

Symmetries in Physics - Fall 2018/19

Bielefeld University

Lecture: Wolfgang Unger

Office: E6-118

wunger@physik.uni-bielefeld.de

Exercise Nr. 2

Discussion on October 22, 14:15-15:45, Room U2-135

Exercises 4) should be handed in **before** the tutorial.

4) Cayley Tables ($3+2+2=7$ points)

a) Fill in the missing elements of the following Cayley tables, each defining a group of order 4:

$\circ$	e	a	b	c
e	.	.	.	.
a	.	e	.	.
b	.	.	e	.
c	.	.	.	.

$\circ$	e	a	b	c
e	.	.	.	.
a	.	b	.	.
b	.	.	e	.
c	.	.	.	.

$\circ$	e	a	b	c
e	.	.	.	.
a	.	c	.	.
b	.	.	c	.
c	.	.	.	.

- b) Are all groups Abelian? Can you tell whether some of the groups above are isomorphic?
- c) Work out the symmetry group of a rectangle (which shall not be a square!) Can you identify one of the above groups with this symmetry group?

5) Symmetry Group of the Triangle ($3+1=4$ points)

- a) Consider the symmetry group of the equilateral triangle.
Show that there is a symmetry transformation for every permutation of the corners of the triangle.
- b) Consider the symmetry group of the equiscale triangle. What permutations are allowed here?

6) Matrix Groups ($2+3+4=9$ points)

- a) Show that the general linear group $\text{GL}(n, \mathbb{C})$ forms a group by explicitly checking the group axioms.
- b) Show that the upper triangular matrices

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

with $a, b, c \in \mathbb{R}^+ \setminus \{0\}$ form a group under the usual matrix multiplication. When is the group Abelian? Does the group structure generalize to higher-dimensional triangular matrices?

- c) Let $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$ be the complex numbers without the 0. Show that the diagonal matrices $\text{Diag}(n, \mathbb{C}_*)$ form an Abelian group under matrix multiplication. Show also that the following map is an endomorphism for $m \in \mathbb{N}$, and an automorphism for $m > 0$:

$$\phi_m : \text{Diag}(n, \mathbb{C}_*) \rightarrow \text{Diag}(n, \mathbb{C}_*), \quad A \mapsto A^m$$

Niels Hendik Abel

(5 August 1802 - 6 April 1829)

Norwegian mathematician who accomplished an amazing amount of brilliant work in his short lifetime. Abel was born on August 5, 1802 in the small village of Findoe, Norway, where his father was minister in the diocese of Christiansand. Abel's life was spent in poverty, caused by the large size of his family (he had six brothers and his father died when he was only eighteen) and the difficult economic situation in Norway at that time. Abel died of tuberculosis at the age of 26 after being forced to live in miserable conditions because of his inability to obtain a university post.



At the age of 16, Abel gave a proof of the binomial theorem valid for all numbers, extending Euler's result which had only held for rationals. At age 19, he showed there is no general algebraic solution for the roots of a quintic equation, or any general polynomial equation of degree greater than four, in terms of explicit algebraic operations. To do this, he invented (independently of Galois) an extremely important branch of mathematics known as group theory, which is invaluable not only in many areas of mathematics, but for much of physics as well. Among his other accomplishments, Abel wrote a monumental work on elliptic functions which, however, was not discovered until after his death. When asked how he developed his mathematical abilities so rapidly, he replied "by studying the masters, not their pupils."

Abel sent a paper on the unsolvability of the quintic equation to Gauss, who proceeded to discard without a glance what he believed to be the worthless work of a crank. In 1825, the Norwegian government funded Abel on a scholarly visit to France and Germany. Abel then traveled to Paris, where he gave an important paper revealing the double periodicity of the elliptic functions, which Legendre later described to Cauchy as "a monument more lasting than bronze" (borrowing a famous sentence by the Roman poet Horatius). However, Cauchy proceeded to misplace the manuscript. In Berlin, In Berlin, Abel met and was befriended by August Crelle, an amateur mathematician who had founded the famous *Journal für die reine und angewandte Mathematik* (Journal for pure and applied mathematics), which had published several papers by Abel.

However, an offer of a professorship in Berlin was not forthcoming for four years, by which time it was too late. A letter from Crelle arrived two days after Abel's death, informing him that he had been offered professorship at the University of Berlin

[From scienceworld.wolfram.com (Eric W. Weisstein)]