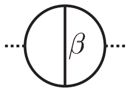
1 The Integral O_1

We work in $D = d - 2\epsilon$ dimensions (d being an integer), and take the integral measure to be

$$\int_p \equiv \left(\frac{e^\gamma}{4\pi}\right)^\epsilon \int \frac{\mathrm{d}^D p}{(2\pi)^D} \quad .$$

Define

$$O_1(\beta) \;\equiv\; \int_p \int_q \frac{1}{p^2(p-k)^2\,[(p-q)^2]^\beta\,(q-k)^2 q^2} \;=\; \quad , \qquad (1)$$



where we are interested in non-integer powers β on the middle rung (for an integer power, this integral can be reduced to a primitive one via traingle relations). A representation of this class of integrals in terms of a double sum over Γ-functions can be found in [1] Eq. (2.19):

$$\begin{aligned} O_1(\beta) \;&=\; \frac{(k^2)^{D-4-\beta}}{(4\pi)^D}\left(\frac{e^\gamma}{4\pi}\right)^{2\epsilon}\frac{\Gamma^3(\lambda)\Gamma(\lambda+1-\beta)}{\Gamma(\beta)\Gamma(2\lambda)}\, 2\sum_{m,n\geq 0}\frac{(-)^m}{m!n!}\,\frac{\Gamma(n+2\lambda)\Gamma(m+n+2-2\lambda+b)}{(n+\lambda)\Gamma(3\lambda-m-1-\beta)\Gamma(m+n+\lambda+1)} \\ &\quad\times\left\{\frac{1}{(n+\beta)(m+n+1-\lambda+\beta)}+\frac{1}{(m+n+1)(m+n+1-\lambda+\beta)}+\frac{1}{(m+n+1)(n+2\lambda-\beta)}\right\} \qquad (2)\end{aligned}$$

Here, $\lambda = \frac{D}{2} - 1$, and the series is convergent for $\beta < \min\{3\lambda-1, 2\lambda, \lambda+2\}$. As a check on the above formula, it is easy to derive the well-known result for $D = 4$ and $\beta = 1$:

$$k^2(4\pi)^4\, O_1(1) \;\stackrel{D=4}{=}\; \sum_{m,n\geq 0}\frac{(-)^m}{m!}\,\frac{2}{(m+n+1)^2\Gamma(1-m)}\left(\frac{2}{n+1}+\frac{1}{m+n+1}\right) \;=\; \sum_{n\geq 0}\frac{6}{(n+1)^3} \;=\; 6\,\zeta(3)\;.$$

For the case of $D = 4 - 2\epsilon$ and $\beta = 3 - \frac{D}{2} = 1 + \epsilon$, an expansion to $O(\epsilon^3)$ can be read from Eq. (14) of [2] (setting $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -1$ and $\alpha_5 = -2$ there).

We shall be interested in the case $D = 3 - 2\epsilon$ and $\beta = 1 - \frac{D}{2} = \epsilon - \frac{1}{2}$. Writing out the integral explicitly for this choice of parameters:

$$\begin{aligned} O_1(\epsilon - \tfrac{1}{2}) \;&\stackrel{3-2\epsilon}{=}\; \frac{e^{2\gamma\epsilon}\,(2-4\epsilon)}{(4\pi)^3(k^2)^{\frac{1}{2}+3\epsilon}}\,\frac{\Gamma^3(\frac{1}{2}-\epsilon)}{\Gamma(\epsilon-\frac{1}{2})}\sum_{m,n\geq 0}\frac{(-)^m}{m!n!}\,\frac{\Gamma(n+1-2\epsilon)\Gamma(m+n+\frac{1}{2}+3\epsilon)}{(n+\frac{1}{2}-\epsilon)\Gamma(1-m-4\epsilon)\Gamma(m+n+\frac{3}{2}-\epsilon)} \\ &\quad\times\left\{\frac{1}{(n-\frac{1}{2}+\epsilon)(m+n+2\epsilon)}+\frac{1}{(m+n+1)(m+n+2\epsilon)}+\frac{1}{(m+n+1)(n+\frac{3}{2}-3\epsilon)}\right\} \qquad (3)\end{aligned}$$

