

Symmetries in Physics - Fall 2018/19

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Exercise Nr. 12

Discussion on January 14, 14:15-15:45, Room U2-135

Exercises 27) should be handed in **before** the tutorial.

27) Regular Representation ($4+4=8$ points)

- a) Determine all 6 matrices of the regular representation of the dihedral group D_3 and check that the relations of its group presentation are fulfilled:

$$\mathcal{R}(c_3)^3 = \mathbb{1}, \quad \mathcal{R}(\sigma)^2 = \mathbb{1}, \quad \mathcal{R}(c_3)\mathcal{R}(\sigma)\mathcal{R}(c_3)\mathcal{R}(\sigma) = \mathbb{1}.$$

Show that the regular representation decomposes into

$$\mathcal{R} = D_1^1 \oplus D_1^2 \oplus D_2 \oplus D_2.$$

28) Properties of Representations ($2+2+2+2=8$ points)

- a) Let $D(g)$ be a representation. Explain why $D^{-1}(g)$ and $D^\dagger(g)$ do not define a representation.
- b) Show that for $U \in \text{SU}(2)$, $\sigma_2 U \sigma_2 = U^*$ (hence $\text{SU}(2)$ is pseudoreal).
- c) Given that a representation D is pseudoreal and $D^*(g) = S D(g) S^{-1}$, show that $S S^* = \lambda \mathbb{1}$ for some $\lambda \in \mathbb{R}$.
- d) Given two representations $D(g)$ and $D'(g)$, show that the tensor product $D(g) \otimes D'(g)$ is indeed a representation.

29) Representations of the Quotient Groups ($2+2=4$ points)

Prove the following theorems:

- a) If a group \mathcal{G} has a non-trivial normal subgroup $\mathcal{N} \triangleleft \mathcal{G}$, then a representation of the quotient group \mathcal{G}/\mathcal{N} is also a representation of \mathcal{G} . This representation of \mathcal{G} is not faithful.
- b) Conversely, if the representation D of the group \mathcal{G} is not faithful, then \mathcal{G} has at least one normal subgroup \mathcal{N} such that D defines a faithful representation on the factor group \mathcal{G}/\mathcal{N} .

Issai Schur

(10 Jan 1865 - 10 Jan 1941)

Although Issai Schur was born in Mogilev on the Dnieper, he spoke German without a trace of an accent [...] In 1894 Schur entered the University of Berlin to read mathematics and physics. [...] In 1901 Schur obtained his doctorate with a thesis which examined rational representations of the general linear group over the complex field. Functions which Schur introduced in his thesis are today called S-functions, where the S stands for Schur. Interest in the results of Schur's thesis continues today [...]. In 1903 Schur became a lecturer at Berlin University and then, from 1911 until 1916, he held a professorship in mathematics at the University of Bonn. He returned to Berlin in 1916 and there he built his famous school and spent most of the rest of his life there. He was promoted to full professor in Berlin in 1919, three years after he returned there, and he held this chair until he was dismissed by the Nazis in 1935.



Schur is mainly known for his fundamental work on the representation theory of groups but he also worked in number theory, analysis and other topics described below. Between 1904 and 1907 he worked on projective representations of groups and group characters. One of the most fundamental results which he discovered at this time is today called Schur's Lemma.

In a series of papers he introduced the concept now known as the 'Schur multiplier'. This is an extremely important abstract concept which arose from the concrete problems that Schur was studying. Much later, in 1949, Eilenberg and Mac Lane defined cohomology groups. They were unaware at that time that the second cohomology group with coefficients in the nonzero complex numbers is the Schur multiplier, and therefore that Schur had made some of the first steps forty years earlier.

Around 1914 Schur's interest in representations of groups was put to one side while he worked on other topics but, around 1925, developments in theoretical physics showed that group representations were of fundamental importance in that subject. Schur returned to work on representation theory with renewed vigour and he was able to complete the programme of research begun in his doctoral dissertation and give a complete description of the rational representations of the general linear group. [...]

From 1933 events in Germany made Schur's life increasingly difficult. [...] On 7 April 1933 the Nazis passed a law which, under clause three, ordered the retirement of civil servants who were not of Aryan descent, with exemptions for participants in World War I and pre-war officials. Schur had held an appointment before World War I which should have qualified him as a civil servant, but the facts were not allowed to get in the way, and he was 'retired'. [...] Schur saw himself as a German, not a Jew, and could not comprehend the persecution and humiliation he suffered under the Nazis. In fact Schur's dismissal was revoked and he was able to carry out some of his duties for a while. By November 1933 when Walter Ledermann took his Staatsexamen he was examined by Schur together with Bieberbach who was wearing Nazi uniform. There were invitations to Schur to go to the United States and to Britain but he declined them all, unable to understand how a German was not welcome in Germany. For example Ledermann obtained a scholarship to go to St Andrews in Scotland in the spring of 1934 and he tried unsuccessfully to persuade Schur to join him in St Andrews.

Schur continued to suffer the humiliation that was heaped on him. [...] Pressure was put on Schur to resign from the Prussian Academy to which he had been honoured to be elected in 1922. [...] Schur left Germany for Palestine in 1939, broken in mind and body, having the final humiliation of being forced to find a sponsor to pay the 'Reichs flight tax' to allow him to leave Germany. Without sufficient funds to live in Palestine he was forced to sell his beloved academic books to the Institute for Advanced Study in Princeton. He died two years later on his 66th birthday.

[From www-history.mcs.st-andrews.ac.uk/Biographies/Schur.html (J. J. O'Connor and E. F. Robertson)]