i^+x_i^-$ appearing as arguments of the logs are always positive.) In writing (3) extreme care has necessarily been taken to ensure the proper analytic structure. For example one can easily check that J naively simplifies to $\frac{1}{2}\log(x^-/x^+)$, but this relation only holds in the regions $\Delta > 0$ or $u_1 + u_2 + u_3 < 1$. We caution the reader that any attempt to use any such naive relations, including the well-known relation between $\mathrm{Li}_n(1/x)$ and $\mathrm{Li}_n(x)$, without careful consideration of the branch structure, voids our warranty on (3).

Besides its great simplicity, two notable features of (3) which set it apart from the DDS formula are manifest symmetry under any permutation of the u_i, and the fact that the expression is valid and readily evaluated for all positive u_i, in particular also outside the unit cube.

DESCRIPTION OF THE ALGORITHM

A Convenient Choice of Variables

The DDS formula is expressed in terms of the classical polylogarithms Li_k as well as a collection of considerably more complicated multiparameter generalizations studied by one of the authors [19] and defined recursively by

$$G(a_k, a_{k-1}, \ldots; z) = \int_0^z G(a_{k-1}, \ldots; t)\frac{dt}{t - a_k} \quad (7)$$

with $G(z) \equiv 1$, of which the harmonic polylogarithms familiar in the physics literature [20] are special cases.

The parameters of the various transcendental functions which appear in the DDS formula involve not just the cross-ratios (1), but also the more complicated combinations $1 - u_i$, $(1 - u_i)/(1 - u_i - u_j)$, $u_i + u_j$, $u_{jkl}^\pm = \frac{1 - u_j - u_k + u_l \pm \sqrt{\Delta}}{2(1 - u_j)u_l}$, and $v_{jkl}^\pm = \frac{u_k - u_l \pm \sqrt{(u_k + u_l)^2 - 4u_ju_ku_l}}{2(1 - u_j)u_k}$. This large collection of variables is redundant in an inefficient way, with many rather complicated algebraic identities amongst them.

Our computation is greatly facilitated by a judicious choice of variables which trivializes all of these algebraic relations. We choose to express the three u_i by six variables z_i valued in $\mathbb{P}^1$ (with an $SL(2, \mathbb{C})$ redundancy) via

$$u_1 = \frac{z_{23}z_{56}}{z_{25}z_{36}}, \quad u_2 = \frac{z_{16}z_{34}}{z_{14}z_{36}}, \quad u_3 = \frac{z_{12}z_{45}}{z_{14}z_{25}}, \quad (8)$$

where $z_{ij} = z_i - z_j$. One virtue of these coordinates is that Δ becomes a perfect square, so that the $u_{jkl}^\pm$ are rational functions of the z_{ij}. (The $v_{jkl}^\pm$ completely drop out as explained in the following subsection.)

We anticipate that for general n the best variables for studying the remainder function will be the momentum twistors of [21]. Indeed the z variables may be thought of as a particular simplification of momentum twistors which is valid for the special case $n = 6$ via the relation $\langle abcd\rangle \propto z_{ab}z_{ac}z_{ad}z_{bc}z_{bd}z_{cd}$. In terms of momentum twistors

$$u_1 = \frac{\langle 1234\rangle\langle 4561\rangle}{\langle 1245\rangle\langle 3461\rangle}, \quad x_1^+ = -\frac{\langle 1456\rangle\langle 2356\rangle}{\langle 1256\rangle\langle 3456\rangle}, \quad \text{etc.} \quad (9)$$

The Symbol of a Transcendental Function

We define a function T_k of transcendentality degree k as one which can be written as a linear combination (with rational coefficients) of k-fold iterated integrals of the form

$$T_k = \int_a^b d\log R_1 \circ \cdots \circ d\log R_k, \quad (10)$$

where a and b are rational numbers, $R_i(t)$ are rational functions with rational coefficients and the iterated integrals are defined recursively by

$$\int_a^b d\log R_1 \circ \cdots \circ d\log R_n = \int_a^b \left(\int_a^t d\log R_1 \circ \cdots \circ d\log R_{n-1}\right) d\log R_n(t). \quad (11)$$

The integrals are taken along paths from a to b. When the R_i are rational functions in several variables the issue of local path independence (or homotopy invariance) is important (see [22]), and we have checked that $R_6^{(2)}$ has this property.

A useful quantity associated with T_k is its symbol, an element of the k-fold tensor product of the multiplicative group of rational functions modulo constants (see [22, sec. 3]). The symbol of the function shown in (10) is

$$\mathrm{symbol}(T_k) = R_1 \otimes \cdots \otimes R_k, \quad (12)$$

and this definition is extended to all functions of degree k by linearity.