Since an analytic solution of the double sum is not (yet?) known, the best we can do here is an expansion in ϵ. Splitting the sum into three regions $\{m = n = 0; m = 0, n > 0; m > 0\}$ and using (below, m, n are positive integers)

$$\Gamma(n+1-2\epsilon) \;\approx\; \Gamma(n+1)e^{2\gamma\epsilon}\left[1 - 2\epsilon\sum_{j=1}^{n}\frac{1}{j} + O(\epsilon^2)\right]$$

$$
\begin{aligned}
\Gamma(n+\frac{1}{2}+3\epsilon) &\approx \Gamma(n+\frac{1}{2})e^{-3\gamma\epsilon}2^{-6\epsilon}\left[1+6\epsilon\sum_{j=1}^{n}\frac{1}{2j-1}+O(\epsilon^2)\right] \\
\Gamma(1-4\epsilon) &\approx e^{4\gamma\epsilon}\left[1+O(\epsilon^2)\right] \\
\Gamma(1-m-4\epsilon) &\approx \frac{(-)^m}{\Gamma(m)\,4\epsilon}\,e^{4\gamma\epsilon}\left[1+O(\epsilon)\right]
\end{aligned}
$$

it is straightforward to expand the double sum in Eq. (3) as

$$
\begin{aligned}
\sum_{m,n}\cdots &= e^{-6\gamma\epsilon}2^{-8\epsilon}\left[-\frac{2}{\epsilon}-\left(\frac{40}{3}-12s_0\right)-(56+2\pi^2-8s_1-24s_2-16s_3)\epsilon+O(\epsilon^2)\right] \\
&= e^{-6\gamma\epsilon}2^{-8\epsilon}\left[-\frac{2}{\epsilon}-12-(16+6\pi^2)\epsilon+O(\epsilon^2)\right]\,. \qquad (4)
\end{aligned}
$$

In the last line, the relations $s_0=\frac{1}{9}$ and $2s_1+6s_2+4s_3=10-\pi^2$ (only checked numerically so far) were used, where

$$
\begin{aligned}
s_0 &= \sum_{n>0}\frac{1}{n(n+1)(2n-1)(2n+3)} \\
s_1 &= \sum_{n>0}\frac{1}{n(n+1)(2n-1)(2n+3)}\,\frac{9-24n-44n^2+16n^3}{n(2n-1)(2n+1)(2n+3)} \\
s_2 &= \sum_{j>0}\sum_{n\geq j}\frac{1}{n(n+1)(2n-1)(2n+3)}\,\frac{2j+1}{j(2j-1)} \\
s_3 &= \sum_{n\geq 0}\sum_{j>n}\frac{3+4j+2n}{j(j+1)(2j+1)(j-n)(2n-1)(2n+3)}
\end{aligned}
$$

Putting together Eqs. (3) and (4), the final result is

$$
O_1(\epsilon-\tfrac{1}{2}) \overset{3-2\epsilon}{=} \frac{1}{(k^2)^{\frac{1}{2}+3\epsilon}}\,\frac{1}{32\pi^2}\left[\frac{1}{\epsilon}+2+\left(\frac{7}{2}\pi^2-12\right)\epsilon+O(\epsilon^2)\right]\,. \qquad (5)
$$

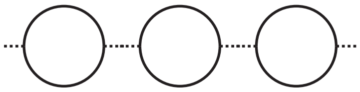
2 An elegant check of the O_1 expansion

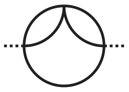
The leading term(s) of the expansion of O_1 in 3 dimensions can be obtained by a completely different method, as outlined below. We shall take it here as a check of the above expansion Eq. (5).

