# Symmetries in Physics WS 2018/19 

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## Chapter 1

## Introduction

### 1.1 The Many Faces of Symmetries

$\rightarrow$ see slides

### 1.1.1 The Notion of Symmetry

### 1.1.2 Symmetries in Nature

### 1.1.3 Symmetries in the Arts

### 1.1.4 Symmetries in Mathematics

### 1.1.5 Symmetries in Physical Phenomena

### 1.2 Symmetries of States and Invariants of Natural Laws

Symmetries ..

- ... are the guiding principles for understanding the laws of nature
- ....also apply to the states found in nature: crystals, molecules, atoms, and sub-nuclear particles.
- ....are important for classical physics, where the dynamics give rise to conservation laws (invariants), and in quantum physics, where the quantum states are related by symmetries

We do not need to know the specific structure of the laws:

- they determine conserved currents and selection rules
- mathematical treatment of symmetries is an important tool for physics

To specify the last statement, we will shortly review classical and quantum physics.

### 1.2.1 Structure of Classical Physics

Let us review the structure of classical mechanics:

- A classical dynamical system can be described by its Lagrangian $\mathcal{L}(q, \dot{q})$, with $q=\left\{q_{i}\right\}_{i=1 \ldots N}$ a set of generalized coordinates
- Typically one chooses coordinates $q$ that reflect the symmetries of the problem (e.g. polar or spherical coordinates)
- From $\mathcal{L}$ we derive the equations of motion via the Euler-Lagrange equation:

$$
\begin{equation*}
\frac{d}{d t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)=\frac{\partial \mathcal{L}}{\partial q} \tag{1.1}
\end{equation*}
$$

- To solve for $q(t)$, one has to solve complicated differential equations, which can only be done exactly in very few cases (integrable system\& ${ }^{11}$ )
- One can however construct a family of solutions:

$$
\begin{equation*}
q(t, \phi)=g_{\phi} q(t) \quad \text { such that } \quad q(t)=q(t, 0) \tag{1.2}
\end{equation*}
$$

- Two simple example are:
- translation symmetry: $q(t, \phi)=q(t)+\phi$
- rotation symmetry: $q(t)=R(\phi) q(0)\left(\right.$ with $\left.R(\phi) R^{T}(\phi)=1\right)$
- The existence of such symmetries follows from symmetries of the Lagrangian itself, that is the invariance under the symmetry transformation

$$
\begin{equation*}
\mathcal{L}(q, \dot{q}) d t=\mathcal{L}\left(g_{\phi} q, g_{\phi} \dot{q}\right) d\left(g_{\phi} t\right) \tag{1.3}
\end{equation*}
$$

- Important: with any continuous symmetry of the Lagrangian we can associate a conserved current (Noether's theorem, see Sec. ??).


### 1.2.2 Structure of Quantum Mechanics

The above observations carry over to quantum mechanics, but here we have additional structure. Let us review the structure of quantum mechanics:

- The states $|\psi\rangle$ form a vector space $\mathcal{H}$ (Hilbert space) equipped with a scalar (inner) product

$$
\begin{equation*}
\mathcal{H} \otimes \mathcal{H} \rightarrow \mathbb{C}:\left(\left|\psi_{1}\right\rangle,\left|\psi_{2}\right\rangle\right) \mapsto\left\langle\psi_{1} \mid \psi_{2}\right\rangle \quad \text { and } \quad\left\langle\psi_{2} \mid \psi_{1}\right\rangle=\left\langle\psi_{1} \mid \psi_{2}\right\rangle^{*} \tag{1.4}
\end{equation*}
$$

- Physical magnitudes are operators on $\mathcal{H}$, i. e. functions $\hat{Q}: \mathcal{H} \rightarrow \mathcal{H}$
- The Hamilton operator $\hat{H}$ is particularly important as it features in the Schrödinger Equation as time evolution operator:

$$
\begin{equation*}
i \hbar \frac{d}{d t}|\psi\rangle=\hat{H}|\psi\rangle \tag{1.5}
\end{equation*}
$$

- The energy eigenstates $\left|\psi_{n}\right\rangle$ are the observable, stable states: $\hat{H}\left|\psi_{n}\right\rangle=E_{n}\left|\psi_{n}\right\rangle$

[^0]- Physical properties are determined by expectation values of Hermitian operators: $\langle\hat{Q}\rangle_{\psi}=$ $\langle\psi| \hat{Q}|\psi\rangle$

Symmetries are present if ...

- ... one finds a set of operators $\left\{\hat{Q}_{i}\right\}$ which commute with themselves $\left[\hat{Q}_{i}, \hat{Q}_{j}\right]=0$ and commute with the Hamilton operator: $\left[\hat{H}, \hat{Q}_{i}\right]=0$
- The operators $\hat{H}, \hat{Q}_{i}$ then have a joint set of eigenstates.
(Linear Algebra: Matrices are simultaneous diagonalizable iff they commute pairwise)
- The degeneracy of energy eigenstates depend on their eigenstates with respect to $Q_{i}$.

Why are there symmetries?

- The degenerate states look "similar"
- Consider the transformations of Hamiltonian:

$$
\begin{equation*}
\hat{H}^{\prime} \equiv e^{i \alpha \hat{Q}_{i}} \hat{H} e^{-i \alpha \hat{Q}_{i}} \quad \text { with } \quad \alpha \in \mathbb{R} \quad \text { leaves } \quad \hat{H} \quad \text { invariant: } \quad \hat{H}^{\prime}=\hat{H} \tag{1.6}
\end{equation*}
$$

- $\langle\psi| \hat{Q}_{i},|\psi\rangle$ is a conserved quantity:

$$
\begin{equation*}
i \hbar \frac{\partial}{\partial t}\langle\psi| \hat{Q}_{i}|\psi\rangle=\langle\psi|\left[\hat{Q}_{i}, \hat{H}\right]|\psi\rangle=0 \tag{1.7}
\end{equation*}
$$

- the Noether Theorem states: there must be a symmetry associated with $\hat{Q}_{i}$
- Symmetries of the Hamiltonian may be broken due to spontaneous symmetry breaking (e.g. ferromagnet)
- Symmetries that are present classically may be broken by quantum effects (quantum anomalies)


### 1.2.3 Role of Mathematics

What is the mathematics needed to describe symmetries?

- Transformations under which $\hat{H}$ is invariant form a group (second chapter).
- Degenerate eigenstates form invariant subspace of $\mathcal{H}$, and are irreducible representations of the group (third chapter)
- there are many general statements about irreducible representations (e. g. on the relation between the number or irreducible representations and the degeneracies observed in the physical system)


### 1.2.4 Examples

Symmetries of crystals and molecules:

- Discrete transformations: $z \mapsto z$ and $z \mapsto-z$ (e.g. a symmetric potential $V(z)$ )
- Amounts to two representations: symmetric and anti-symmetric wave functions (both onedimensional, no degeneracy)
"Dynamics"


Figure 1.1: System with a discrete transformation $z \mapsto z$ and $z \mapsto-z$

Hydrogen atom:

- In absence of external fields: rotation symmetry $\mathrm{SO}(3)$
- The angular momentum is characterized by orbital quantum number $l:(2 l+1)$-dimensional representations, $l=0,1,2, \ldots$
- $(2 l+1)$-fold degeneracy $(m=-l, \ldots l$, lifted in external field $)$
"Dynamics"
$\Delta$ Energy

$\qquad$
$\begin{array}{lllll}1 & 1 & 1 & \mid & \text { "Representation" } \\ 0 & 1 & 2 & 3 & \text { I (orbital quantum number) }\end{array}$

Figure 1.2: Hydrogen atom with (2l+1)-fold degeneracy of its energy states

- The light quarks up, down and strange can be assumed almost mass degenerate
- inner flavor symmetry: continuous compact group $\mathrm{SU}(3)$
- "Eightfold way" (Gell-Mann matrices)


Figure 1.3: Hadronic Multiplet in QCD

Poincaré symmetry in particle physics

- Poincare' symmetry (Lorentz symmetry + translations) form a non-compact group, but it has compact parts
- particles have both continuous (momentum) and discrete (spin) properties


## Chapter 2

## Basics of Group Theory

### 2.1 Axioms and Definitions

### 2.1.1 Group Axioms

Definition: A group $\mathcal{G}=(G, \circ)$ is defined by a set $G$ and a binary operation $\circ$ called group composition or multiplication that fulfills the following four axioms:

$$
\begin{align*}
& \text { closure: for all } g, g^{\prime} \in G: g \circ g^{\prime} \in G  \tag{G.0}\\
& \text { associativity: for all } g, g^{\prime}, g^{\prime \prime} \in G:\left(g \circ g^{\prime}\right) \circ g^{\prime \prime}=g \circ\left(g^{\prime} \circ g^{\prime \prime}\right)  \tag{G.1}\\
& \text { neutral element: there is an element } e \text { such that for all } g \in G: e \circ g=g \circ e=g \tag{G.2}
\end{align*}
$$

inverse element: for all $g \in G$ there is an element $g^{-1}$ such that $g^{-1} \circ g=g \circ g^{-1}=e$

## Remarks:

- The closure axiom Eq. G.0p can be omitted if $\circ: G \times G \rightarrow G$ is taken to be an internal composition.
- The associativity axiom Eq. G.1 implies that parantheses can be omitted.
- The axioms of the existence of a neutral element Eq. (G.2) and inverse element Eq. G.3) can be weakened: we can equally demand that there exists a left neutral $g \circ e=g$ element, or a left inverse element $g^{-1} \circ g=e$.
- The inverse of a product is $\left(g_{1} \circ g_{2} \circ \ldots \circ g_{n}\right)^{-1}=g_{n}{ }^{-1} \circ \ldots g_{2}{ }^{-1} \circ g_{1}{ }^{-1}$.
- The existence of a neutral element, often also called identity element, implies that the set $G$ is not empty.
- If one omits the axioms Eq. G.3 and Eq. G.4 the structure $(G, \circ)$ is called a semigroup (which can be empty)


## Notation:

- Oftentimes we will omit the binary operation, writing $g g^{\prime}$ instead $g \circ g^{\prime}$ for the composition.
- The repeated application of the same element will be shortend by $g^{n}$ with $n \in \mathbb{N}$ (via $g^{0}=e$ and $g^{n+1}=g g^{n}$ )
- We write $g \in \mathcal{G}$ instead $g \in G$ for $\mathcal{G}=(G, \circ)$.

Theorem: Every Group $\mathcal{G}$ has a unique neutral element $e$ and every element $g \in \mathcal{G}$ has a unique inverse element $g^{-1}$.

## Proof:

1. Assume both $e$ and $g$ are neutral elements, then $g \circ g=g$ and $g^{-1} \circ g \circ g=g^{-1} \circ g=e$. Hence $g=e$. We also found that $e$ is its own inverse element.
2. Assume both $g^{\prime}$ and $g^{\prime \prime}$ are inverse elements of $g$, then $g^{\prime} \circ g=e=g \circ g^{\prime \prime}$ and $g^{\prime}=g^{\prime} \circ e=g^{\prime} \circ\left(g \circ g^{\prime \prime}\right)=\left(g^{\prime} \circ g\right) \circ g^{\prime \prime}=g^{\prime \prime}$

Definition: A group $\mathcal{G}$ is called Abelian if the group composition is commutative:

$$
\begin{equation*}
\text { commutativity: for all } g, g^{\prime} \in G: g \circ g^{\prime}=g^{\prime} \circ g \tag{G.4}
\end{equation*}
$$

## Remarks:

- Often the group composition of Abelian groups is denoted with "+"

Definition: The order of a group is the number of the elements of the underlying set: $|\mathcal{G}|=|G|$. If $|\mathcal{G}|<\infty$, the group is called finite. If it is not finite, the set of group elements can be countable or uncountable. Among the uncountable groups we can further distinguish compact groups and non-compact groups (see ??).
For finite groups, one can define the group by writing down all possible compositions in a multiplication table called Cayley table:.

| $\circ$ | $e$ | $g_{2}$ | $g_{3}$ | $\ldots$ | $g_{n}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $e$ | $e$ | $g_{2}$ | $g_{3}$ | $\ldots$ | $g_{n}$ |
| $g_{2}$ | $g_{2}$ | $g_{2} \circ g_{2}$ | $g_{2} \circ g_{3}$ | $\ldots$ | $g_{2} \circ g_{n}$ |
| $g_{3}$ | $g_{3}$ | $g_{3} \circ g_{2}$ | $g_{3} \circ g_{3}$ | $\ldots$ | $g_{3} \circ g_{n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |  | $\vdots$ |
| $g_{n}$ | $g_{n}$ | $g_{n} \circ g_{2}$ | $g_{n} \circ g_{3}$ | $\ldots$ | $g_{n} \circ g_{n}$ |

Table 2.1: Cayley table: every entry is the result of a multiplication of the element denoting a with the element denoting the column.

## Remarks:

- in every row (and in every column) each group element appears exactly once, since $g \circ g^{\prime}=g \circ g^{\prime \prime}$ implies $g^{\prime}=g^{\prime \prime}$
- the inverse elements can be read off from the table by identifying the coordinates of the neutral elements.
- if the group is Abelian, the multiplication table is symmetric
- a rearrangement of the group will produce the same group:

$$
\begin{equation*}
\text { define for any } g \in \mathcal{G}: g \circ \mathcal{G}=\left\{g g^{\prime} \mid g^{\prime} \in G\right\} \text {, then } g \circ \mathcal{G}=\mathcal{G} \tag{2.1}
\end{equation*}
$$

### 2.1.2 Examples: Numbers

Examples of groups in mathematics involving numbers are:
E.1.1) The set $\mathbb{Z}_{2}=\{+1,-1\}$ with the multiplication forms a group $\left(\mathbb{Z}_{2}, *\right)$ with 1 the neutral element and each element its own inverse:

| $*$ | 1 | -1 |
| ---: | ---: | ---: |
| 1 | 1 | -1 |
| -1 | -1 | 1 |

E.1.2) A similar group is obtained by $\tilde{\mathbb{Z}}_{2}=\{0,1\}$ with the addition modulo 2. Here, 0 the neutral element:

| + | 0 | 1 |
| :---: | :---: | :---: |
| 0 | 0 | 1 |
| 1 | 1 | 0 |

E.1.3) the integer numbers with the addition: $(\mathbb{Z},+)$, where the 0 is the neutral element and $-n$ is the inverse of $n$ (countable).
E.1.4) the non-zero rational numbers with the multiplication: ( $\mathbb{Q} \backslash\{0\}, *$ ) with 1 the neutral element and $1 / x$ the inverse element of $x$ (countable)
E.1.5) the unit circle in the complex plane, $\left(\left\{e^{i \phi} \mid \phi \in[0,2 \pi]\right\}, *\right)$, or equivalently $(\{\phi \in[0,2 \pi]\} \mid),+$ $\bmod 2 \pi)$ (uncountable)
E.1.6) the 4 roots of unity under multiplication: $(\{1, i,-1,-i\}, *)$

| $*$ | 1 | $i$ | -1 | $-i$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | $i$ | -1 | $-i$ |
| $i$ | $i$ | -1 | $-i$ | 1 |
| -1 | -1 | $-i$ | 1 | $i$ |
| $-i$ | $-i$ | 1 | $i$ | -1 |

form an Abelian group.

### 2.1.3 Examples: Permutations

We find permutations of sets in many contexts, e. g. in shuffling cards. A permutation $\pi$ of $n$ objects can be denoted by

$$
\left(\begin{array}{cccc}
1 & 2 & \ldots & n  \tag{2.2}\\
\pi(1) & \pi(2) & \ldots & \pi(n)
\end{array}\right)
$$

Let us consider some examples:
E.2.1) With the $3!=6$ permutations $e=\binom{123}{123}, \tau_{12}=\left(\begin{array}{ll}123 \\ 2 & 1\end{array}\right), \tau_{13}=\left(\begin{array}{ll}123 \\ 3 & 2\end{array}\right), \tau_{23}=$ $\binom{123}{13}, \sigma=\left(\begin{array}{ll}1 & 2\end{array}\right)$ and $\bar{\sigma}=\left(\begin{array}{ll}1 & 2 \\ 3 & 1\end{array}\right)$ the set of permutations, where the second row is the permutation of distinctive objects of the first row (denoted by numbers), the set $\left\{e, \tau_{12}, \tau_{23}, \tau_{13}, \sigma, \bar{\sigma}\right\}$ together with the successive execution forms a group:

| $\circ$ | $e$ | $\tau_{12}$ | $\tau_{23}$ | $\tau_{13}$ | $\sigma$ | $\bar{\sigma}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $e$ | $e$ | $\tau_{12}$ | $\tau_{23}$ | $\tau_{13}$ | $\sigma$ | $\bar{\sigma}$ |
| $\tau_{12}$ | $\tau_{12}$ | $e$ | $\sigma$ | $\bar{\sigma}$ | $\tau_{23}$ | $\tau_{13}$ |
| $\tau_{23}$ | $\tau_{23}$ | $\bar{\sigma}$ | $e$ | $\sigma$ | $\tau_{13}$ | $\tau_{12}$ |
| $\tau_{13}$ | $\tau_{13}$ | $\sigma$ | $\bar{\sigma}$ | $e$ | $\tau_{12}$ | $\tau_{23}$ |
| $\sigma$ | $\sigma$ | $\tau_{13}$ | $\tau_{12}$ | $\tau_{23}$ | $\bar{\sigma}$ | $e$ |
| $\bar{\sigma}$ | $\bar{\sigma}$ | $\tau_{23}$ | $\tau_{13}$ | $\tau_{12}$ | $e$ | $\sigma$ |

The $\tau_{i j}$ are transpositions (which are their own inverse elements), and the $\sigma$ is cyclic permutations (with $\bar{\sigma}$ the inverse of $\sigma$. Permutations of more than three objects also involve also other elements. This group is non-Abelian. More aspects of permutations will be discussed in section ?? on the symmetric group.

