Symmetries in Physics - Fall 2018/19

Bielefeld University

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Exercise Nr. 7

Discussion on November 26, 14:15-15:45, Room U2-135

Exercises 17) should be handed in **before** the tutorial.

17) Motions in Euclidean Space *(2+2+2+2=8 points)*

a) Compute the resulting rotation of the product $R(\pi/2, \vec{e}_1)R(\pi/2, \vec{e}_2)R(\pi/2, \vec{e}_1)^\dagger$ in SO(3).

b) Let $(\vec{a}, R)$ be some motion. Compute the same motion in a different basis of the Euclidean space, obtaind by shifting the origin via $\vec{a}_1$ and rotating the Cartesian basis via R_1.

c) Show with your result from b) that the group of translations is indeed an Abelian normal subgroup of E_n.

c) Show that $E^+(n)$ is a normal subgroup of $E(n)$.

18) Euclidean group *(6+2+4=12 points)*

a) Show that the following properties are invariants of the Euclidean motions (Euclidean geometry), implied by the invariance of the Euclidean metric:

$$d(\vec{p}, \vec{q}) = \|\vec{q} - \vec{p}\| = \sqrt{\langle \vec{p} \,|\, \vec{q} \rangle} = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + \ldots + (q_n - p_n)^2} \quad \text{for} \quad \vec{p}, \vec{q} \in E_n$$

- collinearity: three or more points on a line are stil on a line after the motion
- parallelism: two lines which are parallel continue to be parallel after the motion
- barycenters: the center of mass (weighted sum of points) after the motion is the moved center of mass of the moved points.

b) Show that the glide reflections in the Euclidean plane given by $(x, y) \mapsto (x + a, -y)$ form a group and determine the set of fixed points.

c) From the set of Euclidean motions: {translations, rotations around a specific axis, reflection with respect to a specific plane, inversion in a point, glide reflections with respect a specific plane} specifiy all possible pairs of motions that commute with each other.

Euclid of Alexandria
(around 300 BC)

Euclid of Alexandria is the most prominent mathematician of antiquity best known for his treatise on mathematics "The Elements". The long lasting nature of "The Elements" must make Euclid the leading mathematics teacher of all time. However little is known of Euclid's life except that he taught at Alexandria in Egypt. [...] We shall assume in this article that [it is true that Euclid was an historical character, and not a leader of a team of mathematicians ...] Euclid's most famous work is his treatise on mathematics "The Elements". The book was a compilation of knowledge that became the centre of mathematical teaching for 2000 years. Probably no results in "The Elements" were first proved by Euclid but the organisation of the material and its exposition are certainly due to him. In fact there is ample evidence that Euclid is using earlier textbooks as he writes the "Elements" since he introduces quite a number of definitions which are never used such as that of an oblong, a rhombus, and a rhomboid.

"The Elements" begins with definitions and five postulates. The first three postulates are postulates of construction, for example the first postulate states that it is possible to draw a straight line between any two points. These postulates also implicitly assume the existence of points, lines and circles and then the existence of other geometric objects are deduced from the fact that these exist. There are other assumptions in the postulates which are not explicit. For example it is assumed that there is a unique line joining any two points. Similarly postulates two and three, on producing straight lines and drawing circles, respectively, assume the uniqueness of the objects the possibility of whose construction is being postulated.

The fourth and fifth postulates are of a different nature. Postulate four states that all right angles are equal. This may seem "obvious" but it actually assumes that space in homogeneous - by this we mean that a figure will be independent of the position in space in which it is placed. The famous fifth, or parallel, postulate states that one and only one line can be drawn through a point parallel to a given line. Euclid's decision to make this a postulate led to Euclidean geometry. It was not until the 19th century that this postulate was dropped and non-euclidean geometries were studied.

There are also axioms which Euclid calls 'common notions'. These are not specific geometrical properties but rather general assumptions which allow mathematics to proceed as a deductive science. For example:- Things which are equal to the same thing are equal to each other. [...] The Elements is divided into 13 books. Books one to six deal with plane geometry. In particular books one and two set out basic properties of triangles, parallels, parallelograms, rectangles and squares. Book three studies properties of the circle while book four deals with problems about circles and is thought largely to set out work of the followers of Pythagoras. Book five lays out the work of Eudoxus on proportion applied to commensurable and incommensurable magnitudes. [...] Book six looks at applications of the results of book five to plane geometry. Books seven to nine deal with number theory. In particular book seven is a self-contained introduction to number theory and contains the Euclidean algorithm for finding the greatest common divisor of two numbers. [...] Book ten deals with the theory of irrational numbers and is mainly the work of Theaetetus. [...] Books eleven to thirteen deal with three-dimensional geometry. In book eleven the basic definitions needed for the three books together are given. The theorems then follow a fairly similar pattern to the two-dimensional analogues previously given in books one and four. The main results of book twelve are that circles are to one another as the squares of their diameters and that spheres are to each other as the cubes of their diameters. [...] The Elements ends with book thirteen which discusses the properties of the five regular polyhedra and gives a proof that there are precisely five. This book appears to be based largely on an earlier treatise by Theaetetus.

Euclid's Elements is remarkable for the clarity with which the theorems are stated and proved. The standard of rigour was to become a goal for the inventors of the calculus centuries later. [...] Euclid may not have been a first class mathematician but the long lasting nature of "The Elements" must make him the leading mathematics teacher of antiquity or perhaps of all time. [...]

[From www-history.mcs.st-andrews.ac.uk/Biographies/Euclid.html (J. J. O'Connor and E. F. Robertson)]