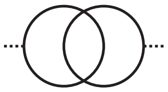
The key observation is that one can compute the leading UV divergence of the (in $3-2\epsilon$ dimensions logarithmically divergent) 4loop vacuum triangle diagram in massless scalar theory

$$
I \equiv \int_{p_1}\int_{p_2}\int_{p_3}\int_{p_4}\frac{1}{p_1^2(p_1-p_4)^2p_2^2(p_2-p_4)^2p_3^2(p_3-p_4)^2} = \qquad (6)
$$

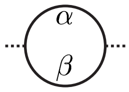
in three different ways. The general strategy can be learned from Appendix C of [3] (where the first of those ways is outlined): Pick one of the four integration momenta to be the outermost integration (which will be represented graphically by a cut in the diagram), perform the other three ones analytically in $\overline{\mathrm{MS}}$, and split UV and IR in the last integral. The three possible 'cuts' are (notation later; superscripts D omitted):

$I \;=$  $\times J \;=\; G^3(1,1) \times J \qquad (7)$

$I \;=$  $\times J \;=\; G^2(1,1)G(4-D,1) \times J \qquad (8)$

$I \;=$  $\times J \;=\; G(1,1)O_1(2-\tfrac{D}{2}) \times J \qquad (9)$

Here, we have readily used the (dimensionless) massless scalar oneloop integral G,

$G(\alpha,\beta) \;\equiv\; \left(k^2\right)^{\alpha+\beta-\frac{D}{2}} \int_p \frac{1}{[p^2]^\alpha\,[(p-k)^2]^\beta} \;=$ 

$$= \frac{e^{\gamma\epsilon}}{(4\pi)^{\frac{D}{2}+\epsilon}} \frac{\Gamma(\alpha+\beta-\frac{D}{2})}{\Gamma(D-\alpha-\beta)} \frac{\Gamma(\frac{D}{2}-\alpha)}{\Gamma(\alpha)} \frac{\Gamma(\frac{D}{2}-\beta)}{\Gamma(\beta)} \tag{10}$$

and factored out the 'last' integral J

$$\begin{aligned} J_{\mathrm{BN}} \;&\equiv\; \int_p \left(p^2\right)^{\frac{3D}{2}-6} \overset{3-2\epsilon}{=} \frac{1}{16\pi^2}\left(\frac{e^\gamma}{4\pi}\right)^\epsilon \left(\frac{1}{\epsilon_{\mathrm{uv}}}-\frac{1}{\epsilon_{\mathrm{ir}}}+8\ln\frac{\Lambda}{\mu}\right) \\ J_{\mathrm{cutoff}} \;&\equiv\; \int_p^{\mu,\Lambda} \left(p^2\right)^{\frac{3D}{2}-6} = \frac{e^{\gamma\epsilon}}{(4\pi)^{\frac{D}{2}+\epsilon}} \frac{2}{\Gamma(\frac{D}{2})} \int_\mu^\Lambda \mathrm{d}p\; p^{4D-13} \\ &\overset{3-2\epsilon}{=} \frac{1}{16\pi^2} \frac{e^{\gamma\epsilon}}{\mu^{8\epsilon}} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}-\epsilon)} \left(1-\left(\frac{\mu}{\Lambda}\right)^{8\epsilon}\right)\frac{1}{\epsilon} \;\approx\; \frac{1}{16\pi^2}\left(8\ln\frac{\Lambda}{\mu}+O(\epsilon)\right) \\ J_{\mathrm{massive}} \;&\equiv\; \int_p \left(p^2+m^2\right)^{\frac{3D}{2}-6} \overset{3-2\epsilon}{=} \frac{1}{16\pi^2} \frac{e^{\gamma\epsilon}}{m^{8\epsilon}} \frac{\Gamma(1+4\epsilon)\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}+3\epsilon)} \frac{1}{\epsilon} \;\approx\; \frac{1}{16\pi^2}\left(\frac{1}{\epsilon_{\mathrm{uv}}}+O(1)\right) \end{aligned}$$

Since the coefficient of the overall UV divergence must be independent of the choice of the order of integration, it follows that

$$G(1+2\epsilon,1)\Big|_{\mathrm{leading}} \overset{3-2\epsilon}{=} G(1,1)\Big|_{\mathrm{leading}} \overset{3-2\epsilon}{=} \frac{1}{8} \tag{11}$$

$$O_1(\epsilon+\tfrac{1}{2})\Big|_{\mathrm{leading}} \overset{3-2\epsilon}{=} G^2(1,1)\Big|_{\mathrm{leading}} \overset{3-2\epsilon}{=} \frac{1}{64} \tag{12}$$