### 2.1.4 Examples: Matrix Groups

An important class of groups are matrix groups:, defined as subsets of all $n \times n$ matrices $\operatorname{Mat}(n, \mathbb{K})$ over a field $\mathbb{K}$ (usually $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$ ), with the matrix multiplication as group composition
E.3.1) the $\operatorname{group} \operatorname{GL}(n, K)=\{A \in \operatorname{Mat}(n, \mathbb{K} \mid \operatorname{det} A \neq 1\}$ of all invertible matrices is called the general linear group
E.3.2) the group $\operatorname{SL}(n, K)=\{A \in \operatorname{Mat}(n, \mathbb{K} \mid \operatorname{det} A=1\}$ of all matrices with determinant 1 is called the special linear group

### 2.1.5 Group Presentation

Definition: A group $\mathcal{G}=(G, \circ)$ can be characterized by a generating set, i.e. a finite set $S \subseteq G$, such that the every element from $(G, \circ)$ can be written as a product of powers of some elements of $a_{i} \in S$ :

$$
\begin{equation*}
\text { for all } g \in \mathcal{G}: g=a_{i_{1}} \ldots a_{i_{n}} \tag{2.3}
\end{equation*}
$$

with the right-hand side a word over the alphabet from $S$. A group presentation $\langle S \mid R\rangle$ is then given by the set $S$ of generators and a set of relations $R$ among these generators. The relations are used to simplify the words.
A minimal generating set $S$ is called a basis. The rank of a group $\operatorname{Rank}(\mathcal{G})$ is the cardinality of such a minimal set.

## Remarks:

- A generating set $S$, and also a basis is not unique.
- The so-called free group, which does not have any relations, is the most general group that can be generated by the generating set.
- Every group has a presentation.(The proof involves free groups.)


## Example:

E.4.1) The example E.1.6 can be expressed in powers of the imaginary unit $i$ only:

| $*$ | 1 | $i$ | -1 | $-i$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | $i^{0}$ | $i^{1}$ | $i^{2}$ | $i^{3}$ |
| $i$ | $i^{1}$ | $i^{2}$ | $i^{3}$ | $i^{0}$ |
| -1 | $i^{2}$ | $i^{3}$ | $i^{0}$ | $i^{1}$ |
| $-i$ | $i^{3}$ | $i^{0}$ | $i^{1}$ | $i^{2}$ |

The set of words is: $\left\{i^{0}, i^{1}, i^{2}, i^{3}\right\}$. Hence its group presentation is $\left\langle i \mid i^{4}=1\right\rangle$. An equivalent presentation with another basis is $\left\langle-i \mid(-i)^{4}=1\right\rangle$
E.4.2) The group of permutations of 3 elements, presented by the basis $\left\{\tau_{12}, \sigma\right\}$ consists of the 6 words $\left\{e, \tau_{12}, \tau_{12} \sigma, \tau_{12} \sigma^{2}, \sigma, \sigma^{2}\right\}$, the group presentation is $\left\langle\tau_{12}, \sigma \mid \tau_{12}^{2}=e, \sigma^{3}=e, \sigma \tau_{12} \sigma^{-2} \tau=e\right\rangle$

### 2.2 Relations Between Group

### 2.2.1 Homomorphism, Isomorphisms

We found in the above example that the Cayley table of the examples () and () are structurally identical (up to renaming the group elements). Here, we want to make this precise:

Definition: A map between two groups $\phi: \mathcal{G} \rightarrow \mathcal{G}^{\prime}, g \mapsto g^{\prime}=\phi(g)$ is called group homomorphism, written $\phi \in \operatorname{Hom}\left(\mathcal{G}, \mathcal{G}^{\prime}\right)$, if the group compositions of $\mathcal{G}^{\prime}$ is consistent with the group composition in $\mathcal{G}^{\prime}$ :

$$
\begin{equation*}
\phi\left(g_{1} \circ g_{2}\right)=\phi\left(g_{1}\right) \circ \phi\left(g_{2}\right) \tag{2.4}
\end{equation*}
$$

If $\phi \in \operatorname{Hom}\left(\mathcal{G}, \mathcal{G}^{\prime}\right)$ is also bijective, it is called isomorphism.
Two groups $\mathcal{G}, \mathcal{G}^{\prime}$ are isomorphic, written $\mathcal{G} \cong \mathcal{G}^{\prime}$, if there exists an isomorphism $\phi: \mathcal{G} \rightarrow \mathcal{G}^{\prime}$
If the map is onto itself, $\phi \in \operatorname{Hom}(\mathcal{G}, \mathcal{G})$, it is called an endomorphism, written $\operatorname{End}(\mathcal{G})$. If an endomorphism $\phi$ is bijective, it is called automorphism, written $\phi \in \operatorname{Aut}(\mathcal{G})$

Theorem: Let $\mathcal{G}$ and $\mathcal{G}^{\prime}$ be groups with neutral elements $e \in \mathcal{G}, e^{\prime} \in \mathcal{G}^{\prime}$ and $\phi \in \operatorname{Hom}\left(\mathcal{G}, \mathcal{G}^{\prime}\right)$. Then $\phi(e)=e^{\prime}$ and $\phi\left(g^{-1}\right)=\phi(g)^{-1}$.
Proof: $\quad$ Since $\phi(g)=\phi(g e)=\phi(g) \phi(e)$ we find $\phi(e)=e^{\prime}$. Also, since $e^{\prime}=\phi\left(g g^{-1}\right)=$ $\phi(g) \phi\left(g^{-1}\right)$ we find $\phi\left(g^{-1}\right)=\phi(g)^{-1}$

Theorem: The group isomorphism $\cong$ is an equivalence relation on the set of all groups. Hence every groups belong to some equivalence classes.

Proof: An equivalence relation is (1) symmetric, (2) reflexive and (3) transitive.
(1) Since $\phi: \mathcal{G}_{1} \rightarrow \mathcal{G}_{2}$ is bijective, there also exists the inverse map $\phi^{-1}: \mathcal{G}_{2} \rightarrow \mathcal{G}_{1}$ such that $\mathcal{G}_{2} \cong \mathcal{G}_{1} \Leftrightarrow \mathcal{G}_{2} \cong \mathcal{G}_{1}$
(2) Every group has as the trivial automorphism the identity map $i d \in \operatorname{Aut}(G)$ with $i d(g)=g$ for all $g \in \mathcal{G}$.
(3) If $\phi_{12}: \mathcal{G}_{1} \rightarrow \mathcal{G}_{2}$ and $\phi_{23}: \mathcal{G}_{2} \rightarrow \mathcal{G}_{3}$ are isomorphisms, then $\phi_{13}=\phi_{23} \circ \phi_{12}:$ $\mathcal{G}_{1} \rightarrow \mathcal{G}_{2}$ is an isomorphism due to the concatination $\phi_{13}(g)=\phi_{23}\left(\phi_{12}(g)\right)$ for all $g \in \mathcal{G}_{1}$

Definition: The kernel of $\phi \in \operatorname{Hom}\left(\mathcal{G}, \mathcal{G}^{\prime}\right)$ with $e^{\prime} \in \mathcal{G}^{\prime}$ the neutral element is the set

$$
\begin{equation*}
\operatorname{Ker}(\phi)=\left\{g \in \mathcal{G} \mid \phi(g)=e^{\prime}\right\} \subseteq G . \tag{2.5}
\end{equation*}
$$

The image of $\phi \in \operatorname{Hom}\left(\mathcal{G}, \mathcal{G}^{\prime}\right)$ is the set

$$
\begin{equation*}
\operatorname{Im}(\phi)=\left\{\phi(g) \mid g=\mathcal{G}^{\prime}\right\} \subseteq G^{\prime} \tag{2.6}
\end{equation*}
$$

## Examples:

E.5.1) For any $a \in \mathbb{Q}$, the map $\phi_{a}:(\mathbb{Q},+) \rightarrow(\mathbb{Q},+)$ with $\phi_{a}(x)=a x$ is an endomorphism. It is also an isomorphism for $a \neq 0$.
E.5.2) The exponential map from $(\mathbb{R},+)$ to $\left(\mathbb{R}_{+}, *\right)$ via $\left.e^{(x+y)}=e^{(x)} e^{( } y\right)$ is an isomorphism. The corresponding inverse map is $\log (x y)=\log (x)+\log (y)$
E.5.3) The determinant is a group homomorphism from $\mathrm{GL}(n, \mathbb{C})$ to $\mathbb{C}$ and the
E.5.4) For any Abelian group $\mathcal{G}$ and any integer number $m \in \mathbb{Z}$, the map $\phi_{m} \in \operatorname{End}(\mathcal{G})$ with $\phi_{m}(g)=g^{m}$. For $\mathcal{G}=(\mathbb{Z},+), \operatorname{Ker}\left(\phi_{m}\right)=\{0\}$ and $\operatorname{Im}\left(\phi_{m}\right)=m \mathbb{Z}$.

### 2.2.2 Automorphism and Conjugation

Theorem: The set of all automorphisms $\operatorname{Aut}(\mathcal{G})$ forms itself a group, with the group composition being the usual composition of maps.
Proof: If $\phi_{1}, \phi_{2} \in \operatorname{Aut}(\mathcal{G})$, then $\phi_{1} \circ \phi_{2} \in \operatorname{Aut}(\mathcal{G})$ due to transitivity. The composition of maps is also associative. The identity map $i d \in \operatorname{Aut}(\mathcal{G})$ is the neutral element. Since an automorphism is an isomorphism, if $\phi \in \operatorname{Aut}(\mathcal{G})$ then the inverse map $\phi^{-1} \in \operatorname{Aut}(\mathcal{G})$.

Definition: For every $g \in \mathcal{G}$ there is an automorphism $\phi_{g} \in \operatorname{Aut}(\mathcal{G})$ called conjugation or innner automorphism defined as follows

$$
\begin{equation*}
\phi_{g}: \mathcal{G} \rightarrow \mathcal{G}: \phi_{g}\left(g^{\prime}\right)=g g^{\prime} g^{-1} \tag{2.7}
\end{equation*}
$$

Remarks: The inner automorphisms of $\mathcal{G}$ form a group denoted by $\operatorname{Inn}(\mathcal{G}) \subseteq \operatorname{Aut}(\mathcal{G})$.

### 2.2.3 Symmetry Group

A symmetry group of an object (image, signal, geometric object, physical state) is the group of all transformations under which the object is invariant with composition as the group operation. This can be formalized by the following

Definition: If $\mathcal{G}$ is a group and $X$ is a set, then a group action is a map

$$
\begin{equation*}
\Phi: \mathcal{G} \times X \mapsto X:(g, x) \mapsto \Phi(g, x) \tag{2.8}
\end{equation*}
$$

that satisfies the following axioms:

$$
\begin{gather*}
\text { identity: for all } x \in X: \Phi(e, x)=x  \tag{2.9}\\
\text { compatibility: for all } g, g^{\prime} \in \mathcal{G}: \Phi\left(g, \Phi\left(g^{\prime}, x\right)\right)=\Phi\left(g \circ g^{\prime}, x\right) \tag{2.10}
\end{gather*}
$$

$\mathcal{G}$ is called a transformation group on $X$.
Theorem: Consider the following map obtained by evaluating the group action for a specific group element $g \in \mathcal{G}$ :

$$
\begin{equation*}
\Phi_{g}: X \rightarrow X, \quad x \mapsto \Phi(g, x) \tag{2.11}
\end{equation*}
$$

Then $\Phi_{g}$ is bijective. The set of all bijective maps on $X$

$$
\begin{equation*}
\operatorname{Sym}(X)=\{f: X \rightarrow X \mid f \quad \text { is bijective }\} \tag{2.12}
\end{equation*}
$$

together with the composition of maps forms the so-called symmetric group on $X$.
Proof: Bijection: The inverse map is $\Phi_{g^{-1}}$ since $\Phi_{g} \Phi_{g^{-1}}=\Phi_{g g^{-1}}=\Phi_{e}=\mathrm{id}_{X}$
The neutral element of $\operatorname{Sym}(X)$ is $\operatorname{id}_{X}: x \mapsto x$ and the inverse element for $f \in S(X)$ is the inverse map $f^{-1}$.

## Remarks:

- If the set $X$ is finite, $S(X)$ is the set of all permutations of $X$.
- The map $g \rightarrow \Phi_{g}$ is a group homomorphism $\mathcal{G} \rightarrow S(X)$, i. e. $\Phi_{g g^{\prime}}=\Phi_{g} \Phi_{g^{\prime}}$.
- Every group homomorphism $\phi \in \operatorname{Hom}(\mathcal{G}, S(X))$ induces a group action of $\mathcal{G}$ on $X$ via $\Phi(g, x)=\phi(g)(x)$.
- Group representations (introduced in the next chapter) are precisely the linear group actions on a vector space: $X=V, \mathrm{GL}(V) \leq S(V)$


## Examples:

E.6.1) The symmetry transformations of the equilateral triangle form the group named $C_{3 v}$. The elements are the identity, the reflections on the three axis $\left\{\sigma_{1 v}, \sigma_{2 v}, \sigma_{3 v}\right\}$ and the rotations around the two center by $120^{\circ}$, and $240^{\circ}:\left\{c_{3}, c_{3}^{2}\right\}$.
E.6.2) Isometries in a Euclidean space.

### 2.2.4 Subgroups and Normal Subgroups, Center

In this and the following sections, we will see how to obtain other groups from a given group. The first idea is to consider subsets of the group which is closed under group composition:


Figure 2.1: The symmetry group of a equilateral triangle $C_{3 v}$

Definition: Let $\mathcal{G}=(G, \circ)$ be a group. A non-empty subset $H \subseteq G$ forms a subgroup, written $\mathcal{H} \leq \mathcal{G}$, if $\mathcal{H}=(H, \circ)$ is itself a group.

## Remarks:

- Any group $\mathcal{G}$ has the trivial subgroups $(e, \circ) \leq \mathcal{G}$ and $\mathcal{G} \leq \mathcal{G}$.
- A proper subgroup $\mathcal{H}$ is a non-trivial subgroup: $(e, \circ)<\mathcal{H}<\mathcal{G}$

Notation: If $\mathcal{H} \leq \mathcal{G}$ and $\mathcal{H}=(H, \circ)$ is defined as a subset: $\mathcal{H}=\{g \in \mathcal{G} \mid$ condition $\}$ it is implied that the subgroup inherits the group composition from $\mathcal{G}$.

## Examples:

E.7.1) The four proper subgroups of the group of permutations of three elements (see example E.2.1) are:

| $\circ$ | $e$ | $\tau_{12}$ |
| :---: | :---: | :---: |
| $e$ | $e$ | $\tau_{12}$ |
| $\tau_{12}$ | $\tau_{12}$ | $e$ |


| $\circ$ | $e$ | $\tau_{23}$ |
| :---: | :---: | :---: |
| $e$ | $e$ | $\tau_{23}$ |
| $\tau_{23}$ | $\tau_{23}$ | $e$ |


| $\circ$ | $e$ | $\tau_{13}$ |
| :---: | :---: | :---: |
| $e$ | $e$ | $\tau_{13}$ |
| $\tau_{13}$ | $\tau_{13}$ | $e$ |


| $\circ$ | $e$ | $\sigma$ | $\bar{\sigma}$ |
| :---: | :---: | :---: | :---: |
| $e$ | $e$ | $\sigma$ | $\bar{\sigma}$ |
| $\sigma$ | $\sigma$ | $\bar{\sigma}$ | $e$ |
| $\bar{\sigma}$ | $\bar{\sigma}$ | $e$ | $\sigma$ |

E.7.2) $\mathrm{SL}(n, \mathbb{K})<\operatorname{GL}(n, \mathbb{K})$ for $\mathbb{K}=\mathbb{Q}, \mathbb{R}, \mathbb{C}$
E.7.3) $\operatorname{Inn}(\mathcal{G}) \leq \operatorname{Aut}(\mathcal{G})$
E.7.4) $(m \mathbb{Z},+)<(\mathbb{Z},+)$, however, since both groups are countable infinite, they are isomorphic, with $\phi_{m}(z)=m z$ an isomorphism.

Theorem: Let $\mathcal{G}=(G, \circ)$ be a a group.
a) A non-empty subset $H \subseteq G$ forms a group under composition $\circ$, i.e.

$$
\begin{equation*}
(H, \circ)=\mathcal{H} \leq \mathcal{G} \quad \Leftrightarrow \quad \text { for all } \quad h_{1}, h_{2} \in \mathcal{H}: h_{1} \circ h_{2}^{-1} \in \mathcal{H} \tag{2.13}
\end{equation*}
$$

b) If $\mathcal{G}$ is finite, then also

$$
\begin{equation*}
(H, \circ)=\mathcal{H} \leq \mathcal{G} \quad \Leftrightarrow \quad \text { for all } \quad h_{1} \circ h_{2} \in \mathcal{H} \tag{2.14}
\end{equation*}
$$

Proof: see textbooks

Theorem: Let $\phi \in \operatorname{Hom}\left(\mathcal{G}, \mathcal{G}^{\prime}\right)$, then

$$
\begin{align*}
\operatorname{Im}(\phi) & \leq \mathcal{G}^{\prime}  \tag{2.15}\\
\operatorname{Ker}(\phi) & \leq \mathcal{G}  \tag{2.16}\\
\mathcal{H} & \leq \mathcal{G} \quad \Rightarrow \quad \phi(\mathcal{H}) \leq \phi(\mathcal{G}) \tag{2.17}
\end{align*}
$$

## Proof:

1. Closure: With $\phi\left(g_{1}\right), \phi\left(g_{2}\right) \in \operatorname{Im}(\phi)$ also $\phi\left(g_{1}\right) \phi\left(g_{2}\right)=\phi\left(g_{1} g_{2}\right) \in \operatorname{Im}(\phi)$.

Inverse Element: With $\phi(g) \in \operatorname{Im}(\phi)$ also $\phi(g)^{-1}=\phi\left(g^{-1}\right) \in \operatorname{Im}(\phi)$
2. Closure: With $g_{1}, g_{2} \in \operatorname{Ker}(\phi)$, i. e. $\phi\left(g_{1}\right)=e^{\prime}$ and $\phi\left(g_{2}\right)=e^{\prime}$ also $\phi\left(g_{1}\right) \phi\left(g_{2}\right)=$ $\phi\left(g_{1} g_{2}\right)=e^{\prime}$, also $g_{1} g_{2} \in \operatorname{Ker}(\phi)$.
Inverse Element: $\phi\left(g g^{-1}\right)=e^{\prime}=\phi(g) \phi\left(g^{-1}\right)$, hence if $g \in \operatorname{Ker}(\phi)$ then $g^{-1} \in$ $\operatorname{Ker}(\phi)$
3. Exercise.

