Symmetries in Physics - Fall 2018/19

Bielefeld University

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Exercise Nr. 6

Discussion on November 19, 14:15-15:45, Room U2-135

Exercises 14) should be handed in **before** the tutorial.

14) Octahedral group *(1+3+4+2+1=11 points)*

a) Explain in your own words why the full octahedral group O_h is both the symmetry group of the octahedron and the cube (i.e. how are the symmetry transformations related, given these Platonic solids are dual to each other?)

b) Write down the chiral octahedral group $O < O_h$ explicity in terms of permutations on the corners, i. e. as s subgroup of S_8 using the cycle notation.

c) Show that the symmetric group S_4 is presented by both

$$\langle \sigma_1, \sigma_2, \sigma_3 \,|\, \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = e, \sigma_1\sigma_3 = \sigma_3\sigma_1, (\sigma_1\sigma_2)^3 = (\sigma_2\sigma_3)^3 = e\rangle$$

$$\text{and} \quad \langle s, t \,|\, s^2 = e, t^3 = e, (st)^4 = e\rangle$$

d) Verify via c) that the chiral octahedral group is isomorphic to S_4.

e) Explain in your own words why the full octahedral group O_h has twice as many group elements as the chiral octahedral group O.

15) Isometries *(4 points)*

Given a group $\mathcal{G}$ and its group action Φ on a set X, show that the relation

$$x \sim y \;\Leftrightarrow\; \exists g \in \mathcal{G} : \Phi(g, x) = y$$

is an equivalence relation and that the orbits are indeed its equivalence classes.

16) 2-dimensional cyrstals *(3+2=5 points)*

Among the symmetries of the Euclidean plane are the discrete infinite symmetry groups of translations and rotations of points that form lattices:

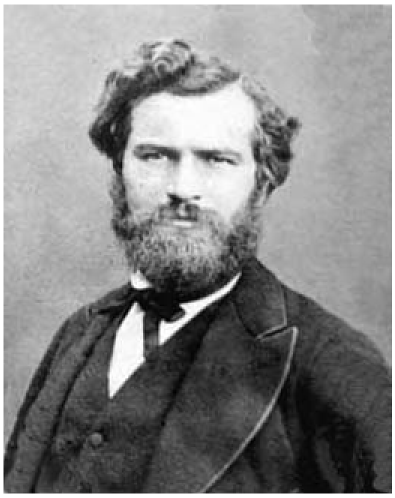
$$\Lambda = \{a_1\vec{v}_1 + a_2\vec{v}_2 \,|\, a_1, a_2 \in \mathbb{Z}\}$$

with $\vec{v}_1$, $\vec{v}_2$ linear independent vectors in $\mathbb{R}^2$.

a) Show that there are 5 types of lattices (classified by symmetries).

b) Discuss the relation of the hexagonal lattice and square lattice to the dihedral group D_3 and D_4.

Marie Ennemond Camille Jordan
(5 January 1838 - 22 January 1922)



Jordan studied at the Lycée de Lyon and at the Collège d'Oullins. He entered the École Polytechnique to study mathematics in 1855. [...] Jordan's doctoral thesis was in two parts with the first part "Sur le nombre des valeurs des fonctions" being on algebra. The second part entitled "Sur des periodes des fonctions inverses des intégrales des différentielles algebriques" was on integrals [...] Jordan married Marie-Isabelle Munet, the daughter of the deputy mayor of Lyon, in 1862. They had eight children, two daughters and six sons. From 1873 he was an examiner at the École Polytechnique where he became professor of analysis on 25 November 1876. He was also a professor at the Collg̀e de France from 1883 although until 1885 he was at least theoretically still an engineer by profession. [...]
Jordan was a mathematician who worked in a wide variety of different areas essentially contributing to every mathematical topic which was studied at that time. [...]

Topology (called analysis situs at that time) played a major role in some of his first publications which were a combinatorial approach to symmetries. [...] He defined a homotopy group of a surface without explicitly using group terminology. Jordan was particularly interested in the theory of finite groups. [...]

To Jordan a group was what we would call today a permutation group; the concept of an abstract group would only be studied later. To give an illustration of the way he tried to build up groups theory we will say a little about his contributions to finite soluble groups. The standard way to define such groups today would be to say that they are groups whose composition factors are abelian groups. Indeed Jordan introduced the concept of a composition series (a series of subgroups each normal in the preceding with the property that no further terms could be added to the series so that it retains that property). The composition factors of a group G are the groups obtained by computing the factor groups of adjacent groups in the composition series. Jordan proved the Jordan-Hölder theorem, namely that although groups can have different composition series, the set of composition factors is an invariant of the group.

Although the classification of finite abelian groups is straightforward, the classification of finite soluble groups is well beyond mathematicians today and for the foreseeable future. Jordan, however, clearly saw this as an aim of the subject, even if it was not one which might ever be solved. He made some remarkable contributions to how such a classification might proceed setting up a recursive method to determine all soluble groups of order n for a given n.

A second major piece of work on finite groups was the study of the general linear group over the field with p elements, p prime. He applied his work on classical groups to determine the structure of the Galois group of equations whose roots were chosen to be associated with certain geometrical configurations. [...]

Jordan's use of the group concept in geometry in 1869 was motivated by studies of crystal structure. He considered the classification of groups of Euclidean motions. His work had gained him a wide international reputation and both Sophus Lie and Felix Klein visited him in Paris in 1870 to study with him. Jordan's interest in groups of Euclidean transformations in three dimensional space influenced Lie and Klein in their own theories of continuous and discontinuous groups. [...] His finiteness theorem showed that for a given c there are only finitely many primitive groups with class c other than the symmetric and alternating groups.

Generalising a result of Fuchs on linear differential equations, Jordan was led to study the finite subgroups of the general linear group of $n \times n$ matrices over the complex numbers. Although there are infinite families of such finite subgroups, Jordan found that they were of a very specific group theoretic structure which he was able to describe. Another generalisation, this time of work by Hermite on quadratic forms with integral coefficients, led Jordan to consider the special linear group of $n \times n$ matrices of determinant 1 over the complex numbers acting on the vector space of complex polynomials in n indeterminates of degree m. [...]

In 1912 Jordan retired from his positions. The final years of his life were saddened, however, because of World War I which began in 1914. Between 1914 and 1916 three of his six sons were killed in the war. [...]

[From www-history.mcs.st-andrews.ac.uk/Biographies/Jordan.html [J. J. O'Connor and E. F. Robertson]