Symmetries in Physics - Fall 2018/19

Bielefeld University

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Exercise Nr. 9

Discussion on December 10, 14:15-15:45, Room U2-135

Exercises 21) should be handed in **before** the tutorial.

21) Angular momentum *(3+2+5=10 points)*

The Hermitian operators J_x, J_y, J_z, which obey the commutation relation

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

are called angular momentum operators (with $\hbar = 1$). The corresponding unitary operators

$$R(\vec{\phi}) = e^{i\vec{\phi}\cdot\vec{J}}, \quad \vec{\phi} \in \mathbb{R}^3$$

describe rotations with $\phi = |\vec{\phi}|$ about the axis $\hat{\phi}$.

a) Given that the rotations R are expressed in a Cartesian basis (rotations about the basis vectors $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$ with $\langle \vec{e}_i | \vec{e}_j \rangle = \delta_{ij}$, find the matrices J_x, J_y, J_z explicitly.
(Hint: Set $\vec{\phi} = \phi \vec{e}_i$ and differentiate with respect to ϕ).

b) Check the identity $\vec{J}^2 = j(j+1)$.

c) The angular momentum operators for a spinor $u \in \mathbb{C}^2$ are given by the Pauli-matrices,

$$J_i^{(1/2)} = \frac{1}{2}\sigma_i$$

The corresponding finite rotations of spinors are $R^{1/2}(\vec{\phi})$. Show that

$$(\vec{J}^{(1/2)})^2 = j(j+1) \quad \text{with} \quad j = \frac{1}{2}.$$

and find explict expressions for $R^{1/2}(\phi\vec{e}_x)$, $R^{1/2}(\phi\vec{e}_y)$ and $R^{1/2}(\phi\vec{e}_z)$.

22) The Campbell Baker Hausdorff Formula *(1+2+2+2+3=10 points)*

Beyond leading order, the product

$$e^{tX}e^{tY} = \exp\left(\sum_{n=1}^{\infty} t^n C_n(X,Y)\right)$$

is determined by the coefficients $C_n(X,Y)$.

a) Expand the exponentials e^{tX} and e^{tY} in a Taylor series.

b) With this, write the product $e^{tX}e^{tY}$ by a double sum of the form $\sum_{s=1}^{\infty}\sum_{m=0}^{s}$.

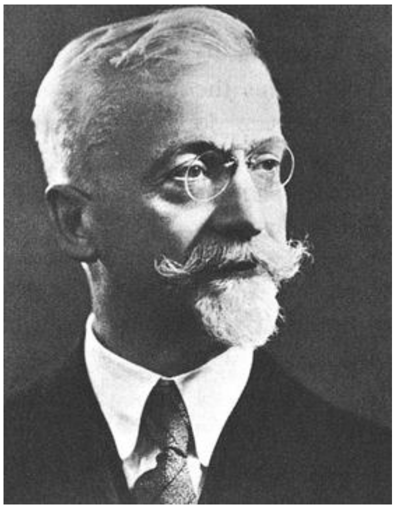
c) Compute the Taylor expansion of $\ln(1+z)$.

d) Use the above formulae to identify z, z^2, z^3, which will allow to compare the coefficients $C_n(X,Y)$ up to $n = 3$

e) Reproduce that $C_1(X,Y) = X + Y$, $C_2(X,Y) = \frac{1}{2}[X,Y]$ and compute $C_3[X,Y]$.

Élie Cartan

(9 April 1869 - 6 May 1951)



[...] Cartan became a student at the École Normale Supérieure in 1888 where he attended courses by the leading mathematicians of the day including Henri Poincaré, Charles Hermite, Jules Tannery, Gaston Darboux, Paul Appell, Émile Picard and Édouard Goursat. Cartan graduated in 1891 and then served for a year in the army before continuing his studies for his doctorate at the École Normale Supérieure. [...] Cartan's doctoral thesis of 1894 contains a major contribution to Lie algebras where he completed the classification of the semisimple algebras over the complex field which Killing had essentially found. However, although Killing had shown that only certain exceptional simple algebras were possible, he had not proved that in fact these algebras exist. This was shown by Cartan in his thesis when he constructed each of the exceptional simple Lie algebras over the complex field. His first papers, published in 1893, were two notes stating his results on simple Lie groups. [...] Cartan published full details of the classification in a third paper which was essentially his doctoral thesis. He obtained his doctorate in 1894 from the Faculty of Science at the Sorbonne. [...]

In Lyons in 1903 he married Marie-Louise Bianconi (1880-1950), the daughter of Pierre-Louis Bianconi who had been a professor of chemistry but had become an inspector in Lyons. Élie and Marie-Louise Cartan had four children [...] In 1903 Cartan was appointed as a professor at the University of Nancy but he also taught at the Institute of Electrical Engineering and Applied Mechanics. He remained there until 1909 when he moved to Paris [...] His appointment in 1909 in Paris was as an assistant lecturer at the Sorbonne but three years later he was appointed to the Chair of Differential and Integral Calculus in Paris. From 1915 to 1918, during World War I, he was drafted into the army where he continued to hold his former rank of sergeant. He was able to continue his mathematical career and, at the same time, work in the military hospital attached to the École Normale Supérieure. He was appointed as Professor of Rational Mechanics in 1920, and then Professor of Higher Geometry from 1924 to 1940. He retired in 1940 but did not stop teaching at this point for he went on to teach at the École Normale Supérieure for girls. Cartan worked on continuous groups, Lie algebras, differential equations and geometry. His work achieved a synthesis between these areas. He added greatly to the theory of continuous groups which had been initiated by Lie. After the work of his thesis on finite continuous Lie groups, he later classified the semisimple Lie algebras over the real field and found all the irreducible linear representations of the simple Lie algebras. He turned to the theory of associative algebras and investigated the structure for these algebras over the real and complex field. Joseph Wedderburn would complete Cartan's work in this area.

He then turned his attention to representations of semisimple Lie groups. His work is a striking synthesis of Lie theory, classical geometry, differential geometry and topology which was to be found in all Cartan's work. He applied Grassmann algebra to the theory of exterior differential forms. [...] From 1916 onwards he published mainly on differential geometry. Klein's 'Erlanger Programme' was seen to be inadequate as a general description of geometry by Weyl and Veblen, and Cartan was to play a major role. He examined a space acted on by an arbitrary Lie group of transformations, developing a theory of moving frames which generalises the kinematical theory of Darboux. In fact this work led Cartan to the notion of a fibre bundle although he does not give an explicit definition of the concept in his work.

Cartan further contributed to geometry with his theory of symmetric spaces which have their origins in papers he wrote in 1926. In these he developed ideas first studied by Clifford and Cayley and used topological methods developed by Weyl in 1925. [...] Cartan discovered the theory of spinors in 1913. These are complex vectors that are used to transform three-dimensional rotations into two-dimensional representations and they later played a fundamental role in quantum mechanics. [...] —

[From www-history.mcs.st-andrews.ac.uk/Biographies/Cartan.html (J. J. O'Connor and E. F. Robertson)]