Symmetries in Physics - Fall 2018/19

Bielefeld University

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Exercise Nr. 10

Discussion on December 17, 14:15-15:45, Room U2-135

Exercises 23) should be handed in **before** the tutorial.

23) Lorentz group *(2+2+5=9 points)*

The 4-dimensional matrix Λ is a Lorentz matrix if $\quad \Lambda^T G \Lambda = G, \quad G = \mathrm{diag}(1,-1,-1,-1)$.

a) Show that the set $\mathrm{O}(1,3)$ of such Lorentz matrices froms a subgroup: $\mathrm{O}(1,3) < \mathrm{GL}(4,\mathbb{R})$.

b) Show that for $\Lambda \in \mathrm{O}(1,3)$, the determinant is $\det\Lambda = \pm 1$.

c) Show that the set of proper Lorentz transformations form a normal subgroup:

$$\mathrm{SO}(1,3) \equiv \mathrm{O}_+(1,3) = \{\Lambda \in \mathrm{O}(1,3) \mid \det\Lambda = 1\} \lhd \mathrm{O}(1,3).$$

24) Poincaré group *(6+5=11 points)*

The Poincaré group is a semidirect product of translations and Lorentz group: $\mathbb{P} = \mathbb{R}^{1,3} \rtimes O(1,3)$.

a) Let a_1, a_2 be translations, Λ_1, Λ_2 Lorentz transformations. Verify that the group product is then given by

$$(a_1,\Lambda_1)\circ(a_2,\Lambda_2) = (a_1+\Lambda_1 a_2, \Lambda_1\Lambda_2),$$

hence the group of Lorentz transformations is unaffected by the group of translations, whereas the group of translations is affected by the Lorentz group. Also argue on the basis of the corresponding Lie algebra: Verify that

$$[M_{\mu\nu}, P_\rho] = g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu,$$

to infer that this commutator is non-vanishing (hence $\mathbb{P}$ is not a direct product), and yields an element of the algebra of the translation group $\mathbb{R}^{1,3}$.

Notation: P_μ are the four generators of time and space translations, i.e. the components of $(H,\vec{P})$, and the anti-symmetric tensor M_{ij} is the set of generators of $\mathrm{SO}(1,3)$, defined via the infinitesimal Lorentz transformations

$$\Lambda(\delta\omega) = \mathbb{1} + \frac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu}, \qquad \Lambda = \exp(-i\omega_{\mu\nu}M^{\mu\nu}),$$

and its components are $K_i \equiv M^{i0}$ the generators of the Lorentz boost,

$$K_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; K_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; K_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \; [K_i,K_j] = -i\epsilon_{ijk}J_k$$

and $J_i \equiv \frac{1}{2}\epsilon_{ijk}M^{jk}$ the generators of the rotations

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \; J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \; J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; [J_i,J_j] = i\epsilon_{ijk}J_k$$

b) Let us consider the non-relativistic limit of the proper Lorentz group SO$(1,3)$.
It is sufficient to take the non-relativistic limit of the Lorentz boosts in some direction $\vec{e_i}$:

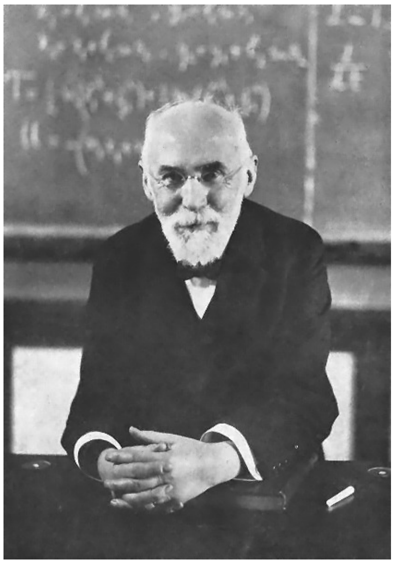
$$L_i(\beta) = e^{i\,\phi(\beta)K_i}, \qquad \phi = \operatorname{arctanh}(\beta), \qquad \beta = \frac{v}{c}.$$

Show that the group of boosts becomes non-compact in the non-relativistic limit $c \to \infty$, i.e. calculate the limit $\lim_{c\to\infty} L_i(\beta, c)$ and verify that the Lorentz transformations carry over to the Galilei transformations by letting Lorentz boost act on a space-time vector $(ct, x, y, z)^T$. Such a limit of groups is called a **group contraction**, the Galilean boost generators are obtained from the non-relativistic limit of the Lorentz boost generators. Its commutator vanishes in contrast to that of the Lorentz boosts generators:

$$C_i = \lim_{c\to\infty} \frac{K_i(c)}{ic}, \qquad [C_i, C_j] = 0$$

The compact group SO(1,3) is then reduced to the non-compact group $\mathbb{R}^3 \rtimes$ SO(3) called homogeneous Galilei group.

Hendrik Antoon Lorentz
(18 July 1853 - 4 February 1928)



Dutch physicist and joint winner (with Pieter Zeeman) of the Nobel Prize for Physics in 1902 for his theory of electromagnetic radiation, which, confirmed by findings of Zeeman, gave rise to Albert Einstein's special theory of relativity. In his doctoral thesis at the University of Leiden (1875), Lorentz refined the electromagnetic theory of James C. Maxwell of England so that it more satisfactorily explained the reflection and refraction of light. He was appointed professor of mathematical physics at Leiden in 1878. His work in physics was wide in scope, but his central aim was to construct a single theory to explain the relationship of electricity, magnetism, and light. Although, according to Maxwell's theory, electromagnetic radiation is produced by the oscillation of electric charges, the charges that produce light were unknown. Since it was generally believed that an electric current was made up of charged particles, Lorentz later theorized that the atoms of matter might also consist of charged particles and suggested that the oscillations of these charged particles (electrons) inside the atom were the source of light.

If this were true, then a strong magnetic field ought to have an effect on the oscillations and therefore on the wavelength of the light thus produced. In 1896 Zeeman, a pupil of Lorentz, demonstrated this phenomenon, known as the Zeeman effect, and in 1902 they were awarded the Nobel Prize.

Lorentz' electron theory was not, however, successful in explaining the negative results of the Michelson-Morley experiment, an effort to measure the velocity of the Earth through the hypothetical luminiferous ether by comparing the velocities of light from different directions. In an attempt to overcome this difficulty he introduced in 1895 the idea of local time (different time rates in different locations). Lorentz arrived at the notion that moving bodies approaching the velocity of light contract in the direction of motion. The Irish physicist George Francis FitzGerald had already arrived at this notion independently (see Lorentz-FitzGerald contraction, and in 1904 Lorentz extended his work and developed the Lorentz transformations. These mathematical formulas describe the increase of mass, shortening of length, and dilation of time that are characteristic of a moving body and form the basis of Einsteinâs special theory of relativity. In 1912 Lorentz became director of research at the Teyler Institute, Haarlem, though he remained honorary professor at Leiden, where he gave weekly lectures.

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