While the first relation is trivially fulfilled, as can be checked from Eq. (10), the second one serves as the abovementioned check on the integral O_1, which is not known analytically. Setting $\beta = 2-\frac{D}{2}$ as well as $D = 3-2\epsilon$ in the generic recursion relation

$$O_1(\beta) \;=\; \frac{2}{D-2-2\beta}\left\{(2+\beta-D)O_1(\beta-1)+(3D-6-4\beta)G(\beta,1)G(2+\beta-\tfrac{D}{2},1)\right\} \tag{13}$$

which follows from partial integration (see e.g. Eq. (4.29) in [4]), Eq. (12) translates into

$$\left(O_1(\epsilon-\tfrac{1}{2})-2\,\frac{1-10\epsilon}{1-6\epsilon}\,G(\tfrac{1}{2}+\epsilon,1)\,G(1+2\epsilon,1)\right)\Big|_{\text{leading}}\overset{3-2\epsilon}{=}\frac{\epsilon}{16}\;.$$

Using Eq. (10), one therefore obtains

$$O_1(\epsilon-\tfrac{1}{2})\quad\overset{3-2\epsilon}{=}\quad\frac{1}{(k^2)^{\frac{1}{2}+3\epsilon}}\,\frac{1}{32\pi^2}\left[\frac{1}{\epsilon}+2+\left(\frac{7}{2}\pi^2-12\right)\epsilon+O(\epsilon^2)\right]\;, \tag{14}$$

in complete agreement with Eq. (5). It should be stressed, though, that for higher accuracy one always has to resort to an expansion of Eq. (3), since the seemingly simpler case $O_1(\frac{1}{2}+\epsilon)$ lies outside the convergence region of the series representation. Nevertheless, it is somewhat more economical to choose $O_1(\frac{1}{2}+\epsilon)$ as the 3D basic[1] O_1-integral and derive all other cases by recursively using Eq. (13).

3 *Integrating by symmetry (to be written)*

Translating arguments of [2] (and Honkonen..) to momentum space as well as 3d, get (at least) one term further in the expansion without much work:

$$O_1(\epsilon+\tfrac{1}{2})\quad\overset{3-2\epsilon}{=}\quad\frac{1}{(k^2)^{\frac{3}{2}+3\epsilon}}\left[\frac{1}{64}+\frac{\epsilon}{16}\left(\frac{21}{\pi^2}\zeta(3)-\ln 2\right)+O(\epsilon^2)\right] \tag{15}$$

Use Eq. (13) to translate to $O_1(\epsilon-\frac{1}{2})$.

References

[1] K.G. Chetyrkin, A.L. Kataev and F.V. Tkachov, *New approach to evaluation of multiloop Feynman integrals: The Gegenbauer polynomial x-space technique*, Nucl.Phys.**B174** (1980) 345.

[2] D.I. Kazakov, *The method of uniqueness, a new powerful technique for multiloop calculations*, Phys.Lett.**B133** (1983) 406.

[3] E. Braaten and A. Nieto, *Effective field theory approach to high-temperature thermodynamics*, Phys.Rev.**D51** (1995) 6990.

[4] K.G. Chetyrkin and F.V. Tkachov, *Integration by parts: The algorithm to calculate β-functions in 4 loops*, Nucl.Phys.**B192** (1981) 159.

[1]Since it contains no leading 'trivial' terms which could be rewritten in terms of G-functions. Note that in 4D the corresponding unique choice is $O_1(1+\epsilon)$. Let me also note here that for these arguments, one inspects the recursion as well as the possible divergence structure of the G-functions: 3D $G(a+\epsilon,1)G(a+\frac{1}{2}+2\epsilon,1)$ starts with $\frac{1}{\epsilon}$ for $a=\frac{1}{2}$,1 and with a constant otherwise; 4D $G(a+\epsilon,1)G(a+2\epsilon,1)$ starts with $\frac{1}{\epsilon^2}$ for a=1,2 and with a constant otherwise.