Symmetries in Physics - Fall 2018/19

Bielefeld University

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Exercise Nr. 13

Discussion on January 21, 14:15-15:45, Room U2-135

Exercises 31) should be handed in **before** the tutorial.

30) Orthogonality Relation *(2+3=5 points)*

Let $g \mapsto D_1(g)$ and $g \mapsto D_2(g)$ be two irreducible (finite-dimensional) representations of a finite group $\mathcal{G}$, with D_i acting on V_i ($i = 1, 2$). Choose a basis for V_i such that $(D_1(g))_{ab}$ and $(D_2(g))_{cd}$ are the representation matrices. Then use Schur's Lemma to show:

a) If D_1 and D_2 are inequivalent, then

$$\forall a, b, c, d: \sum_{g \in \mathcal{G}} (D_1(g^{-1}))_{ab} (D_2(g))_{cd} = 0.$$

b) If D_1 and D_2 are equivalent, then

$$\frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} (D_1(g^{-1}))_{ab} (D_2(g))_{cd} = \frac{1}{\dim(V_1)} \delta_{ad} \delta_{bc}.$$

31) Character Tables *(5+5=10 points)*

a) Determine the character table of symmetric group S_4.

a) Determine the character table of the dihedral group D_4.

32) Irreducible Representations of the Cubic Group *(5 points)*

The cubic group (chiral octahedral group O with 24 elements) has five conjugacy classes:

1. trivial class $\{e\}$
2. the class C_2 with π rotations around the axex connecting opposite faces
3. the class C_3 with $2\pi/3$ rotations about the body diagonals
4. the class C_2' with π rotations about the axes connecting opposite edges
5. the class C_4' with $\pm\pi/2$ rotations about the axes connecting opposite faces

What are the dimensions of the five irreducible representations?

Ferdinand Georg Frobenius
(26 Oct 1849 - 3 Aug 1917)

[...] he received his doctorate (awarded with distinction) in 1870 supervised by Weierstrass. In 1874, after having taught at secondary school level first at the Joachimsthal Gymnasium then at the Sophienrealschule, he was appointed to the University of Berlin as an extraordinary professor of mathematics. [...] Frobenius was only in Berlin for a year before he went to Zürich to take up an appointment as an ordinary professor at the Eidgenössische Polytechnikum. For seventeen years, between 1875 and 1892, Frobenius worked in Zürich. He married there and brought up a family and did much important work in widely differing areas of mathematics. [...]

In the last days of December 1891 Kronecker died and, therefore, his chair in Berlin became vacant. Weierstrass, strongly believing that Frobenius was the right person to keep Berlin in the forefront of mathematics, used his considerable influence to have Frobenius appointed. However, for reasons which we shall discuss in a moment, Frobenius turned out to be something of a mixed blessing for mathematics at the University of Berlin. [...]

In his work in group theory, Frobenius combined results from the theory of algebraic equations, geometry, and number theory, which led him to the study of abstract groups. He published "Uber Gruppen von vertauschbaren Elementen" in 1879 (jointly with Stickelberger, a colleague at Zürich) which looks at permutable elements in groups. This paper also gives a proof of the structure theorem for finitely generated abelian groups. In 1884 he published his next paper on finite groups in which he proved Sylow's theorems for abstract groups (Sylow had proved his theorem as a result about permutation groups in his original paper). The proof which Frobenius gives is the one, based on conjugacy classes, still used today in most undergraduate courses. In his next paper in 1887 Frobenius continued his investigation of conjugacy classes in groups which would prove important in his later work on characters. In the introduction to this paper he explains how he became interested in abstract groups, and this was through a study of one of Kronecker's papers. It was in the year 1896, however, when Frobenius was professor at Berlin that his really important work on groups began to appear. In that year he published five papers on group theory and one of them "Über die Gruppencharactere" on group characters is of fundamental importance. [...] This paper on group characters was presented to the Berlin Academy on July 16 1896 and it contains work which Frobenius had undertaken in the preceding few months. In a series of letters to Dedekind, the first on 12 April 1896, his ideas on group characters quickly developed. Ideas from a paper by Dedekind in 1885 made an important contribution and Frobenius was able to construct a complete set of representations by complex numbers. It is worth noting, however, that although we think today of Frobenius's paper on group characters as a fundamental work on representations of groups, Frobenius in fact introduced group characters in this work without any reference to representations. In was not until the following year that representations of groups began to enter the picture, and again it was a concept due to Frobenius. Hence 1897 is the year in which the representation theory of groups was born. Over the years 1897-1899 Frobenius published two papers on group representations, one on induced characters, and one on tensor product of characters. In 1898 he introduced the notion of induced representations and the Frobenius Reciprocity Theorem. It was a burst of activity which set up the foundations of the whole of the machinery of representation theory.

In a letter to Dedekind on 26 April 1896 Frobenius gave the irreducible characters for the alternating groups A_4, A_5, the symmetric groups S_4, S_5 and the group PSL(2,7) of order 168. He completely determined the characters of symmetric groups in 1900 and of characters of alternating groups in 1901, publishing definitive papers on each. He continued his applications of character theory in papers of 1900 and 1901 which studied the structure of Frobenius groups. [...]

[From www-history.mcs.st-andrews.ac.uk/Biographies/Frobenius.html
(J. J. O'Connor and E. F. Robertson)]