Definition: Let $\mathcal{H} \leq \mathcal{G}$. The conjugate subgroup for some $g \in \mathcal{G}$ is defined by $\phi_{g}(\mathcal{H})=g H g^{-1}=$ $\left\{g h g^{-1} \mid h \in \mathcal{H}\right\}$.

Theorem: For any $\mathcal{H} \leq \mathcal{G}$ and $g \in \mathcal{G}$ the conjugate subgroup is isomorphic to $\mathcal{H}: \phi_{g}(\mathcal{H}) \cong \mathcal{H}$.
Proof: The conjugation is an inner automorphism. Hence $\operatorname{Im}\left(\phi_{g}(\mathcal{H})\right) \leq \mathcal{G}$ and $\phi_{g}$ is bijective.

Definition: A subgroup $\mathcal{N} \leq \mathcal{H}$ is called normal subgroup (or invariant subgroup), denoted by
$\mathcal{N} \unlhd \mathcal{G}$, if it is invariant under conjugation:

$$
\begin{equation*}
\text { for all } g \in \mathcal{G}: \phi_{g}(\mathcal{N})=g \mathcal{N} g^{-1}=\mathcal{N} \tag{2.18}
\end{equation*}
$$

Every group $\mathcal{G}$ has the trivial normal subgroups $(e, \circ)$ and $\mathcal{G}$.
A proper normal subgroup is not trivial: $(e, \circ) \triangleleft \mathcal{N} \triangleleft \mathcal{G}$.
Property: If $\mathcal{G}$ is Abelian, then every subgroup is normal.

Definition: A group $\mathcal{G}$ is called simple if it has only the trivial normal subgroups.
A group $\mathcal{G}$ is called semisimple if it does not contain any proper Abelian normal subgroup.

Theorem: Let $\phi \in \operatorname{Hom}\left(\mathcal{G}, \mathcal{G}^{\prime}\right)$. Then

$$
\begin{align*}
\operatorname{Ker}(\phi) & \unlhd \mathcal{G}  \tag{2.19}\\
\mathcal{N} & \unlhd \mathcal{G} \quad \Rightarrow \quad \phi(\mathcal{N}) \unlhd \phi(\mathcal{G}) \tag{2.20}
\end{align*}
$$

Proof: (1) Since $\operatorname{Ker}(\phi) \leq \mathcal{G}$, we only have to show the invariance under conjugation: Let $n \in \operatorname{Ker}(\phi)$, then for all $g \in \mathcal{G}: \phi\left(g h g^{-1}\right)=\phi(g) \phi(h) \phi\left(g^{-1}\right)=\phi(g) e^{\prime} \phi\left(g^{-1}\right)=e^{\prime}$. Hence $g h g^{-1} \in \operatorname{Ker}(\phi)$
(2) Exercise

If every element of a subgroup $\mathcal{H} \leq \mathcal{G}$ commutes with all elements of $\mathcal{G}$, then $\mathcal{H}$ is an Abelian normal subgroup. The largest such Abelian normal subgroup gives rise to the following

Definition: The center of a group of $\mathcal{G}$ is the subgroup $\mathcal{Z}_{\mathcal{G}}=\{z \in \mathcal{G} \mid \forall g \in \mathcal{G}: z g=g z\} \leq \mathcal{G}$ which are those group elements which commute with all other elements.

Theorem: The center $\mathcal{Z}_{\mathcal{G}}$ forms a normal subgroup: $\mathcal{Z} \unlhd \mathcal{G}$. In fact, every element $z \in \mathcal{Z}_{\mathcal{G}}$ is invariant under conjugation.
Proof:

$$
z g=g z \quad \Rightarrow \quad g^{-1} z g=z
$$

### 2.2.5 Cosets, Quotient Groups

Definition: Let $\mathcal{H} \leq \mathcal{G}$ be a subgroup.
The set $g \mathcal{H}=\{g h \mid h \in \mathcal{H}\} \subseteq \mathcal{G}$ is called a left coset.
The set $\mathcal{H} g=\{h g \mid h \in \mathcal{H}\} \subseteq \mathcal{G}$ is called a right coset.
One can define an equivalence relation for left (right) cosets:
$a \sim_{l} b \Leftrightarrow b \in a \mathcal{H}\left(a \sim_{r} b \Leftrightarrow b \in \mathcal{H} a\right)$.
Hence $\mathcal{G}$ can be decomposed into disjoint sets:

$$
\begin{equation*}
\mathcal{G}=\mathcal{H} \dot{\cup} g_{2} \mathcal{H} \dot{\cup} \ldots \dot{\cup} g_{k} \mathcal{H} \quad \text { or } \quad \mathcal{G}=\mathcal{H} \dot{\cup} \mathcal{H} g_{2}^{\prime} \dot{\cup} \ldots \dot{\cup} \mathcal{H} g_{k}^{\prime} \tag{2.21}
\end{equation*}
$$

with $k=[\mathcal{G}: \mathcal{H}]$ is called the index.

## Remarks:

- In general left and right cosets give rise to two different decompositions
- We denote the equivalence class of $g \mathcal{H}$ by $[g]_{\mathcal{H}}$ and of $\mathcal{H} g$ by $\mathcal{H}[g]$, with $g$ some representative. Then $a \sim_{l} b$ amounts to $[a]_{\mathcal{H}}=[b]_{\mathcal{H}}$. and $a \sim_{r} b$ to $\mathcal{H}_{\mathcal{H}}[a]=_{\mathcal{H}}[b]$.
- For Abelian groups $\mathcal{G}$, the left cosets are identical to the right cosets: $[g]_{\mathcal{H}}={ }_{\mathcal{H}}[g]$
- If $\mathcal{N} \unlhd \mathcal{G}$, then $[g]_{\mathcal{N}}=\mathcal{N}[g]$.

Theorem: (Lagrange) Let $\mathcal{G}$ be a finite group and $\mathcal{H} \leq \mathcal{G}$. Then the index is the given by $k=|\mathcal{G}| /|\mathcal{H}| \in \mathbb{N}_{*}$, or equivalently, $|\mathcal{G}|=k|\mathcal{H}|$
Proof: $\quad$ All cosets have the same order: for all $g \in \mathcal{G}:|\mathcal{H}|=|g \mathcal{H}|=|\mathcal{H} g|$

| $\|\mathcal{G}\|$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\|\mathcal{H}\|$ | - | - | - | 2 | - | 2,3 | - | 2,4 | 3 | 2,5 |

Table 2.2: Number of elements in all possible proper subgroups $\{e\}<\mathcal{H}<\mathcal{G}$.
Definition: Let $\mathcal{N} \unlhd \mathcal{G}$ be a normal subgroup. Then the set of all cosets form a group, called the quotient group (or factor group): $\mathcal{G} / \mathcal{N}=\left(\left\{[g]_{\mathcal{N}} \mid g \in \mathcal{G}\right\}, \circ\right.$ ) with the group composition $\circ: \quad[g]_{\mathcal{N}} \circ\left[g^{\prime}\right]_{\mathcal{N}}=\left[g g^{\prime}\right]_{\mathcal{N}}$.
The group is closed since $[g]_{\mathcal{N}} \circ\left[g^{\prime}\right]_{\mathcal{N}}=(g \mathcal{N})\left(g^{\prime} \mathcal{N}\right)=g g^{\prime} \mathcal{N} g^{\prime-1} g^{\prime} \mathcal{N}=g g^{\prime} \mathcal{N}=\left[g g^{\prime}\right]_{\mathcal{N}}$ The identity is $[e]_{\mathcal{N}}$ and the inverse of $[g]_{\mathcal{N}}$ is $\left[g^{-1}\right]_{\mathcal{N}}$.

Theorem: (1. Isomorphism Theorem) Let $\phi \in \operatorname{Hom}\left(\mathcal{G}, \mathcal{G}^{\prime}\right)$, then $\mathcal{G} / \operatorname{Ker}(\phi) \cong \mathcal{G}^{\prime}$. Conversely, for every $\mathcal{N} \unlhd \mathcal{G}$ there is a homomorphism $\phi \in \operatorname{Hom}(\mathcal{G}, \mathcal{G} / \mathcal{N})$. Hence the set of all homomorphisms is solely determined by $\mathcal{G}$.
Proof: See textbooks.

### 2.2.6 Conjugacy Class

Definition: Two elements $a, b \in \mathcal{G}$ are called conjugate, $a \sim b$, if there is a $g \in \mathcal{G}$ such that $g a g^{-1}=b$. Conjugation is an equivalence relation. The equivalence relation allows to define the conjugacy classes

$$
\begin{equation*}
K_{a}=\left\{g a g^{-1} \mid g \in \mathcal{G}\right\} \tag{2.22}
\end{equation*}
$$

The group $\mathcal{G}$ can then be written as a disjoint union of the conjugacy classes:

$$
\begin{equation*}
\mathcal{G}=K_{a_{1}} \dot{\cup} \ldots \dot{\cup} K_{a_{n}} \tag{2.23}
\end{equation*}
$$

Every center element is a representative of one such class. Let $\left|K_{a}\right|$ be the number of elements in $K_{a}$, then

$$
\begin{equation*}
|\mathcal{G}|=\sum_{i}\left|K_{a_{i}}\right| . \tag{2.24}
\end{equation*}
$$

## Property:

1. If $\mathcal{G}$ is Abelian, every element $a \in \mathcal{G}$ is only conjugate to itself, $\left|K_{a}\right|=1$
2. The conjugacy classes of the group of all permutations on $n$ elements are given by their cycle structure (see below).

Definition: Let $S \subseteq \mathcal{G}$ be a subset. Then

$$
\begin{equation*}
\mathcal{C}_{\mathcal{G}}(S)=\left\{g \in \mathcal{G} \mid \forall a \in S: g a g^{-1}=a\right\} \leq \mathcal{G} \tag{2.25}
\end{equation*}
$$

is called centralizer, and

$$
\begin{equation*}
\mathcal{N}_{\mathcal{G}}(S)=\left\{g \in \mathcal{G} \mid g S g^{-1}=S\right\} \leq \mathcal{G} \tag{2.26}
\end{equation*}
$$

is called normalizer, which both form subgroup. For the singleton set $S=\{a\}$ both definitions agree.

## Remarks:

- $\mathcal{C}_{\mathcal{G}}(S) \unlhd \mathcal{N}_{\mathcal{G}}(S)$.
- $\mathcal{Z}_{\mathcal{G}}=\mathcal{C}_{\mathcal{G}}(\mathcal{G})$ if and only if $\mathcal{G}$ is Abelian (and then $\mathcal{Z}_{\mathcal{G}}=\mathcal{G}$ ).
- If $S=\mathcal{H} \leq \mathcal{G}$, then the normalizer of a subgroup $\mathcal{H}$ is the the largest subgroup of $\mathcal{G}$ in which $\mathcal{H}$ is normal: $\mathcal{H} \unlhd \mathcal{N}_{\mathcal{G}}(\mathcal{H})$.

Theorem: Let $K_{a}$ be a conjugacy class.

1. $\left|K_{a}\right|$ divides $|\mathcal{G}|$.
2. All elements of $K_{a}$ have the same order.

## Proof:

1. $\left|K_{a}\right|=\left[\mathcal{G}: \mathcal{N}_{a}\right]=|\mathcal{G}| /\left|\mathcal{N}_{a}\right|$
2. if $a^{n}=e$ and $b=g a g^{-1}$ then $b^{n}=\left(g a g^{-1}\right)^{n}=g a^{n} g^{-1}=e$.


### 2.2.7 Direct Products, Semidirect Products

Definition: Given two groups $\mathcal{G}_{1}, \mathcal{G}_{2}$, one can form a new group $G=\mathcal{G}_{1} \times \mathcal{G}_{2}$ defined as the set of all pairs with the component-wise group composition:

$$
\begin{align*}
\mathcal{G}_{1} \times \mathcal{G}_{2}=\left(\left\{\left(g_{1}, g_{2}\right) \mid g_{1} \in \mathcal{G}_{1}, g_{2} \in \mathcal{G}_{2}\right\}, \circ=\left(\circ_{1}, \circ_{2}\right)\right)  \tag{2.27}\\
\text { i.e. for all } g, h \in \mathcal{G}_{1} \times \mathcal{G}_{2}: g \circ h \equiv\left(g_{1}, g_{2}\right) \circ\left(h_{1}, h_{2}\right)=\left(g_{1} \circ_{1} h_{1}, g_{2} \circ_{2} h_{2}\right) \tag{2.28}
\end{align*}
$$

## Remarks:

- The identity in $\mathcal{G}_{1} \times \mathcal{G}_{2}$ is $e=\left(e_{1}, e_{2}\right)$, the inverse of $g=\left(g_{1}, g_{2}\right)$ is $g^{-1}=\left(g_{1}^{-1}, g_{2}^{-1}\right)$.
- The direct $\mathcal{G}_{1} \times \mathcal{G}_{2}$ product isomorphic to $\mathcal{G}_{2} \times \mathcal{G}_{1}$.
- By induction, the above definition can $=$ easily generalized direct products with more than two components due to associativity of the direct product:

$$
\mathcal{G}=\left(\ldots\left(\mathcal{G}_{1} \times \mathcal{G}_{2}\right) \times \ldots\right) \times \mathcal{G}_{n}=\mathcal{G}_{1} \times \mathcal{G}_{2} \times \ldots \times \mathcal{G}_{n} .
$$

- The order is $\left|\mathcal{G}_{1} \times \mathcal{G}_{2}\right|=\left|\mathcal{G}_{1}\right|\left|\mathcal{G}_{2}\right|$.
- $\left(\mathcal{G}_{1}, e_{2}\right) \unlhd \mathcal{G}_{1} \times \mathcal{G}_{2}$ and $\left(e_{1}, \mathcal{G}_{2}\right) \unlhd \mathcal{G}_{1} \times \mathcal{G}_{2}$.


## Examples:

E.8.1) "Kleinsche Vierergruppe" $K_{4}=Z_{2} \times Z_{2}$

| $\left(+2,+_{2}\right)$ | $(0,0)$ | $(1,0)$ | $(0,1)$ | $(1,1)$ |
| :---: | :---: | :---: | :---: | :---: |
| $(0,0)$ | $(0,0)$ | $(1,0)$ | $(0,1)$ | $(1,1)$ |
| $(1,0)$ | $(1,0)$ | $(0,0)$ | $(1,1)$ | $(0,1)$ |
| $(0,1)$ | $(0,1)$ | $(1,1)$ | $(0,0)$ | $(1,0)$ |
| $(1,1)$ | $(1,1)$ | $(0,1)$ | $(1,0)$ | $(0,0)$ |

E.8.2) The vector space $\left(\mathbb{R}^{n},+\right)$ under vector addition is a $n$-fold direct product of $(\mathbb{R},+)$.

Definition: If $\mathcal{N}_{1} \unlhd \mathcal{G}$ and $\mathcal{N}_{2} \unlhd \mathcal{G}$ and $\mathcal{N}_{1} \cap \mathcal{N}_{2}=\{e\}$ are normal subgroups, then $\mathcal{G}=\mathcal{N}_{1} \times \mathcal{N}_{2}$ is called an inner direct product and $\mathcal{G} / \mathcal{N}_{1}=\mathcal{N}_{2}, \mathcal{G} / \mathcal{N}_{2}=\mathcal{N}_{1}$

## Remarks:

- The above definition generalizes to the inner direct product of normal subgroups

$$
\mathcal{G}=\mathcal{N}_{1} \times \mathcal{N}_{2} \times \ldots \times \mathcal{N}_{k}
$$

provided that their mutual intersections is $e$

- An element $g \in \mathcal{G}$ can then be written as $g=n_{1} n_{2} \ldots n_{k}$ which (up to permutations of the indices) is unique decomposition.
- In general, it is not true that taking the tensor product is the inverse of taking quotient group: $\mathcal{G} \neq \mathcal{G} / \mathcal{N} \times \mathcal{N}$ (e.g. $A_{3} \unlhd S_{3}$ and $S_{3} / A_{3}=Z_{2}$, but $A_{3} \times Z_{2}=Z_{6}$ is Abelian whereas $S_{3}$ is non-Abelian (and hence cannot be isomorphic). Clearly, $Z_{2}$ is not a normal subgroup.

Definition: Let $\mathcal{N}$ and $\mathcal{H}$ be groups, and $\phi: \mathcal{H} \rightarrow \operatorname{Aut}(\mathcal{N})$ be a group homomorphism. The set $G=\mathcal{N} \times \mathcal{H}$ forms a group $\mathcal{G}=(G, *)$ called semi-direct product if the group composition on $*$ is defined via

$$
\begin{equation*}
\left(n_{1}, h_{1}\right) *\left(n_{2}, h_{2}\right)=\left(n_{1} \circ_{1} \phi\left(h_{1}\right)\left(n_{2}\right), h_{1} \circ_{2} h_{2}\right) . \tag{2.29}
\end{equation*}
$$

The semi-direct product is denoted by $\mathcal{G}=\mathcal{N} \rtimes \mathcal{H}$ or more specifically $\mathcal{G}=\mathcal{N} \rtimes_{\phi} \mathcal{H}$.

## Remarks:

- Similar to inner direct products, we can introduce inner semi-direct products.
- $\left(\mathcal{N}, e_{2}\right) \unlhd \mathcal{N} \rtimes \mathcal{H}$ and $\left(e_{1}, \mathcal{H}\right) \leq \mathcal{N} \rtimes \mathcal{H}$
- If also $\left(e_{1}, \mathcal{H}\right) \unlhd \mathcal{G}$, then it is a direct product. This is the case if there exists an identity id $\in \operatorname{Hom}(\mathcal{H}, \operatorname{Aut}(\mathcal{N}))\left(\right.$ because $\left(e_{1}, \operatorname{Ker}(\mathrm{id})\right) \unlhd G$ due to the Isomorphism theorem).


