Symmetries in Physics - Fall 2018/19

Bielefeld University

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Exercise Nr. 8

Discussion on December 3, 14:15-15:45, Room U2-135

Exercises 19) should be handed in **before** the tutorial.

19) The Lie Group SU(2) ***(2+2+1+3=8 points)***

a) Show that if α, β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$, then the matrix

$$U = \begin{pmatrix} \alpha & -\beta* \\ \beta & \alpha^* \end{pmatrix}$$

is in SU(2).

b) Show that every element in SU(2) can be written as in a).

c) Explain why SU(2) can be viewed as a 3-dim sphere in $\mathbb{C}^2 = \mathbb{R}^4$ and why it is hence simply connected.

d) Consider the linear combinations of the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(x, \sigma) = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}$$

Show that $x^\mu = \frac{1}{2}\mathrm{tr}[\bar{\sigma}^\mu(x, \sigma)]$ with $\bar{\sigma}_0 = \sigma_0$ and $\bar{\sigma}_i = -\sigma_i$. Argue why this defines a bijective map from $\mathbb{R}^4$ to the linear space of all 2-dimensional Hermitian matrices.

20) The Lie groups $\mathrm{U}(n)$ **and** $\mathrm{O}(n)$ ***(2+3+3+4=12 points)***

a) Show that $\mathrm{U}(n)$ is isomorphic to $\mathrm{SU}(n) \times \mathrm{U}(1)/C_n$. Hint: calculate the kernel of the map

$$\phi : \mathrm{SU}(n) \times \mathrm{U}(1) \to \mathrm{U}(n), \quad (U, e^{i\lambda}) \mapsto e^{i\lambda} U$$

b) Determine the group centers of $\mathrm{SU}(n)$, $\mathrm{O}(n)$ and $\mathrm{SO}(n)$. Hint: the center elements are multiples of the identity.

c) What manifolds are the cosets of $\mathrm{SO}(n)/\mathrm{SO}(n-1)$ and $\mathrm{SU}(n)/\mathrm{SU}(n-1)$?

d) The set of matrices

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

constitute a linear space $\mathbb{R}^4$ with scalar product $\langle A_1 | A_2 \rangle = \mathrm{tr}(A_1^\dagger A_2)$.
Show that for $U_1, U_2 \in \mathrm{SU}(2)$ as parameterized above but with $|\alpha^2| + |\beta^2| = 1$, the following map

$$R(U_1, U_2)A = U_1 A U_2^\dagger$$

is linear and preserves the scalar product. With this, show that R defines a group homomorphism $\mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{O}(4)$. Is R surjective?

Marius Sophus Lie
(17 December 1842 - 18 February 1899)

Sophus Lie's father was Johann Herman Lie, a Lutheran minister. His parents had six children and Sophus was the youngest of the six. [...] At university [in Oslo] Lie studied a broad science course. There was certainly some mathematics in this course, and Lie attended lectures by Ludwig Sylow in 1862. [...] he graduated in 1865 without having shown any great ability for the subject, or any great liking for it. [...] It was during the year 1867 that Lie had his first brilliant new mathematical idea. [...] Lie wrote a short mathematical paper in 1869, which he published at his own expense, based on the inspiration which had struck him in 1867. [...] Lie was awarded a scholarship to travel and meet the leading mathematicians. Setting off near the end of the year 1869, Lie went to Prussia and visited Göttingen and then Berlin. In Berlin he met Kronecker, Kummer and Weierstrass. [...] Most important to Lie, however, was the fact that in Berlin he met Felix Klein. [...]

In the spring of 1870 Lie and Klein were together again in Paris. [...] Jordan seems to have succeeded in a way that Sylow did not, for Jordan made Lie realise how important group theory was for the study of geometry. Lie started to develop ideas which would later appear in his work on transformation groups. He began to discuss with Klein these new ideas on groups and geometry and he would collaborate later with Klein in publishing several papers. This joint work had as one of its outcomes Klein's characterisation of geometry in his Erlangen Program of 1872 as properties invariant under a group action. While in Paris Lie discovered contact transformations. These transformations allowed a 1-1 correspondence between lines and spheres in such a way that tangent spheres correspond to intersecting lines. [...] In 1871 Lie became an assistant at Christiania, having obtained a scholarship, and he also taught at Nissen's Private Latin School in Christiania where he had been a pupil himself. He submitted a dissertation On a class of geometric transformations (written in Norwegian) for his doctorate which was duly awarded in July 1872. The dissertation contained ideas from his first results published in Crelle's Journal and also the work on contact transformations, a special case of these transformations being a transformation which maps a line into a sphere, which he had discovered while in Paris.

It was clear that Lie was a remarkable mathematician and the University of Christiania reacted in a very positive way, creating a chair for him in 1872. [...] He married Anna Birch and they would have three children, one daughter and two sons.

[...] This led to combining the transformations in a way that Lie called an infinitesimal group, but which is not a group with our definition, rather what is today called a Lie algebra. It was during the winter of 1873-74 that Lie began to develop systematically what became his theory of continuous transformation groups, later called Lie groups leaving behind his original intention of examining partial differential equations. [...] Engel then was appointed to Leipzig and, when Klein left the chair at Leipzig in 1886, Lie was appointed to succeed him. The collaboration between Engel and Lie continued for nine years culminating with their joint major publication "Theorie der Transformationsgruppen" in three volumes between 1888 and 1893. This was Lie's major work on continuous groups of transformations.

[...] Towards the end of the 1880s Lie's relationship with Engel broke down. In 1892 the lifelong friendship between Lie and Klein broke down and the following year Lie publicly attacked Klein [...] Perhaps an indication of Lie's love for his homeland is the fact that he continued to hold his chair in Christiania from his first appointment in 1872, being officially on leave while holding the chair in Leipzig. However his health was already deteriorating when he returned to a chair in Christiania in 1898, and he died of pernicious anaemia in February 1899 soon after taking up the post. [...] —

[From www-history.mcs.st-andrews.ac.uk/Biographies/Lie.html (J. J. O'Connor and E. F. Robertson)]