## Examples:

E.9.1) The group of rigid movements (the isometries) in 3-dimensional space is a semidirect product of translations and rotations, with $\phi: O(3) \rightarrow \operatorname{Aut}\left(\mathbb{R}^{3}\right)$ :

$$
\mathcal{E}_{3}=\mathbb{R}^{3} \rtimes O(3):(\vec{a}, R) *\left(\vec{a}^{\prime}, R^{\prime}\right)=\left(\vec{a}+R^{\prime} \vec{a}^{\prime}, R R^{\prime}\right)
$$

E.9.2) The symmetry group of the regular polygon is the semidirect product of discrete rotation and reflection: $D_{n}=C_{n} \rtimes Z_{2}$ (see below).
E.9.3) $\operatorname{Aut}(\mathcal{G})=\operatorname{Inn}(\mathcal{G}) \rtimes \operatorname{Out}(\mathcal{G})$, i. e. $\operatorname{Out}(\mathcal{G})=\operatorname{Aut}(\mathcal{G}) / \operatorname{Inn}(\mathcal{G})$ is in general not normal in $\operatorname{Aut}(\mathcal{G})$

### 2.3 Finite Groups

We will give an overview over some families of finite groups.

### 2.3.1 Cyclic Groups

The cyclic groups $C_{n}$ are Abelian groups of order $n$ that are generated by one element: $C_{n}=$ $\left\langle a \mid a^{n}=e\right\rangle$ The cyclic group are Abelian, since $g^{k} g^{l}=g^{k+l}=g^{l} g^{k}$.

We have already encountered the cyclic subgroup $C_{2} \cong\left(\mathbb{Z},+_{2}\right)$ and $C_{4} \cong\left\langle i \mid i^{4}=1\right\rangle$. We also have encountered direct products of cyclic groups: $K_{4}=C_{2} \times C_{2}$

Theorem: Every group $\mathcal{G}$ has at least one cyclic subgroup.
Proof: $\quad$ Choose any element $g \in \mathcal{G}$ and define the subgroup generated by $g: \mathcal{H}=\left\{g^{k} \mid k \in \mathbb{Z}\right\} \leq$ $\mathcal{G}$. $\mathcal{H}$ is cyclic. If $\mathcal{H}=\mathcal{G}$ then $\mathcal{G}$ is itself cyclic.

Definition: The cyclic subgroup generated by one element $g \in \mathcal{G}$ is denoted by $(g) \leq \mathcal{G}$. The order of the element $g$ is the order of the subgroup $(g)$.

Theorem: (1) A finite cyclic group is isomorphic to $\left(\mathbb{Z} / n \mathbb{Z},+_{n}\right)$
(2) An infinite cyclic group is isomorphic to $(\mathbb{Z},+)$

Proof: Exercise

Theorem: If $\mathcal{G}$ with the order $|\mathcal{G}|$ being a prime number $p$, then it is a cylic group. If the orders of two cyclic groups $C_{n}$ and $C_{m}$ are coprime, i.e. the only number that divides both $m$ and $n$ is 1 , then $C_{n} \times C_{m} \cong C_{n m}$
Proof: $\quad$ The subgroup generated by any element $g \in \mathcal{G}$ has index $k=|\mathcal{G}| /|\mathcal{H}|$. Since $p$ can only be divided by 1 and itself, $(g)=\{e\}$ or $(g)=\mathcal{G}$.

### 2.3.2 Dihedral Group

The dihedral groups $D_{n}$ are the group of geometrical transformations that leave a regular $n$-gon invariant. Besides the rotations $r$ that are described by $C_{n}$, there are also reflections $s$ at the $n$ symmetry axis (dihedron: "solid with two faces")

## Remarks:

- $\left|D_{n}\right|=2 n$ ( $n$ rotations and $n$ reflections)
- $D_{n}=C_{n} \rtimes Z_{2}$ since $C_{n} \unlhd D_{n}$, and for $n=2(2 k+1), k \in N a t: D_{n}=D_{n} / 2 \times Z_{2}$
- $D_{n} \leq O(2)$
- group presentation: $\left\langle r, s \mid r^{n}=s^{2}=(s r)^{2}=1\right\rangle$, i. e. $s r s=r^{-1}$ (mirror of rotation looks like inverse rotaiton)
- matrix form: $r_{1}=\left(\begin{array}{cc}\cos \frac{2 \pi}{n} & -\sin 2 \pi n \\ \sin \frac{2 \pi}{n} & \cos 2 \pi n\end{array}\right)$ with $r_{j}=r_{1}^{j}$ and $s_{0}=\left(\begin{array}{ll}1 & 0 \\ 0 & -1\end{array}\right)$ with $s_{j}=r_{j} s_{0}$
- the center is only $\{e\}$ if $n$ is odd, but $e, r^{n / 2}$ if $n$ is even
- conjugacy classes:
- if $n$ is odd, all reflections are conjugate to each other, hence $(n+3) / 2$ conjugacy classes
- if $n$ is even there are two conjugacy classes (reflections through vertices or through edges),
hence $(n+6) / 2$ conjugacy classes


### 2.3.3 The Symmetric Group

The symmetric group $\operatorname{Sym}(X)$ of a set $X$ has already been introduced in Sec. 2.2 .3 as the set of all bijetive maps on $X$.
Definition: Let $|X|=n$ be finite, then the set of all permutations on $X$ together with the composition of functions forms the symmetric group $S_{n}$. Usually, $X=\{1,2, \ldots n\}$ are called letters and $p \in S_{n}$ permutes the letters.

Examples have already been discussed in Sec. 2.1.3 and in example E.7.1).

## Remarks:

- $S_{n}$ is is the symmetry group of the regular $n$-simplex (point, line segment, triangle, tetrahedron, ...), i.e. rotations and reflections of these geometrical objects centered in a $n-1$ dimensional Euclidean space, and $S_{n}<O(n-1)$.
- $S_{n}$ is non-Abelian for $n>2$ and $\left|S_{n}\right|=n$ !
- Every permutation can be decomposed into disjoint cycles.
- We introduce the cycle notation for the elements from $\pi \in S_{n}$ : Each cycle is denoted by a bracket and starts with some (not yet used) element $a \in X$ and applies the permutation $\pi$ until $\pi^{k}(a)=a$, with $k$ the cycle length. $\pi$ is then the product of independent cycles. For example, some of the elements of $S_{4}$ :

| standard notation | cyclic notation |
| :---: | :---: |
| $\left(\begin{array}{l}1234 \\ 1342 \\ 1234 \\ 2143\end{array}\right)$ | $(1)(234)$ |

- group multiplication can be easily realized, e. g. $(234) \circ(234)=(243)$ or $(234) \circ(123)=(13)(24)($ from right to left $)$.
- it is not necessary that the set $X$ is ordered, $X=\{\square, \boxtimes, \boxtimes, \boxplus, \boxminus\}$ and $\pi=(\boxtimes \boxplus)(\boxminus \square)(\square)$ is well defined permutation

Theorem: Every permutation is a commutative product of disjoint cycles, and the decomposition into cycles is unique up to the order of cycles of equal length The cycle notation can be ordered by length.

Notation: Cycles of length 1 are fixed points of the permutation and can be omitted.

Definition: A transposition is a permutation $\pi$ with only a two-cycle: $\pi=\tau_{k l}=(k l)$
An even permutation can be written as a product of an even number of transpositions
An odd permutation can be written as a product of an odd number of transpositions
An involution is a permutation that does not cant any cycles of length $>2$. A derangement is a permutation $\pi$ with no fixed points

Theorem: Any cycle can be written as a product of transpositions.
Proof: $\quad\left(i_{1} i_{2} \ldots i_{k}\right)=\left(i_{1} i_{k}\right) \ldots\left(i_{1} i_{3}\right)\left(i_{1} i_{2}\right)$

## Remarks:

- The inverse of a single cycle is the cycle in reverse order, shifting the cycle yields the same cycle.
- The symmetric group $S_{n}$ is generated by any transposition $\tau=(12)$ together with a cyclic permutation $\sigma=(12 \ldots n)$ (but a basis consists of $n-1$ elements).
- the product of two even permutations is even, and the inverse of an even permutation is even: the sign or signature of a permutation $\sigma(\pi) \in S_{n}$ is a group homomorphism $\sigma: S_{n} \rightarrow \mathbb{Z}_{2}=$ $(\{+1,-1\}, \cdot)$ such that $\sigma(\pi)=1$ for $\pi$ even and $\sigma(\pi)=-1$ for $\pi$ odd.
- graphical representation in terms of directed graph, or permutation matrix (see representation theory)

Theorem: (Cayley theorem): Every finite group of order $n$ is isomorphic to a subgroup of the symmetric groups $S_{n}$.
Proof: $\quad$ For $|\mathcal{G}|=n$, a group homomorphism $\phi: \mathcal{G} \rightarrow S_{n}, g \mapsto \pi_{g}$ exists, and $\operatorname{Im}(\phi) \leq S_{n}$. Due to the rearrangement theorem, $\operatorname{Im}(\phi) \cong \mathcal{G}$.

## Remarks:

- The subgroup $\left\{\pi_{g} \mid g \in \mathcal{G}\right\} \leq S_{n}$ is usually a very small subgroup: $n \ll n$ !

We will now discuss the conjugacy classes of $S_{n}$ :
Definition: The cyle type of a permutation is the unordered set of cycles lengths $\left\{\ell_{i}\right\}_{i=1, \ldots k}$ with $k$ the number of cycles.

## Remarks:

- The cycle type can be represented as in integer partition:

Theorem: Two permutations of they same type are in the same conjugacy class
Proof: The cycle type of a permutation is invariant under conjugation: Let $\nu_{\ell}$ be the number of cycles of length $\ell$, then $\nu_{1}+2 \nu_{2}+\ldots+n \nu_{n}=n$. If two permutation are of same cycle type, then they have the same set of $\nu_{i}$, e. g. (138)(45)(26)(7) and (146)(35)(78)(2) both have $\left(\nu_{1}, \nu_{2}, \nu_{3}, \nu_{4}, \ldots \nu_{8}\right)=(1,2,1,0, \ldots, 0)$.

## Remarks:

- In general, the product of two permutations of the same type does not have the same type.
- Example that conjugation does not change the cycle type: Let $g=(123)(45)(6)$ and $a=$ $(25)=a^{-1}$, then $a g a^{-1}=(25)(123)(45)(6)(25)=(153)(24)(6)$ interchanges only 2 and 5 in the cycles.

Some futher properties of $S_{3}$ :

- The centralizers are:
$\mathcal{N}_{e}=S_{3}, \mathcal{N}_{\tau_{12}}=\left\{e, \tau_{12}\right\}, \mathcal{N}_{\tau_{13}}=\left\{e, \tau_{13}\right\}, \mathcal{N}_{\tau_{23}}=\left\{e, \tau_{23}\right\}$, which have indices $\left[S_{3}: \mathcal{N}_{e}\right]=1,\left[S_{3}: \mathcal{N}_{\tau_{12}}\right]=2,\left[S_{3}: \mathcal{N}_{e}\right]=1,\left[S_{3}: \mathcal{N}_{e}\right]=1,\left[S_{3}: \mathcal{N}_{e}\right]=1$, $\left[S_{3}: \mathcal{N}_{e}\right]=1 \mathcal{N}_{\sigma}=\mathcal{N}_{\sigma}=A_{3}$
- The conjugacy classes are: $K_{e}=\{e\}, K_{\sigma}=K_{\bar{\sigma}}=\{\sigma, \bar{\sigma}\}, K_{\tau_{12}}=K_{\tau_{13}}=K_{\tau_{23}}=\left\{\tau_{12}, \tau_{13}, \tau_{23}\right\}$


### 2.3.4 The Alternating Group

Definition: The alternating group $A_{n}$ is the subgroup of all even permutation of $S_{n}$, i.e. $A_{n}=$ $\operatorname{Ker}(\sigma) \leq S_{n}$ with $\sigma: S_{n} \rightarrow Z_{2}$ the signature.

## Remarks:

- $\left|A_{n}\right|=\frac{n}{2}$ and $A_{3} \cong C_{3}$ and $A_{n}$ is non-Abelian for $n>3$.
- $A_{n}$ is generated by the set of 3-cycles, which can be written as a product of two 2 -cycles: $(a b c)=(a c)(a b)$
- An example of an alternating group: the sliding puzzle. The 15-piece sliding puzzle has $\left|A_{1} 6\right|$ possible configurations.

Theorem: The alternating group is a normal subgroup: $A_{n} \unlhd S_{n}$.
Proof: Exercise

### 2.3.5 Other Permutation Groups and Wreath Product

Since due to Cayleys theorem, all finite groups $\mathcal{G}$ are isomorphic to a subgroup of $S_{n}$, they can be also viewed as permutation groups.

## Remarks:

- It is rather non-trivial to identify the smallest $n$ such that $\mathcal{G} \leq S_{n}$, and often $n$ can be chosen much smaller than $|\mathcal{G}|$. There are no general results.
- The Klein group is isomorphic to $K_{4} \cong\{e,(12),(34),(12)(34)\}<S_{4}$
- Pauli group: finite matrix group, $G_{1}=\left\{ \pm I, \pm i I, \pm \sigma_{1}, \pm i \sigma_{1}, \pm \sigma_{2}, \pm i \sigma_{2}, \pm \sigma_{3}, \pm i \sigma_{3}\right\}=\left\langle\sigma_{1}, \sigma_{2}, \sigma_{3}\right\rangle$ (generated by the Pauli matrices) acts on 1 qubit, and $\left|G_{1}\right|=16$, central product $G_{1} \cong$ $\left(C_{4} \times C_{2}\right) \rtimes C_{2} \cong C_{4} \circ D_{4}$ (it is generated by $C_{4}$ and $D_{4}$ and every element of $C_{4}$ commutes with every element of $D_{4}$.

Other finite groups isomorphic to subgroups of the symmetric group can be defined via the following

Definition：Let $\mathcal{G}$ and $\mathcal{H}$ be（finite）groups and $\mathcal{H}$ acting on a（finite）set $X$ ，then the wreath product $\mathcal{G} 2_{X} \mathcal{H}$ is a group given by $\mathcal{G}^{X} \rtimes H$（with the direct product $\mathcal{G}^{X}$ indexed by $X$ is called the base）．The action $\mathcal{H}$ on $X$ extends to an action on the base via $h\left(g_{x}\right)=g_{h^{-1}(x)}$ ．For $\mathcal{H}=S_{n}$ or $\mathcal{H}=C_{n}$ ，often $X=\{1, \ldots n\}$ is implied．The order is given by $|\mathcal{G}|^{\mid X}|\mathcal{H}|$ ．

## Examples：

E．10．1）Rubik＇s Cube group： $\mathcal{R}<S_{48}$ with $|\mathcal{R}|=2^{2} 73^{1} 45^{3} 7^{2} 11$ and is generated by the basic moves $\{\mathrm{F}, \mathrm{B}, \mathrm{U}, \mathrm{D}, \mathrm{L}, \mathrm{R}\}$ with $\mathcal{R} \leq C_{4}^{6} \times\left(C_{3} \backslash S_{8}\right) \times\left(C_{2} \backslash S_{12}\right)$（ 6 faces， 8 corners， 12 edges）

E．10．2）（finite）lamplighter group：$Z_{2}$ 〔 $Z_{n}$ with $\mid Z_{2}$ 〕 $Z_{n} \mid=2^{n} n\left(Z_{2}\right.$ 〕 $\left.Z_{2} \cong D_{4}\right)$
E．10．3）the generalized symmetric group $C_{m} \backslash S_{n}$ with base $C_{m}{ }^{n}$
E．10．4）the hyperoctahedral group $C_{2}$ 乙 $S_{n}$ is the the special case $m=2$ ，consisting of the signed permutations and is the symmetry group of a hypercube（dual to the hyperoctahdron）．

E．10．5）The octahedral group（rotations and reflections of a cube）is $O_{h}=C_{2}$ 乙 $S_{3}=C_{2}^{3} \rtimes S_{3}$ has order 48，and the chiral octahedral group $O<O_{h}$（only rotations）is $O \cong S_{4}$ and has order 24．For other platonic groups，see Sec．？？on point groups．

## 2．3．6 Classification of finite simple groups

Every finite groups can be decomposed into a finite number of finite simple groups：．
Definition：A composition series of a group $\mathcal{G}$ is a finite sequence of normal subgroups：

$$
\begin{equation*}
\{e\}=\mathcal{N}_{0} \triangleleft \mathcal{N}_{1} \triangleleft \mathcal{N}_{2} \triangleleft \ldots \triangleleft \mathcal{N}_{n}=\mathcal{G} \tag{2.30}
\end{equation*}
$$

such that each quotient group $\mathcal{N}_{i+1} / \mathcal{N}_{i}$（the so－called composition factors），with $i \in\{0, \ldots n-1\}$ ，is simple．The composition length is given by $n$ ．

## Remarks：

－Example：$\{e\} \triangleleft A_{3} \triangleleft S_{3}$ ．
－Every finite group has a composition series，two different composition series have the same length．
－Infinite groups do not always have composition series，e．g．$(\mathbb{Z},+)$ has an infinite sequence of normal subgroups $\ldots \triangleleft 8 \mathbb{Z} \triangleleft 4 \mathbb{Z} \triangleleft 2 \mathbb{Z} \triangleleft \mathbb{Z}$ ．

Theorem：（Jordan－Hölder Theorem）A composition series of a group $\mathcal{G}$ is essentially unique： Two different composition series have the same composition length and the factors $\mathcal{N}_{i+1} / \mathcal{N}_{i}$ are the same，up to a permutation of the factors．

## Proof：

## Remarks：

－example：

$$
\begin{array}{lll}
C_{1} \triangleleft C_{2} \triangleleft C_{6} \triangleleft C_{12} & \text { with } & C_{2} / C_{1} \cong C_{2}, C_{6} / C_{2} \cong C_{3}, C_{12} / C_{6}=C_{2} \\
C_{1} \triangleleft C_{2} \triangleleft C_{4} \triangleleft C_{12} & \text { with } & C_{2} / C_{1} \cong C_{2}, C_{4} / C_{2} \cong C_{2}, C_{12} / C_{4}=C_{3} \\
C_{1} \triangleleft C_{3} \triangleleft C_{6} \triangleleft C_{12} & \text { with } & C_{3} / C_{1} \cong C_{3}, C_{6} / C_{3} \cong C_{2}, C_{12} / C_{6}=C_{2} \tag{2.33}
\end{array}
$$

All finite simple groups can be classified into four families:
Theorem: Every finite simple group belongs the following groups:

1. cyclic groups of prime order
2. alternating groups of degree at least 5
3. groups of Lie type
4. one of 27 sporadic groups

Proof: The proof spreads over 500 articles (1955-1983), with more than 100 mathematicians contributing, about 10000 pages

### 2.3.7 List of finite groups

| order | \# groups | \# Abelian | \# non-Abelian |
| :--- | :--- | :--- | :--- |
| 1 | 1 | $1\left(C_{1} \cong S_{1} \cong A_{2}\right)$ | 0 |
| 2 | 1 | $1\left(C_{2} \cong S_{2} \cong D_{1}\right)$ | 0 |
| 3 | 1 | $1\left(C_{3} \cong A_{3}\right)$ | 0 |
| 4 | 2 | $2\left(C_{4}, D_{2} \cong K_{4}=C_{2}^{2}\right)$ | 0 |
| 5 | 1 | $1\left(C_{5}\right)$ | 0 |
| 6 | 2 | $1\left(C_{6} \cong C_{3} \times C_{2}\right)$ | $1\left(D_{3} \cong S_{3}\right)$ |
| 7 | 1 | $1\left(C_{7}\right)$ | 0 |
| 8 | 5 | $3\left(C_{8}, C_{4} \times C_{2}, C_{2}^{3}\right)$ | $2\left(D_{4}, Q_{8} \cong D_{i c}\right)$ |
| 9 | 2 | $2\left(C_{9}, C_{3}^{2}\right)$ | 0 |
| 10 | 2 | $1\left(C_{10} \cong C_{5} \times Z_{2}\right)$ | $1\left(D_{5}\right)$ |

Table 2.3: List of finite non-isomorphic groups $\mathcal{G}$ up to order $|\mathcal{G}| \leq 10$

## Remarks:

- the number of non-isomorphic groups grows rapidly by order (depending on prime number decomposition of the order):

| order | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 32 | 64 | 128 |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| Abelian | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 3 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 5 | 7 | 11 | 15 |
| non-Abelian | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 2 | 0 | 1 | 0 | 3 | 0 | 1 | 0 | 9 | 44 | 256 | 2313 |
| all | 1 | 1 | 1 | 2 | 1 | 2 | 1 | 5 | 2 | 2 | 1 | 5 | 1 | 2 | 1 | 14 | 51 | 267 | 2328 |

Table 2.4: List of finite non-isomorphic groups $\mathcal{G}$ up to order $|\mathcal{G}| \leq 10$

- number of distinct Abelian groups of order $p^{n}$ is given by integer partitions of $n$ (independent of $p$ )
- the smallest non-Abelian group is $D_{3} \cong S_{3}$
- the smallest non-cyclic simple group is $A_{5}$ with $\mid A_{5}=60$
- the first group that is not simple is $D i h_{2} \cong K_{4}=Z_{2}^{2}$
- the dicyclic groups $D i c_{n}$ is an extension of the cyclic group


### 2.3.8 List of finite simple groups

| Cyclic | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| Non-Cyclic | 60 | 168 | 360 | 504 | 660 | 1092 | 2448 | 2520 | 3420 | 4080 | 5616 | 6048 |
| $\ldots$ |  |  |  |  |  |  |  |  |  |  |  |  |
| Sporadic | 7920 | 95040 | 175560 | $\ldots$ |  |  |  |  |  |  |  |  |

Table 2.5: List of the orders of simple groups. In red: alternating groups $A_{n}$, in blue: classical groups of Lie type, black: other groups of Lie type

## Remarks:

- There are few non-isomorphic groups of same order, e.g. $A_{8} \cong \mathrm{PSL}_{4}(2)$ and $\mathrm{PSL}_{3}(4)$ both have order 20160 .
- the largest sporadic finite simple group is the monster group M with order $|M|=808017424794512875886459904961710757005754368000000000 \approx 8 \times 10^{53}$ (and may be relevant for string theory due to the "monstrous moonshine"


### 2.4 Countably Infinite Groups

We have already encountered the infinite cyclic group (and direct products thereof). Here are some more examples, most of them are discrete subgroups of continuous groups:
E.11.1) Infinite dihedral group $D_{\infty}=\left\langle a, x \mid x^{2}=e, x a x^{-1}=a^{-1}\right\rangle$
E.11.2) Discrete Heisenberg group $H_{3}(\mathbb{Z})$ : set of matrices $\left(\begin{array}{lll}1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1\end{array}\right)$ with $a, b, c \in \mathbb{Z}$ (continous Heisenberg group: used to show equivalence of Heisenberg and Schrödinger picture).
E.11.3) Frieze groups and Wallpaper groups: discrete subgroups of the isometry group of the Euclidean plane
E.11.4) Crystallographic groups: discrete subgroups of the isometry group of the Euclidean space.

### 2.5 Continuous Groups

### 2.5.1 Isometries in the Euclidean Space

Isometries are important in classical mechanics as the symmetry groups of movements of rigid bodies. In order to discuss the properties of the isometries in Euclidean space, let us consider the following types of group actions:
Definition: A group action $\Phi$ is faithful if for all $\Phi_{g} \in \operatorname{Sym}(X)$ only $\Phi_{e}=$ id is the identity, $\Phi$ is free if for $g \neq e$ there is no fixed point: $\forall x \in X: \Phi_{g}(x) \neq x$, and $\Phi$ is transitive if for any pair $x, y \in X$ there is a group element $g$ with $\Phi_{g}(x)=y$.

To understand continuous symmetry groups, the following definition is helpful:

Definition: The orbit of a point $x \in X$ under the action $\Phi$ is the set

$$
\begin{equation*}
B(x)=\left\{y=\Phi_{g}(x) \mid g \in \mathcal{G}\right\} \subset X \tag{2.34}
\end{equation*}
$$

They are the equivalence classes of the equivalence relation $x \sim y$ iff $\exists g: \Phi_{g}(x)=y$.
The stablilizer or isotropy group of $x \in X$ is

$$
\begin{equation*}
N_{x}=\left\{g \in \mathcal{G} \mid \Phi_{g}(x)=x\right\} \leq \mathcal{G} \tag{2.35}
\end{equation*}
$$

Theorem: For $\mathcal{G}$ acting on the set $X$ transitive, all stabilizers $N_{x}$ are conjugate to each other.
Proof: Consider any two elements $x, y \in X$. Then since there is a $g^{\prime} \in \mathcal{G}: \Phi_{g^{\prime}}(x)=y$, it follows

$$
\Phi_{g}(x)=x \Leftrightarrow \Phi_{g^{\prime} g g^{\prime-1}}(y)=\Phi_{g^{\prime} g}(x)=\Phi_{g^{\prime}}(x)=y .
$$

Hence $N_{y}=g^{\prime} N_{x} g^{\prime-1}$.

## Remarks:

- if $X=\mathcal{G}$, then one can define the adjoint action $\operatorname{Ad}_{g}(a)=g a g^{-1}$ with $a, g \in \mathcal{G}$.

In the following we will consider the Euclidean group $E(n)$, which is a group of motions for the $n$-dimensional metric space given by the Euclidean metric. $E(n)$ is a subgroup of the group affine transformations $(\vec{y}=A \vec{x}+\vec{b}$, with $A \in \mathrm{GL}(n, \mathbb{R}))$ and consists of translations $T(n) \cong\left(\mathbb{R}^{n},+\right)$ and rotations $\mathrm{O}(n)$, and $E(n)=T(n) \rtimes O(n)$. This has been discussed for $d=3$ in example E.9.3)

## Remarks:

- The Euclidean group preserves the metric, i.e. the distance between 2 points
- The affine group preserves collinearity, parallelism, convexity, ratios of lengths, barycenters


## Rotations in space

Let $X=\mathbb{R}^{n}$ be the $n$-dimensional vector space and $\left\{\vec{e}_{1}, \ldots \vec{e}_{n}\right\}$ a Cartesian basis, i. e. $\left(\vec{e}_{i}, \vec{e}_{j}\right)=\delta_{i j}$. Then a rotation $R=\left(R_{i j}\right)$ (expressed as a matrix) rotates the original basis into another basis

$$
\begin{equation*}
\vec{e}_{i}^{\prime}=\sum_{j} R_{i j} \vec{e}_{j} \tag{2.36}
\end{equation*}
$$

Theorem: The matrix of a rotation with respect to a Cartesian basis fulfills the conditions:

$$
\begin{equation*}
R R^{T}=\mathbb{1} \quad \text { i.e. } \quad R^{T}=R^{-1} \tag{2.37}
\end{equation*}
$$

and the composition of two rotations is a rotation, hence the rotations form a continuous group, called the orthogonal group $\mathrm{O}(n)$.

Scale about origin
$\left[\begin{array}{ccc}W & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & 1\end{array}\right]$

Rotate about origin Shear in $x$ direction
Shear in y direction

Reflect about origin

## Reflect about $x$-axis

Reflect about y-axis


Figure 2.2: Affine Transformations.

Proof: The rotations $R$ do not change length of angles of vectors, and hence the scalar product is invariant:

$$
\begin{equation*}
\left(\vec{e}_{i}^{\prime}, \vec{e}_{j}^{\prime}\right)=\left(\sum_{k} R_{i k} \vec{e}_{k}, \sum_{l} R_{j l} \vec{e}_{l}\right)=\sum_{k} R_{i k} R_{j k}=\sum_{k} R_{i k} R_{k j}^{T}=\left(R R^{T}\right)_{i j} \stackrel{!}{=}\left(\vec{e}_{i}, \vec{e}_{j}\right)=\delta_{i j} \tag{2.38}
\end{equation*}
$$

This is also true for the composition of two rotations:

$$
\begin{equation*}
\left(R_{1} R_{2}\right)\left(R_{1} R_{2}\right)^{T}=R_{1} R_{2} R_{2}^{T} R_{1}^{T}=\mathbb{1} \tag{2.39}
\end{equation*}
$$

## Remarks:

- The rotations are exactly those isometries (distance-preserving transformations of the Euclidean space) that leave the origin fixed: it is the symmetry group of the $n-1$-sphere.
- The entries $R_{i j}$ depend on the choice of basis
- One can distinguish between active transformations: $\vec{x} \mapsto \vec{x}^{\prime}=R \vec{x}$, and passive transformations: $x_{i} \vec{e}_{i}=x_{i}^{\prime} \vec{e}_{i}{ }^{\prime}$. What transformation is assumed depends on the context ("inertial systems").
- For the rotation matrix, $\operatorname{det} R R^{T}=1$, hence $\operatorname{det} R= \pm 1$.

Definition: The rotations with $\operatorname{det} R=1$ preserve the orientation are called proper rotations and form the special orthogonal group $\mathrm{SO}(n) \leq \mathrm{O}(n)$ The rotations with $\operatorname{det} R=-1$ are called improper rotations and reverse the orientation.
Likewise, the special Euclidean group is $E^{+}(n)=T(n) \rtimes \mathrm{SO}(n)$, which defines the direct isometries. Motions with improper rotations are called indirect isometries.

## Remarks:

- viewed as matrix groups, $\mathrm{O}(n) \leq \mathrm{GL}(n, \mathbb{R})$ and $\mathrm{SO}(n) \leq \mathrm{SL}(n, \mathbb{R})$.
- The following subgroups have the indices $[\mathrm{O}(n): \mathrm{SO}(n)]=2,\left[E(n): E^{+}(n)\right]=2$.
- $\mathrm{O}(n)$ can be generated by reflections: two reflections give a rotation.
- $\operatorname{SO}(2) \cong U(1)($ see E.1.5)

Theorem: (Euler) If $n$ is odd, every proper rotation has a 1-dimensional subspace of fixed points. If $n$ is even, every improper rotation has a 1-dimensional subspace of fixed points.
Proof: $\quad$ Solve for $R \vec{n}=\vec{n}$ such that $\vec{n} \neq 0$. Then $\vec{n}$ defines the fixed axis. A solution exists iff $R$ has some eigenvalue $\lambda=1$ i. e. if $\operatorname{det}(R-\mathbb{1})=0$.

$$
\begin{align*}
\operatorname{det}(R-\mathbb{1}) & =\operatorname{det}\left(R^{-1}-\mathbb{1}\right)=\operatorname{det}\left(R^{-1}(\mathbb{1}-R)\right)=(-1)^{n} \operatorname{det} R \operatorname{det}(R-\mathbb{1})  \tag{2.40}\\
& \begin{cases}=0 & n \text { odd and } \operatorname{det} R=+1 \\
=0 & n \text { even and } \operatorname{det} R=-1 \\
\neq 0 & \text { otherwise }\end{cases} \tag{2.41}
\end{align*}
$$

## Remarks:

- In 3 dimensions, every rotation has an axis.
- For every proper rotation the Cartesian basis can be chosen such that the rotation has the form

$$
R\left(\vec{e}_{3}, \phi\right)=\left(\begin{array}{ccc}
\cos \phi & \sin \phi & 0  \tag{2.42}\\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{array}\right) \quad \text { and } \quad R\left(\vec{e}_{1}\right) R\left(\vec{e}_{2}\right) R\left(\vec{e}_{1}\right)^{T}=R\left(\vec{e}_{3}\right)
$$

- A rotation can also be parameterized by the Euler angles:

$$
\begin{equation*}
R(\phi, \theta, \psi)=R\left(\vec{e}_{1}, \phi\right) R\left(\vec{e}_{2}, \theta\right) R\left(\vec{e}_{3}, \psi\right) \tag{2.43}
\end{equation*}
$$

- An improper rotation $R$ has in all dimensions an eigenvalue -1 .


## Properties of the Euclidean group

A motion (geometry) is the group action

$$
\begin{equation*}
(\vec{a}, R): \vec{x} \mapsto \vec{x}^{\prime}=R \vec{x}+\vec{a} \quad \text { with } \quad \vec{a} \in \mathbb{R}^{n}, R \in O(n) \tag{2.44}
\end{equation*}
$$

The composition of motions in $E_{n}$ is

$$
\begin{equation*}
\left.\left(\vec{a}^{\prime}, R^{\prime}\right) \dot{(\vec{a}}, R\right)=\left(R^{\prime} \vec{a}+\vec{a}, R^{\prime} R\right) \tag{2.45}
\end{equation*}
$$

The inverse motion is

$$
\begin{equation*}
(\vec{a}, R)^{-1}=\left(-R^{-1} \vec{a}, R^{-1}\right) \tag{2.46}
\end{equation*}
$$

## Remarks:

- Motions with $\vec{a} \neq \overrightarrow{0}$ have no fixed points.
- The conjugacy classes of rotations have simple representations known as normal forms and are extended to $E_{n}$. Based on this, one can distinguish the following types of motions in 3 dimensions [degrees of freedom in brackets]:
(a) identity [0] (direct, $\operatorname{det}(R)=1$ and $\operatorname{tr}(R)=3$ )
(b) translation [3] (direct, $\operatorname{det}(R)=1)$
(c) proper rotations [5] (direct, $\operatorname{det}(R)=1$ )
(d) screw displacement $[6](\operatorname{direct}, \operatorname{det}(R)=1)$
(e) reflection in a plane [3] (indirect, $\operatorname{det}(R)=-1$ and $\operatorname{tr}(R)<1$ )
(f) glide plane operation [5] (indirect, $\operatorname{det}(R)=-1$ and $\operatorname{tr}(R)=1$ )
(g) improper rotation [6] (indirect, $\operatorname{det}(R)=-1$ )
(h) inversion in some point [3] (indirect, $\operatorname{det}(R)=-1$ and $\operatorname{tr}(R)=-3$ )


## Further Subgroups of $E(n)$

An important class of finite subgroups of $\mathrm{O}(n)$, which are always point groups:
Theorem: Every finite subgroup $\mathcal{G}<E_{n}$ has a fixed point.
Proof: Let $x \in \mathbb{R}^{n}$ be the coordinates of a fixed point in the Euclidean space and $B(\vec{x})=$ $\left\{R_{i} \vec{x}+a_{i} \mid\left(a_{i}, R_{i}\right) \in \mathcal{G}\right\}$ its orbit under the group action. For a finite group $\mathcal{G}$, the orbit is a finite set, $B(\vec{x})=\left\{\vec{x}_{1}, \ldots \vec{x}_{n}\right\}$ with $\vec{x}_{i}=R_{i} \vec{x}+\vec{a}_{i}$. The center

$$
\begin{equation*}
\vec{x}=\frac{1}{n} \sum_{i=1}^{n} \vec{x}_{i} \tag{2.47}
\end{equation*}
$$

is a fixed point, $g \vec{x}=\vec{x}$ for any $g \in \mathcal{G}$, since they only permute the elements of the orbit.

## Remarks:

- $d=1$ : identity and reflection group, $C_{1} \cong S O(1), D_{1} \cong O(1)$
- $d=2$ : rosette groups, $C_{n} \leq S O(2), D_{n} \leq O(1)$. With crystallographic restriction theorem: $n=1,2,3,4,6$.
- $d=3$ : molecular point groups, which can be distinguished in axial groups (cylinder, Frieze patterns) $C_{n}, S_{2 n}, C_{n h}, C_{n v}, D_{n}, D_{n d}, D_{n h}$. and polyhedral groups $T, T_{d}, T_{h}, O, O_{h}, I, I_{h}$. With crystallographic restriction theorem: 32 crystallographic point groups.

Examples for countably infinite subgroups are the discrete space groups which are the symmetry groups of configurations in space, with $\vec{y}=M \vec{x}+D$ and $D$ is a lattice vector (generating a lattice). The lattice dimension $l$ can be smaller than the dimension of the Euclidean space $n$.

- $(1,1)$ : one-dim. line groups
- $(2,1)$ : two-dim. line groups (Frieze group)
- $(2,2)$ : Wallpaper group
- $(3,1)$ : three-dim. line groups (Rod groups)
- $(3,2)$ : Layer Groups
- $(3,3):$ Space Groups


### 2.5.2 Lie Groups

## Definition and Properties

Continuous groups can have the additional property that the group elements can be mapped onto each other by a continuous map. The elements of a continuous group $g \in \mathcal{G}$ can be parameterized my a set of parameters $\alpha=\left(\alpha_{1}, \ldots \alpha_{n}\right), g=g(\alpha)$ such that

$$
\begin{equation*}
\forall \alpha, \beta \in \mathbb{R}^{n} \exists \gamma \in \mathbb{R}^{n}: g(\alpha) g(\beta)=g(\gamma) \quad \gamma=m(\alpha, \beta) \tag{2.48}
\end{equation*}
$$

The number o parameters $n$ is called the dimension of the group. For example, $U(1)$ is parameterized by $e^{i \phi}$. Every $O(n)$ matrix can be parameterized by $n(n-1) / 2$ parameters
Definition: An $n$-dimensional manifold $M$ is a topological space with the additional properties

1. $M$ is Hausdorffsch (i.e. every 2 points can have disjoint neighborhoods),
2. it has a countable basis (basis $\mathcal{B}$ : open sets such that every subset of $M$ is a union of sets from $\mathcal{B}$ ),
3. it is locally Euclidean (for every $p \in M$ there is a neighborhood $U$ of $p$ and a (continuous=topology preserving bijective map) homeomorphism

$$
\phi: U \mapsto \phi(U) \subset \mathbb{R}^{n}
$$

The map $\phi$ is called chart, and the collection of all charts $\left\{\phi_{a} \mid \in A\right\}$ is called atlas if the set of all charts covers the full manifold, $M=\bigcup_{a \in A} \operatorname{Ker} \phi_{a}$.
For two charts $\phi_{\alpha}, \phi_{\beta}$ which have a non-empty intersection of their kernels
$U_{\alpha \beta}=U_{\alpha} \cap U_{\beta}$, one can define a transition map (coordinate transformation), which is a homeomorphism between to open sets in $\mathbb{R}^{n}$ :

$$
\begin{equation*}
\phi_{\alpha \beta}=\phi_{\beta} \circ \phi_{\alpha}^{-1}: \phi_{\alpha}\left(U_{\alpha \beta}\right) \mapsto \phi_{\beta}\left(U_{\alpha \beta}\right) \tag{2.49}
\end{equation*}
$$

A manifold is differentiable if $M$ can be covered by maps which have smooth transition maps. In this case the transition maps are diffeomorphisms.‘

## Remarks:

- many examples of manifolds are obtained by identifying a surface in a higher-dimensional space: e. g. the surface of a any solid
- likewise for differentiable manifolds, which should then not have corners or edges: e. g. circle, torus, sphere, ellipsoid (algebraic geometry)
- compact manifolds have a finite "volume" (characterization by open sets and finite covers)
- closed manifolds are compact and have no boundary.
- manifolds can be further classified by dimension and Euler characteristics (handles, etc. ), fundamental group ("winding numbers"), homotopy and other invariants (e.g. orientability).
- any n-Sphere $S^{n}$ is a n-dimensional differentiable manifold embedded in $\mathbb{R}^{n+1}$ by the condition $\|\vec{x}\|=1$. As a manifold, $\mathrm{SO}(n) \cong S^{n-1}$ (closed manifold) and $\mathrm{SO}(2) \cong U(1) \cong S^{1}$ is the circle.
- $G L(n, \mathbb{R})$ is non-compact

Definition: A Lie group is a group that is at the same time a differentiable manifold, such that the group multiplication $\left(g_{1}, g_{2}\right) \mapsto g_{1} g_{2}$ and the inverse $g \mapsto g^{-1}$ are both smooth (continuous and differentiable) maps.
A linear Lie group is a subgroup of the linear groups $\mathrm{GL}(n \mathbb{R})$ or $\mathrm{GL}(n \mathbb{C})$ (which are also Lie groups, but not $\operatorname{GL}(n, \mathbb{Q}))$.
Since it is a manifold, every group element $g$ in a neighborhood $U$ has chart $\phi$ such that $g \mapsto \alpha \in \phi(U) \subset \mathbb{R}^{n}$. With respect to the local coordinates given by the charts, a continuous group is a Lie group if

$$
\begin{equation*}
(\alpha, \beta) \mapsto m(\alpha, \beta) \quad \text { and } \quad g^{-1}(\alpha)=g(\operatorname{inv}(\alpha)) \tag{2.50}
\end{equation*}
$$

such that the maps $m$ and inv are differentiable.

A Lie group can be decomposed into connected components:
Definition: Two elements of a Lie group are path connected if there is a path ...
A Lie group is connected if all elements are path connected (there is only one connected component).
A Lie group is simple connected if every closed path (loop) can be continuously deformed into a point.

## Remarks:

- Path-connectedness is an equivalence relation
- If $\mathcal{G}_{0}$ is a connected component of $\mathcal{G}$ that contains the neutral element, then $\mathcal{G}_{0}$ is a normal subgroup and $\mathcal{G} / \mathcal{G}_{0}$ is the group of the connected components.
- $\mathrm{O}(n)$ has two connected components.
- $\mathrm{SO}(2)$ is a connected Lie group, but not simple connected, it is homeomorphic to $S^{1}$.
- $\mathrm{SO}(3)$ is a connected Lie group, but not simple connected, it is homeomorphic to $R P^{3}$.

Definition: A Lie subgroup is a subgroup which is also a submanifold.
A Lie normal subgroup is a normal subgroup which is also a Lie subgroup.
Theorem: (Cartan's Theorem) A (normal) subgroup of a Lie groups is a Lie (normal) subgroup iff it is (topologically) closed.
Proof:
Theorem: If $\mathcal{G}$ is a connected Lie group and $U$ an open neighborhood of $e$, then $\mathcal{G}$ is generated by $\langle U\rangle$. It follows that for a connected Lie group, a discrete normal subgroup is in the center of the group.

## Proof:

## Remarks:

- If $\mathcal{G}_{1}, \mathcal{G}_{2}$ are Lie groups, then $\mathcal{G}_{1} \times \mathcal{G}_{2}$ is a Lie group.
- If $\mathcal{N}$ is a Lie normal subgroup of $\mathcal{G}$, then $\mathcal{G} / \mathcal{N}$ is a Lie group.
- $\mathrm{SO}(n)$ does not have a discrete normal subgroup $(\mathrm{SO}(3)$ is connected and the center is trivial).


## Unitary groups

Consider a $n$-dimensional complex vector space $\mathbb{C}^{n}$. By choosing a basis $\left(\vec{e}_{1}, \ldots \vec{e}_{n}\right)$, every vector $\vec{r}=x_{1} \vec{e}_{1}+x_{2} \vec{e}_{2}$ is given by $\vec{x}=\left(x_{1}, \ldots, x_{n}\right)$ and every linear map $A$ acts on the vector via $\vec{x} \mapsto A \vec{x}$, i.e. $A \in \mathrm{GL}(n, \mathbb{C})$.

Definition: A unitary matrix $U \in \mathrm{GL}(n, \mathbb{C})$ is characterized by the property that the Hermitian scalar product is invariant:

$$
\begin{equation*}
(\vec{x}, \vec{y})=\sum_{i=1}^{n} \bar{x}_{i} y_{i}, \quad \text { then } \quad(U \vec{x}, U \vec{y})=(\vec{x}, \vec{y}) \tag{2.51}
\end{equation*}
$$

A unitary matrix is unitary if

$$
\begin{equation*}
U^{\dagger} U=\mathbb{1}, \quad \text { i.e. } \quad U^{\dagger}=U^{-1} \tag{2.52}
\end{equation*}
$$

and the set of unitary matrices forms the unitary group $\mathrm{U}(n)$.
The special unitary group $\mathrm{SU}(n)$ is a Lie normal subgroup of $\mathrm{U}(n)$ which is also a subgroup of $\operatorname{SL}(n, \mathbb{C})$ : the additional requirement $\operatorname{det} U=1$ holds.

Let us consider $\mathrm{U}(2)$ and $\mathrm{SU}(2)$ in some detail:

- Since the columns of a matrix $U \in \mathrm{U}(2)$ are orthogonal, we can parameterize it as follows:

$$
U=\left(\begin{array}{cc}
a \lambda & \lambda \bar{b} \\
-b \lambda & \lambda \bar{a}
\end{array}\right)
$$

- Since the columns have length $1,|a|^{2}+|b|^{2}=1$ and hence $|\lambda|=|\operatorname{det} U|=1 . \mathrm{U}(2)$ is a 4-dimensional Lie group:

$$
S U(2)=\{U \in U(2) \mid \operatorname{det} U=1\}=\left\{U=\left.\left(\begin{array}{cc}
a & \bar{b}  \tag{2.53}\\
-b & \bar{a}
\end{array}\right)| | a\right|^{2}+|b|^{2}=1\right\}
$$

is a normal subgroup. It is also a Lie normal subgroup as it is closed.

- The parameterization of the group is given by the homeomorphism

$$
\begin{equation*}
S^{3}=\left\{\alpha=\left(\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}\right) \mid \sum_{i} \alpha_{i}^{2}=1\right\} \rightarrow \mathrm{SU}(2) \tag{2.54}
\end{equation*}
$$

hence $\mathrm{SU}(2)$ as a topological manifold is a sphere, and hence is simple connected and $C^{\infty}$ (exercise).

- A matrix $U \in U(2)$ is parameterized by $e^{i \alpha} U^{\prime}$ with $U^{\prime} \in \mathrm{SU}(2)$. However, $e^{i \alpha} U^{\prime}$ and $e^{-i \alpha} U^{\prime}$ are in the same coset since $-\mathbb{1} \in S U(2)$. Hence

$$
\begin{equation*}
\mathrm{U}(2) / \mathrm{SU}(2) \cong\left\{e^{i \alpha} \mid e^{i \alpha} \sim e^{-i \alpha}\right\}=\mathrm{U}(1) / Z(2) \tag{2.55}
\end{equation*}
$$

- $\mathrm{U}(1) \cong\left\{\left.\left(\begin{array}{cc}e^{i \alpha} & 0 \\ 0 & e^{-i \alpha}\end{array}\right) \right\rvert\, \alpha \in[0,2 \pi]\right\}<S U(2)$
- Any unitary matrix $U$ can be diagonalized: $U=V D V^{\dagger}, V \in \mathrm{SU}(2)$, and hence two matrices are conjugated if the have the same Eigenvalues. Since the order does not matter, two matrices $U_{1}, U_{2} \in \mathrm{SU}(2)$ are conjugated $U_{1} \sim U_{2}$ if $\operatorname{tr}\left[U_{1}\right]=\operatorname{tr}\left[U_{2}\right]$.
- The center of $\operatorname{SU}(2)$ is $\{+\mathbb{1},-\mathbb{1}\}$, other diagonal matrices are not invariant under $\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$

Theorem: The center of $\mathrm{SU}(2)$ is a normal subgroup $\{+\mathbb{1},-\mathbb{1}\} \triangleleft \mathrm{SU}(2)$ and the quotient group is $\mathrm{SU}(2) /\{+\mathbb{1},-\mathbb{1}\} \cong \mathrm{SO}(3)$, i.e. there is a continuous group homomorphism

$$
\begin{equation*}
R: \mathrm{SU}(2) \rightarrow \mathrm{SO}(3), \quad R(\mathbb{1})=\mathbb{1}, \quad R\left(U_{1} U_{2}\right)=R\left(U_{1}\right) R\left(U_{2}\right) \tag{2.56}
\end{equation*}
$$

and $\operatorname{Ker}(R)=\{+\mathbb{1},-\mathbb{1}\}$.
Proof: Let $\vec{a} \cdot \vec{\sigma}=\sum_{i=1}^{3} a_{i} \sigma_{i}$ be a linear combination of the Pauli matrices $\sigma_{i}$, which span all traceless Hermitian matrices (exercise) and $\operatorname{det}(\vec{a} \cdot \vec{\sigma})=-\vec{a}^{2}$. Then there exists a vector $\vec{b} \in \mathbb{R}^{3}$ with

$$
\begin{equation*}
U(\vec{a} \cdot \vec{\sigma}) U^{-1}=\vec{b} \cdot \vec{\sigma} \tag{2.57}
\end{equation*}
$$

Since

$$
\begin{equation*}
\operatorname{det}\left(U(\vec{a} \cdot \vec{\sigma}) U^{-1}\right)=\operatorname{det}(\vec{a} \cdot \vec{\sigma})=-\vec{a}^{2}=\operatorname{det}(\vec{b} \cdot \vec{\sigma})=-\vec{b}^{2} \tag{2.58}
\end{equation*}
$$

the linear map $\vec{a} \mapsto \vec{b}$ is length preserving and hence a rotation depending on $U$ exists:

$$
\begin{align*}
U(\vec{a} \cdot \vec{\sigma}) U^{-1} & =(R(U) \vec{a}) \cdot \vec{\sigma}, \quad R^{T}(U) R(U)=\mathbb{1}  \tag{2.59}\\
\left(R\left(U_{1} U_{2}\right) \vec{a}\right) \cdot \vec{\sigma} & =U_{1} U_{2}(\vec{a} \cdot \vec{\sigma}) U_{2}^{\dagger} U_{1}^{\dagger}=U_{1}\left(\left(R\left(U_{2}\right) \vec{a}\right) \cdot \vec{\sigma}\right) U_{1}^{\dagger}=\left(R\left(U_{1}\right) R\left(U_{2}\right) \vec{a}\right) \cdot \bar{a} \tag{2.60}
\end{align*}
$$

valid for any $\vec{a} \in \mathbb{R}^{3}$. Moreover, $\operatorname{det} R(U)=1$, hence $\operatorname{Im}(R) \cong S U(3)$ are the proper rotations.

Matrix Lie groups can be characterized by their invariance: $A^{\dagger} \eta A=\eta$, with $\eta$ the metric tensor. Definition: The Symplectic group is defined by the invariance of the skew-symmetric bilinear form

$$
\mathrm{Sp}(2 n, \mathbb{K})=\left\{M \in \mathrm{GL}(2 n, \mathbb{K}) \mid M^{T} J M=J\right\}, \quad J=\left(\begin{array}{cc}
0 & \mathbb{1}_{n}  \tag{2.61}\\
\mathbb{1}_{n} & 0
\end{array}\right)
$$

## Remarks:

- $\operatorname{Sp}(2, \mathbb{R}) \cong \mathrm{SL}(2, \mathbb{R})$ because for $n=1, M^{T} J M=(\operatorname{det} M) J=J$, hence $\operatorname{det} M=1$.
- $\operatorname{Sp}(2 n, \mathbb{R})$ is non-compact, connected and simple and diffeomorphic to $\mathrm{U}(n) \times \mathbb{R}^{n(n+1)}$
- $\operatorname{Sp}(2 n, \mathbb{C})$ is non-compact, simply connected and simple
- $\operatorname{Sp}(n)=\operatorname{Sp}(2 n, \mathbb{C}) \cap \mathrm{U}(2 n) \cong \mathrm{U}(n, \mathbb{H})$ is compact.

We conclude by listing important "classical" Lie groups: These are the continuous matrix groups, which are Lie groups, since

- they are manifolds and even metric spaces, as a metric is induced by the Frobenius norm:

$$
\begin{equation*}
d(A, B)^{2}=\sum_{i, j=1}^{n}\left|a_{i j}-b_{j i}\right|^{2}=\operatorname{tr}(A-B)^{\dagger} \operatorname{tr}(A-B)=\|A-B\|_{\text {Forb }} \tag{2.62}
\end{equation*}
$$

- the matrix multiplication is continuous since the matrix elements of $A B$ are polynomials of that of $A$ and $B$
- the inverse is continuous since the matrix elements of $A^{-1}$ are rational functions of those of $A$ (Cramer's rule).

| Group | Condition | Dimension |  |
| :---: | :---: | :---: | :---: |
| $\mathrm{GL}(n, \mathbb{R})$ | $\operatorname{det} A \neq 0$ | $n^{2}$ |  |
| $\mathrm{GL}(n, \mathbb{C})$ | $\operatorname{det} A \neq 0$ | $2 n^{2}$ |  |
| $\mathrm{SL}(n, \mathbb{R})$ | $\operatorname{det} A=1$ | $n^{2}-1$ |  |
| $\mathrm{SL}(n, \mathbb{C})$ | $\operatorname{det} A=1$ | $2\left(n^{2}-1\right)$ |  |
| $\mathrm{O}(n)$ | $R^{T} R=\mathbb{1}$ | $\frac{n(n-1)}{2+}$ |  |
| $\mathrm{U}(n)$ | $U^{\dagger} U=\mathbb{1}$, | $n^{2}$ |  |
| $\mathrm{SO}(n)$ | $R^{T} R=\mathbb{1}, \operatorname{det} R=1$ | $\frac{n(n-1)}{2+}$ |  |
| $\mathrm{SU}(n)$ | $U^{\dagger} U=\mathbb{1}, \operatorname{det} U=1$ | $n^{2}-1$ |  |
| $\mathrm{Sp}(2 n, \mathbb{R})$ | $M^{T} J M=J$ | $n(2 n+1)$ |  |
| $\mathrm{Sp}(2 n, \mathbb{C})$ | $M^{T} J M=J$ | $2 n(2 n+1)$ |  |
| $J$ |  |  |  |
| $\operatorname{Cartan}$ | $\operatorname{Group}$ | Dimension |  |
| $A_{n}$ | $\operatorname{SL}(n+1, \mathbb{R}), \mathrm{SU}(n+1)$ | $n(n+2)$ |  |
| $B_{n}$ | $\mathrm{SO}(2 n+1)$ | $n(2 n+1)$ |  |
| $C_{n}$ | $\mathrm{Sp}(2 n+1)$ | $n(2 n+1)$ |  |
| $D_{n}$ | $\operatorname{SO}(2 n)$ | $n(2 n-1)$ |  |

Table 2.6: Top: List of matrix groups with their dimensions as Lie Groups. Bottom: Cartan classification (see Lie algebras).

### 2.5.3 Lie Algebras

Theorem: Let $K$ be a connected component of a $d$-dimensional matrix Lie group $\mathcal{G}$ that contains the neutral element. Then every matrix $A \in K$ can be written as

$$
\begin{equation*}
A=\exp \left(i \sum_{a=1}^{d} \theta^{a} T^{a}\right), \quad \theta^{a} \quad \in \mathbb{R} \quad T^{a} \in \operatorname{Mat}(n, \mathbb{K}) \text { with } \quad T^{a} \eta-\eta T^{a}=0 \tag{2.63}
\end{equation*}
$$

The matrices $T^{a}$ are called generators and the vector space obtained by the basis $T^{a}$ is called Lie algebra.

Proof: Define $\theta=\sum_{a} \theta^{a} T^{a}$ and the exponential map

$$
\begin{equation*}
\mathbb{R}^{d} \rightarrow \operatorname{Mat}(n, \mathbb{C}), \quad\left\{\theta_{a}\right\} \mapsto e^{i \theta} \tag{2.64}
\end{equation*}
$$

We now show that $e^{i \theta} \in K$ : Let $f(\lambda)=e^{i \lambda \theta}, \lambda \in[0,1]$ and $c(\lambda)=f(\lambda)^{\dagger} \eta f(\lambda)$. Then

$$
\begin{equation*}
\frac{d}{d \lambda} c(\lambda)=-i e^{-i \lambda \theta^{\dagger}} \underbrace{\left(\theta^{\dagger} \eta-\eta \theta\right)}_{=0} e^{i \lambda \theta}=0 \tag{2.65}
\end{equation*}
$$

Hence $c(\lambda)=c(0)$ for all $\lambda$, and thus $f(\lambda) \in K$ for all $\lambda$.
Moreover, the exponential map is surjective: (proof via the path connectedness)

## Remarks:

$$
\begin{equation*}
\operatorname{det} A=\prod_{k=1}^{d} e^{i \lambda_{k}}=e^{i \operatorname{tr}[\theta]}=1 \quad \Rightarrow \quad \operatorname{tr}\left[T^{a}\right]=1 \tag{2.66}
\end{equation*}
$$

with $\lambda_{k}$ the eigenvalues of $\theta$.

- A large transformation can be decomposed into many small transformations:

$$
\begin{equation*}
f(1)=f(1 / 2) f(1 / 2)=\lim _{N \rightarrow \infty} f(1 / N)^{N}=\lim _{N \rightarrow \infty}\left(1+\frac{\theta}{N}+\mathcal{O}\left(1 / N^{2}\right)\right)^{N}=e^{i \theta} \tag{2.67}
\end{equation*}
$$

- in the Lie algebra, both the sum $\theta+\theta^{\prime}$ and the product is defined: consider two elements $A=e^{i \theta^{a} T^{a}} \in K, B=e^{i \phi^{a} T^{a}} \in K$, then $A B=e^{i \xi^{a} T^{a}} \in K$.
The series expansion up to second order yields:

$$
\begin{array}{r}
{\left[\mathbb{1}+i \theta^{a} T^{a}-\frac{1}{2} \theta^{a} \theta^{b} T^{a} T^{b}\right]\left[\mathbb{1}+i \phi^{a} T^{a}-\frac{1}{2} \phi^{a} \phi^{b} T^{a} T^{b}\right]=\mathbb{1}+i \xi^{a} T^{a}-\frac{1}{2} \xi^{a} \xi^{b} T^{a} T^{b}+\mathcal{O}(\theta, \phi, \xi)^{3}} \\
\mathbb{1}+i\left(\theta^{a}+\phi^{a}\right) T^{a}-\frac{1}{4}\left(\theta^{a}+\phi^{a}\right)\left(\theta^{b}+\phi^{b}\right)\left\{T^{a} T^{b}\right\}-\frac{1}{2} \theta^{a} \phi^{b}\left[T^{a}, T^{b}\right]=\mathbb{1}+i \xi^{a} T^{a}-\frac{1}{4} \xi^{a} \xi^{b}\left\{T^{a}, T^{b}\right\}+\mathcal{O}(\theta, \phi, \xi)^{3} \tag{2.69}
\end{array}
$$

since $T^{a} T^{b}=\frac{1}{2}\left\{T^{a}, T^{b}\right\}+\frac{1}{2}\left[T^{a}, T^{b}\right]$ and $\xi^{a} \xi^{b} T^{a} T^{b}=\frac{1}{2}\left\{T^{a}, T^{b}\right\}$. This can only be fulfilled if $\theta^{a} \phi^{a}\left[T^{a}, T^{b}\right] \propto i \xi^{a} T^{a}$.

Definition: The Lie algebra $\mathfrak{g}$ is characterized by the so-called structure constants $f^{a b c}$ via $\left[T^{a}, T^{b}\right]=i f^{a b c} T^{c}$. with the binary operation:

$$
\begin{equation*}
[., .]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \tag{2.71}
\end{equation*}
$$

the Lie-bracket (commutator). They are real-valued.

## Remarks:

- The Lie bracket defines a multiplication of lie algebra elements and has the following properties:
- bilinearity: $[a X+b Y, Z]=a[X, Z]+b[Y, Z]$
- alternativity for all $X \in \mathfrak{g}:[X, X]=0$
- antisymmetry: $[X, Y]=-[Y, X]$ (implied by bilinearity and alternativity)
- Jacobi identity: $[[X, Y], Z]+[[Y, Z], X]+[[Z, X], Y]=0$
- A linear combination of $T^{a}$ can be a basis of $\mathfrak{g}$, it is orthogonal if $\operatorname{tr}\left[T^{a} T^{b}\right] \propto \delta_{a b}$
- In most cases (e.g. for the matrix groups), the generators can be normalized: $\operatorname{tr}\left[T^{a} T^{b}\right]=\frac{\delta_{a b}}{2}$
- The basis $\left\{T_{a}\right\}$, as it is complete, fulfills a completeness condition. For $\operatorname{SU}(n)$ :

$$
\begin{equation*}
\sum_{a=1}^{n^{2}} T_{i j}^{a} T_{k l}^{a}=\frac{1}{2}\left(\delta_{i l} \delta_{j k}-\frac{1}{n} \delta_{i j} \delta_{k l}\right) \tag{2.72}
\end{equation*}
$$

- $\operatorname{tr}\left(T^{a} T^{b}\right)=\frac{\delta_{a b}}{2}$ and $\left[T^{a}, T^{b}\right]=i f^{a b c} T^{c}$ implies $f^{a b c}=-2 i \operatorname{tr}\left(\left[T^{a}, T^{b}\right] T^{c}\right)$. Hence $f^{a b c}$ is anti-symmetric.
- From the Jacobi identity it follows that

$$
\begin{equation*}
\forall a, b, c, e \in\{1, \ldots d\}: \quad f^{a b d} f^{c d e}+f^{b c d} f^{a d e}+f^{c a d} f^{b d e}=0 \tag{2.73}
\end{equation*}
$$

Definition: If there is a generator $T^{a}$ which commutes with all other generators, $T_{a}$ defines an Abelian sub-algebra. Otherwise the algebra $\mathfrak{g}$ is called semi-simple.

## Example:

1. $\mathfrak{s u}(2)$ is semi-simple and has dimension 3 , with

- with $\sigma^{a}$ the Pauli matrices, the generators are

$$
\begin{equation*}
T^{a}=\frac{\sigma^{a}}{2} \quad \text { with } \quad \operatorname{tr}\left(T^{a} T^{b}\right)=\frac{\delta^{a b}}{2}, \quad\left(T^{a}\right)^{\dagger}=T^{a}, \quad \operatorname{tr}\left[T^{a}\right]=0 \tag{2.74}
\end{equation*}
$$

- the Lie bracket is $\left[T^{a}, T^{b}\right]=i \epsilon^{a b c} T^{c}$ with $\epsilon^{a b c} T$ the Levi-Civita tensor


### 2.5.4 Lie Algebras of Matrix Lie Groups

The matrix Lie groups as subgroups of $\operatorname{GL}(n, \mathbb{C})$ can be characterized by relations among the generators of their Lie algebras: Let $t: \mathbb{R} \rightarrow \mathfrak{g l}(n, \mathbb{C}), t \mapsto g(t)$ be a curve with $g(0)=0$. Then the corresponding tangential vector at the neutral element

$$
\begin{equation*}
X=\left.\frac{d g}{d t}\right|_{t=0} \tag{2.75}
\end{equation*}
$$

is a complex $n \times n$ matrix: $\mathfrak{g l}(n, \mathbb{C})=\operatorname{Mat}(n, \mathbb{C})$. For the subgroups, additional constraints follow, e. g. for $\operatorname{SO}(n)$ with $X=\dot{R}(0)$

$$
\begin{equation*}
0=\left.\frac{d}{d t}\left(R^{T}(t) R(t)\right)\right|_{t=0}=X^{T}+X \tag{2.76}
\end{equation*}
$$

| Group | Lie Algebra | Generators | Dimension |
| :---: | :---: | :---: | :---: |
| $\mathrm{GL}(n, \mathbb{C})$ | $\mathfrak{g l}(n, \mathbb{C})$ | $X$ complex | $2 n^{2}$ |
| $\mathrm{GL}(n, \mathbb{R})$ | $\mathfrak{g l}(n, \mathbb{R})$ | $X$ real | $n^{2}$ |
| $\mathrm{SL}(n, \mathbb{C})$ | $\mathfrak{s l}(n, \mathbb{C})$ | $\operatorname{tr}(X)=0$ complex | $2 n^{2}-2$ |
| $\mathrm{SL}(n, \mathbb{R})$ | $\mathfrak{s l}(n, \mathbb{R})$ | $\operatorname{tr}(X)=0$ real | $n^{2}-1$ |
| $\mathrm{U}(n)$ | $\mathfrak{u}(n, \mathbb{C})$ | $X+X^{\dagger}=0$ complex | $n^{2}$ |
| $\mathrm{SU}(n)$ | $\mathfrak{s u}(n, \mathbb{C})$ | $X+X^{\dagger}=0, \operatorname{tr}(X)$ complex | $n^{2}-1$ |
| $\mathrm{O}(n), \mathrm{SO}(n)$ | $\mathfrak{s o}(n, \mathbb{C})$ | $X+X^{T}=0$ real | $n(n-1) / 2$ |
| $\mathrm{Sp}(2 n, \mathbb{C})$ | $\mathfrak{s p}(2 n, \mathbb{C})$ | $J X+X^{\dagger} J=0$ complex | $2 n(2 n+1)$ |
| $\mathrm{Sp}(2 n, \mathbb{R})$ | $\mathfrak{s p}(2 n, \mathbb{C})$ | $J X+X^{T} J=0$ real | $n(2 n+1)$ |

Table 2.7: List of the classical groups and their corresponding Lie algebras.

## Chapter 3

## Applications of Group Theory in Physics

We will now turn to continuous symmetries similar to the matrix groups Lie groups we already encountered. We will see that the these space-time symmetries arise from invariance principles. In contrast to space-time as considered in general relativity, we here will restrict ourselves to models of space and time which are independent of the matter distribution. The models are the GalileiNewton model which has an absolute time, and the Einstein-Poincar'e model, where time intervals depend on the reference frame. The first is limiting case a of the latter.

A system in which the law of inertia holds for bodies on which no external forces act is called an inertial system. Such bodies are at rest or at constant velocity.

### 3.1 The Galilei Group

Definition: In the Galilei-Newton model, two inertial systems $I, I^{\prime}$ are related by Galileien symmetry: the relative motions of their origin $O$ and $O^{\prime}$ are given by

1. by a time-independent spatial translation $\vec{a}$ and/or
2. by a constant velocity $\vec{u}$.

The position vector $\vec{r}(t)=\vec{r}(0)+\vec{v} t$ in inertial system $I$ is then related to the position vector in $I^{\prime}$ by

$$
\begin{equation*}
\vec{r}^{\prime}(t)=(\vec{r}(0)+\vec{a})+(\vec{v}+\vec{u}) t . \tag{3.1}
\end{equation*}
$$

This Abelian group of transformations has six parameters. It can be enlarged by also taking into account that the origins in time are also transformed by a shift $t^{\prime}=t+\tau$ and by rotating a Cartesian basis of $I$ into a Cartesian basis of $I^{\prime}$ via a rotation matrix $R \in \mathrm{O}(n)$.

## Remarks:

- We only change the description of the movement of a rigid body (passive transformation).
- An example is whether I describe movements within the ship that is at constant speed from the inertial system "Land" or "Ship"

Definition: The Galilei transformations has 10 parameters, which form a 10-dimensional continuous group Gal(3) with group multiplication given by

$$
\begin{equation*}
\left(\tau_{1}, \vec{a}_{1}, \vec{u}_{1}, R_{1}\right) \circ\left(\tau_{2}, \vec{a}_{2}, \vec{u}_{2}, R_{2}\right)=\left(\tau_{1}+\tau_{2}, \vec{a}_{1}+R_{1} \vec{a}_{2}+\vec{u}_{1} \tau_{2}, \vec{u}_{1}+R_{1} \vec{u}_{2}, R_{1} R_{2}\right) \tag{3.2}
\end{equation*}
$$

Theorem: The Galilei group $\operatorname{Gal}(3)$ has two connected components and

$$
\begin{equation*}
\operatorname{Gal}(3)=\operatorname{SGal}(3) \dot{\cup} \sigma \operatorname{SGal}(3) \tag{3.3}
\end{equation*}
$$

with $\sigma$ a reflection at some plane, and $\operatorname{SGal}(3)$ the special Galilei group.
Proof: This is evident since the subgroup $O(3)$ has two connected components and all other subgroups have only one connected component.

Theorem: The Galilei group is semi-direct product:

$$
\begin{equation*}
\operatorname{Gal}(3)=\mathbb{R}^{4} \rtimes E_{3} \tag{3.4}
\end{equation*}
$$

with $\mathbb{R}^{4}$ the group of translations $(\tau, \vec{a})$ in time and space, and $E_{3}$ the group of elements ( $\vec{u}, R$ ). Also,

$$
\begin{equation*}
\operatorname{SGal}(3)=\mathbb{R}^{4} \rtimes\left(\mathbb{R}^{3} \rtimes \mathrm{SO}(3)\right) \tag{3.5}
\end{equation*}
$$

Proof: $\quad$ First note that $E_{3}=\mathbb{R}^{3} \rtimes \mathrm{O}(3)=\left\{\left(\vec{u}, R \mid \vec{u} \in \mathbb{R}^{3}, R \in \mathrm{O}(3)\right)\right\}$ is not the group of isometries $(\vec{a}, R)$, but of velocity $\vec{u}$ and rotations $R$. Moreover, the time and space translations form a normal subgroup $\operatorname{Gal}(3)$, but $E_{3}$ does not.

## Remarks:

- The Galilean group can be viewed as a subgroup: $\operatorname{Gal}(3)<\operatorname{GL}(5, \mathbb{R})$ via the identification with a matrix acting on a 5 -vector whose first 4 coordinates are the space-time coordinates.

$$
\left(\begin{array}{c}
\vec{x}^{\prime}  \tag{3.6}\\
t^{\prime} \\
1
\end{array}\right)=\left(\begin{array}{ccc}
R & \vec{u} & \vec{a} \\
0 & 1 & \tau \\
0 & 0 & 1
\end{array}\right)\left(\begin{array}{c}
\vec{x} \\
t \\
1
\end{array}\right)=\left(\begin{array}{c}
R \vec{x}+\vec{u} t+\vec{a} \\
t^{\prime}+\tau \\
1
\end{array}\right)
$$

- important subgroups are:

1) the uniformly special transformations: $\quad\{g \in \operatorname{Gal}(3) \mid \tau=0, \vec{a}=0\} \cong E_{3}$,
2) the shifts of origin: $\quad\{g \in \operatorname{Gal}(3) \mid \vec{u}=0, R=\mathbb{1}\} \cong\left(\mathbb{R}^{4},+\right)$,
3) rotations of the reference frame: $\quad\{g \in \operatorname{Gal}(3) \mid \tau=0, \vec{a}=0, \vec{v}=0\} \cong \mathrm{SO}(3)$,
4) uniform frame motions: $\quad\{g \in \operatorname{Gal}(3) \mid \tau=0, \vec{a}=0, R=\mathbb{1}\} \cong\left(\mathbb{R}^{3},+\right)$

### 3.2 The Lorentz Group and the Poincaré Group

### 3.2.1 Lorentz Transformations and the The Lorentz Group

The 4 -vectors $x=\left(x^{\mu}\right)=\binom{c t}{\vec{x}}$ in the Minkowski space given by the metric $\left(g_{\mu \nu}\right)=\operatorname{diag}(1,-1,-1,-1)$ describe events. The Lorentz-invariant distance between two events $\xi=\left(\xi_{m} u\right)=y-x$ is given by $d(x, y)=(\xi, x i)$, where the scalar product in Minkowksi space is

$$
\begin{equation*}
(\xi, \eta)=\sum_{\mu \nu} g_{\mu \nu} \xi^{\mu} \eta^{\nu} \tag{3.7}
\end{equation*}
$$

### 3.2.2 Inhomogenous Lorentz Transformations and the Poincaré Group

 [coming soon]
## Definition:

$$
\begin{equation*}
\mathrm{iO}(1,3)=i O(1,3)_{+}^{\uparrow} \dot{\cup} \mathrm{iO}(1,3)_{-}^{\uparrow} \dot{U} \mathrm{iO}(1,3)_{+}^{\downarrow} \dot{\cup} \mathrm{iO}(1,3)_{-}^{\downarrow} \tag{3.8}
\end{equation*}
$$

### 3.3 Lagrange and Hamilton Formalism of Relativistic Field Theory

We now want to apply the space-time symmetries to relativistic field theories. In order to do so, we need to review classical field theory.

### 3.3.1 Lagrange Formalism

The action in terms of the Lagranian density is

$$
\begin{equation*}
S=\int d^{4} x \mathcal{L}(x) \equiv \int d t d^{3} \vec{x} \mathcal{L}(t, \vec{x}) \tag{3.9}
\end{equation*}
$$

Note that in contrast to Newtonian mechanics of points, the Lagrange function now also depneds on the space coordinates. We set $\hbar=1, c=1$.

We will make use of the functional derivative, here for a generic field $\phi(x)$ :

$$
\begin{align*}
S[\phi+\delta \phi] & \left.\left.=S[\phi]+\delta S+\mathcal{O}\left((\delta \phi)^{2}\right)\right)=S[\phi]+\int d^{4} x \frac{\partial S}{\partial \phi(x)} \delta \phi(x)+\mathcal{O}\left((\delta \phi)^{2}\right)\right)  \tag{3.10}\\
\delta S & =\int d^{4} x\left(\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi+\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)} \partial_{\mu} \delta \phi\right)=\int d^{4} x\left(\frac{\partial \mathcal{L}}{\partial \phi}-\partial \mu \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)}\right) \delta \phi+\int d^{4} x \partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)} \delta \phi\right) \tag{3.11}
\end{align*}
$$

From the principle of least action $\delta S=0$ the Euler-Lagrange equations for a continuous field that falls off fast enough to zero at spatial infinity such that the bounary term on the right hand side vanishes:

$$
\begin{equation*}
\frac{\partial \mathcal{L}}{\partial \phi}-\partial \mu \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)}=0 \tag{3.12}
\end{equation*}
$$

Consider the Poincaré transformation $x^{\prime}=\Lambda x+a$. From the invariance of the Lagrangian, the corresponding transformations on the fields are implied:

- for a scalar field, $\phi^{\prime}\left(x^{\prime}\right)=\phi(x)$
- for a vector field, $A_{\mu}^{\prime}\left(x^{\prime}\right)=\Lambda_{\mu}^{\alpha} A_{\alpha}(x)$ (e.g. the vector field $A_{\mu}=\left(A_{0}, \vec{A}\right)$ of electrodynamics)
- for a tensor field, $F_{\mu \nu}^{\prime}\left(x^{\prime}\right)=\Lambda_{\mu}^{\alpha} \Lambda_{\nu}^{\beta} F_{\alpha \beta}(x)$ (e.g. the field strength tensor $F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}$ of electrodynamics)
- a general tensor of rank $p$ transforms as $\Phi_{\mu_{1} \ldots \mu_{p}}^{\prime}\left(x^{\prime}\right)=\mathcal{S}_{\mu_{1} \ldots \mu_{p}}^{\alpha_{1} \ldots \alpha_{p}} \Phi_{\alpha_{1} \ldots \alpha_{p}}(x), \mathcal{S}_{\mu_{1} \ldots \mu_{p}}^{\alpha_{1} \ldots \alpha_{p}}=\Lambda_{\mu_{1}}^{\alpha_{1}} \ldots \Lambda_{\mu_{p}}^{\alpha_{p}}$

Theorem: The fields of a relativistic field theory transform with homomorphism of the Lorentz group: the map $\Lambda \mapsto \mathcal{S}(\Lambda)$ is a homomorphism from the Lorentz group $\Lambda$ to the group of invertible $\mathcal{S}$.
Proof: The product of two Poincaré transformations is

$$
\begin{equation*}
\Phi^{\prime \prime}\left(x^{\prime \prime}\right)=\mathcal{S}\left(\Lambda^{\prime}\right) \Phi^{\prime}\left(x^{\prime}\right)=\mathcal{S}\left(\Lambda^{\prime}\right) \mathcal{S}(\Lambda) \Phi(x)=\mathcal{S}\left(\Lambda \Lambda^{\prime}\right) \Phi(x) \tag{3.13}
\end{equation*}
$$

hence $\mathcal{S}(\mathbb{1})=\mathbb{1}$ and $\mathcal{S}\left(\Lambda^{\prime} \Lambda\right)=\mathcal{S}\left(\Lambda^{\prime}\right) \mathcal{S}(\Lambda)$.

### 3.3.2 Hamilton Formalism

The transition from the Lagrangian to the Hamiltonian formulation is obtained via the Lengendre tranformation from the velocity of the field to the conjugate momentum of the field $\phi$ :

$$
\begin{equation*}
\dot{\phi}=\partial_{t} \phi(x) \mapsto \pi(x)=\frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}, \quad H=\int d \vec{x} \pi(\vec{x}) \dot{\phi}(\vec{x})-L, \quad L=\int d \vec{x} \mathcal{L}\left(\phi, \partial_{\mu} \phi\right) \tag{3.14}
\end{equation*}
$$

Definition: The fundamental Poisson brackets are

$$
\begin{equation*}
\{\phi(\vec{x}), \phi(\vec{y})\}=0, \quad\{\pi(\vec{x}), \pi(\vec{y})\}=0, \quad\{\phi(\vec{x}), \pi(\vec{y})\}=\delta(\vec{x}-\vec{y}) . \tag{3.15}
\end{equation*}
$$

The Poisson bracket on Functionals $F[\phi, \pi], G[\phi, \pi]$ acting on the phase space is

## Remarks:

$$
\begin{equation*}
\{F, G\}=\int d \vec{x}\left(\frac{\delta F}{\delta \phi(\vec{x})} \frac{\delta G}{\delta \pi(\vec{x})}-\frac{\delta F}{\delta \pi(\vec{x})} \frac{\delta F}{\delta \phi(\vec{x})}\right) \tag{3.16}
\end{equation*}
$$

- The Poisson bracket is bilinear, anti-symmetric and fulfills the product rule

$$
\begin{equation*}
\{F, G H\}=G\{F, H\}+\{F, G\} H \tag{3.17}
\end{equation*}
$$

and the Jacobi identity

$$
\begin{equation*}
\{F,\{G, H\}\}+\{H,\{F, G\}\}+\{G,\{H, F\}\}=0 \tag{3.18}
\end{equation*}
$$

- the Hamiltonian equations of motion are:

$$
\begin{equation*}
\dot{\phi}(\vec{x})=\{\phi(\vec{x}), H\}=\frac{\delta H}{\delta \pi(\vec{x})}, \quad \dot{\pi}(\vec{x})=\{\pi(\vec{x}), H\}=-\frac{\delta H}{\delta \phi(\vec{x})} \tag{3.19}
\end{equation*}
$$

### 3.4 Noether Theorem

### 3.4.1 Inner Symmetries

Consider a complex scalar field with the Lagrangian

$$
\begin{equation*}
\mathcal{L}=\partial_{\mu} \phi^{*} \partial^{\mu} \phi-V\left(\phi^{*} \phi\right) \tag{3.20}
\end{equation*}
$$

which is invariant under the phase transformation

$$
\begin{equation*}
\phi(x) \mapsto e^{i \alpha} \phi(x), \quad \phi^{*}(x) \mapsto e^{-i \alpha} \phi(x), \tag{3.21}
\end{equation*}
$$

More generally, we consider fields with values in a vector space with scalar product $\phi(x) \in \mathbb{C}^{n}$ that is invariant under a unitary transformation:

$$
\begin{equation*}
(U \phi(x), U \phi(x))=(\phi(x), \phi(x)), \quad U \in \mathrm{U}(n) \tag{3.22}
\end{equation*}
$$

If we assume that $U$ does not depend on $x$,

$$
\begin{equation*}
U \partial_{\mu} \phi=\partial_{\mu} U \phi \tag{3.23}
\end{equation*}
$$

then

$$
\begin{equation*}
\left.\mathcal{L}=\left(\partial_{\mu} \phi, \partial^{\mu} \phi\right)-V((\phi, \phi))\right) \tag{3.24}
\end{equation*}
$$

is invariant under $\phi(x) \mapsto U \phi(x)$. For a unitary matrix $U=\exp (i X)$, the matrix $X$ is Hermitian, and
Theorem: (Noethers First Theorem )
For a Lagrangian that is invariant under an inner symmetry,

$$
\begin{equation*}
\phi \mapsto U \phi \simeq \phi+i X \phi \equiv \phi+\delta_{X} \phi, \quad X=X^{\dagger} \tag{3.25}
\end{equation*}
$$

the sol called Noether current

$$
\begin{equation*}
J_{X}^{\mu}=\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)} \delta_{X} \phi \tag{3.26}
\end{equation*}
$$

is conserved:

$$
\begin{equation*}
\partial_{\mu} J_{X}^{\mu}=\partial_{0} J_{X}^{0}+\vec{\nabla} \cdot \vec{J}_{X}=0 \tag{3.27}
\end{equation*}
$$

Proof: Since the Lagrangian is invariant:

$$
\begin{align*}
0 & =\delta_{X} \mathcal{L}=\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)} \partial_{\mu}\left(\delta_{X} \phi\right)+\frac{\partial \mathcal{L}}{\partial \phi} \delta_{X} \phi  \tag{3.28}\\
& =\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)} \partial_{\mu}\left(\delta_{X} \phi\right)+\left(\partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)}\right)\right) \delta_{X} \phi=\partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)} \delta_{X} \phi\right) \tag{3.29}
\end{align*}
$$

where in the second step the Euler-Lagrange equation was used.
Theorem: If the Noether current vanishes at spatial infinity fast enough, the Noether current implies a conserved charge called Noether charge:

$$
\begin{equation*}
Q_{X}=\int_{x_{0}} d \vec{x} J_{X}^{0}=\int_{x_{0}} d \vec{x} \pi(x) \delta_{X} \phi(x), \quad \quad \frac{d}{d t} Q_{X}=0 \tag{3.30}
\end{equation*}
$$

Proof: The spatial volume integral of the space derivative is transformed into a surface integral that vanishes

$$
\begin{equation*}
0=\int d \vec{x} \partial_{\mu} J_{X}^{\mu}=\frac{\partial}{\partial x_{0}} \int d \vec{x} \partial_{\mu} J_{X}^{0}+\int d \vec{x} \vec{\nabla} \cdot \vec{J}_{X}=\frac{\partial}{\partial x^{0}} \int d \vec{x} J_{X}^{0}+\underbrace{\oint d \vec{f} \cdot \vec{J}_{X}}_{=0} \tag{3.31}
\end{equation*}
$$

## Remarks:

- The charges $Q_{X}$ act on the fields via the Poisson brackets

$$
\begin{align*}
\left\{\phi(\vec{x}), Q_{X}\right\} & =\int d \vec{y}\left\{\phi(\vec{x}), \pi(\vec{x}) \delta_{X} \phi(\vec{y})\right\}=\delta_{X} \phi(\vec{x})  \tag{3.32}\\
\left\{\pi(\vec{x}), Q_{X}\right\} & =\int d \vec{y}\left\{\pi(\vec{x}), \pi(\vec{x}) \delta_{X} \phi(\vec{y})\right\}=\delta_{X} \pi(\vec{x}) \tag{3.33}
\end{align*}
$$

## Example:

1. Consider again the complex scalar field which is invariant under the $U(1)$ phase transformation:

$$
\begin{equation*}
\delta_{\alpha} \phi=i \alpha \phi, \quad \quad \delta_{\alpha} \phi^{*}=-i \alpha \phi \tag{3.34}
\end{equation*}
$$

Then the Noether current and Noether charge is:

$$
\begin{equation*}
J_{\alpha}^{\mu}=\alpha J^{\mu}, \quad J^{\mu}=i\left(\partial_{\mu} \phi^{\dagger} \phi-\phi^{\dagger} \partial_{\mu} \phi\right), \quad Q=i \int_{x_{0}} d \vec{x}\left(\pi_{\phi}, \phi-\pi_{\phi^{*}} \phi^{*}\right) \tag{3.35}
\end{equation*}
$$

This charge generates the symmetry:

$$
\begin{equation*}
\{\phi, Q\}=i \phi, \quad\left\{\phi^{*}, Q\right\}=-i \phi^{*} \tag{3.36}
\end{equation*}
$$

### 3.4.2 Noether Theorem for Translations

[coming soon]

## Chapter 4

## Basics of Representation Theory

### 4.1 Definitions

Definition: A representation of a group $\mathcal{G}$ on a linear space $V$ is a group homomorphism

$$
\begin{align*}
& D: \mathcal{G} \rightarrow \mathrm{GL}(V), \quad \quad g \mapsto D(g), \quad \text { i.e. } \\
& D\left(g_{1} g_{2}\right)=D\left(g_{1}\right) D\left(g_{2}\right), \quad D(e)=\mathbb{1}, \quad D\left(g^{-1}\right)=D(g)^{-1} \tag{4.1}
\end{align*}
$$

## Remarks:

- after the choice of a basis, the linear space $V$ can be considered to be a vector space $\mathbb{K}^{n}$ and $\operatorname{GL}(V)=\operatorname{GL}(n, \mathbb{K})$
- the representations are linear group actions and as such can also be faithful, free and transitive
- a faithful representation has $D: \mathcal{G} \rightarrow \operatorname{Im}(\mathcal{G}) \leq \mathrm{GL}(V)$ and $\operatorname{Ker}(D)=e \in \mathcal{G}$, hence all group structure is preserved
- there is always the trivial representation $D(g)=\mathbb{1}$ for all $g \in \mathcal{G}$, where all group structure is lost
- for any representation $D(g)$, another (one-dimensional) representation is given by $\operatorname{det} D(g)$


## Example:

- representations of the dihedral group $D_{3}$ : it is sufficient to determine the matrices for a rotation $c_{3}$ and a reflection $\sigma$, as this is a basis for the group presentation:

1. there are two inequivalent one-dimensional representations, the trivial representation $D_{1}^{1} \equiv 1$ and the alternating (or sign) representation $D_{1}^{2}$ with $D_{1}^{2}\left(c_{3}\right)=1, D_{1}^{2}(\sigma)=-1$
2. the two-dimensional representation $D_{2}$ is

$$
D_{2}\left(c_{3}\right)=\left(\begin{array}{cc}
\cos \frac{2 \pi}{3} & -\sin \frac{2 \pi}{3} \\
\sin \frac{2 \pi}{3} & \cos \frac{2 \pi}{3}
\end{array}\right), \quad D_{2}(\sigma)=\left(\begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array}\right)
$$

3. the three-dimensional (natural) representation $D_{3}$ is a permutation representation:

$$
D_{3}\left(c_{3}\right)=\left(\begin{array}{lll}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right), \quad D_{3}(\sigma)=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{array}\right)
$$

The representations are only defined up to basis transformations:
Definition: Two representations $D, D^{\prime}$ are equivalent if they are related by conjugation:

$$
\begin{equation*}
D^{\prime}(g)=S D(g) S^{-1} \tag{4.2}
\end{equation*}
$$

The matrices $D(g)$ and $D^{\prime}(g)$ denote the same linear transformation in a different basis for $V: e_{i}^{\prime}=S_{i j} e_{j}$ such that

$$
\begin{equation*}
\vec{x} \mapsto D(g) \vec{x} \quad \text { and } \quad \vec{y}=S \vec{x} \mapsto S D(g) \vec{x}=S D(g) S^{-1} \vec{y}=D^{\prime}(g) \vec{y} \tag{4.3}
\end{equation*}
$$

The group properties of equivalent representations are equal and hence they can be identified.

Definition: The regular representation $g \mapsto \mathcal{R}(g)$ is the $|\mathcal{G}|$-dimensional representation which acts on the group itself by translation. We will consider left-regular representations $h \mapsto \mathcal{R}(g) h$ for all $h \in \mathcal{G}$ (but one can also define right-regular representations). For finite groups it can be read off from the Cayley table if ordered such that $e$ is on the diagonal.

Theorem: The regular representation is an orthogonal faithful representation of $\mathcal{G}$ into the matrix group $\mathrm{SO}\left(|\mathcal{G}|, \mathbb{Z}_{2}\right)$
Proof: $\quad$ Since $\mathcal{R}_{k j}\left(g_{i}\right)=1$ iff $g_{k} \circ g_{j}^{-1}=g_{i}$, also $g_{i} \circ g_{j}=\sum_{p=1}^{n} \mathcal{R}_{p j}\left(g_{i}\right) g_{p}$. Due to associativity:

$$
\begin{align*}
g_{i} \circ\left(g_{j} \circ g_{k}\right) & =\sum_{p, q} \mathcal{R}_{p k}\left(g_{j}\right) \mathcal{R}_{q p}\left(g_{i}\right) q_{q} \\
=\left(g_{i} \circ g_{j}\right) \circ g_{k} & =\sum_{q} \mathcal{R}_{q k}\left(g_{j} \circ g_{j}\right) q_{q} \\
\Rightarrow \quad \mathcal{R}\left(g_{i}\right) \mathcal{R}\left(g_{j}\right) & =\mathcal{R}\left(g_{i} \circ g_{j}\right) \tag{4.4}
\end{align*}
$$

## Example:

- the regular representation of the dihedral group $D_{3}$ is six-dimensional and

$$
\mathcal{R}\left(c_{3}\right)=\left(\begin{array}{cccccc}
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0
\end{array}\right), \quad \mathcal{R}(\sigma)=\left(\begin{array}{cccccc}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{array}\right)
$$

- Homework: Show that this representation is faithful (isomorphic to $D_{3}$ ).


## Remarks:

- The regular representation contains all irreducible representations of a group (see Sec. 4.5.1).


### 4.2 Reducible and Irreducible Representations

Definition: A representation $D$ is called irreducible if the linear space $V$ has no proper subspace that is invariant under all $D(g)$. Otherwise the representation is called reducible.

## Example:

- The three-dimensional representation of the dihedral group $D_{3}$ is reducible and can be written as a direct sum of irreduclbe subspaces: $D_{3}=D_{1}^{1} \oplus D_{2}, \operatorname{dim} D_{3}=\operatorname{dim} D_{1}^{1}+\operatorname{dim} D_{2}$. To see this, consider the following basis vectors:

$$
\begin{align*}
\vec{f}_{1} & =\frac{1}{\sqrt{3}}\left(\vec{e}_{1}+\vec{e}_{2}+\vec{e}_{3}\right) \\
\vec{f}_{2} & =\frac{1}{\sqrt{2}}\left(\vec{e}_{2}-\vec{e}_{3}\right) \\
\vec{f}_{3} & =\frac{1}{\sqrt{6}}\left(-2 \vec{e}_{1}+\vec{e}_{2}+\vec{e}_{3}\right) \tag{4.5}
\end{align*}
$$

which are obtained via $S^{T}=S^{-1}=\frac{1}{\sqrt{6}}\left(\begin{array}{ccc}\sqrt{2} & 0 & -2 \\ \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & -\sqrt{3} & 1\end{array}\right)$ and $\vec{f}_{i}=\left(S^{-1}\right)_{i j} \vec{e}_{i}$. Then with $D_{3}^{\prime}=S D_{3} S^{-1}$

$$
D_{3}^{\prime}\left(c_{3}\right)=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \frac{2 \pi}{3} & -\sin \frac{2 \pi}{3} \\
0 & \sin \frac{2 \pi}{3} & \cos \frac{2 \pi}{3}
\end{array}\right), \quad D_{3}^{\prime}(\sigma)=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{array}\right)
$$

- the regular representation of the dihedral group $D_{3}$ is reducible and decomposes into $\mathcal{R}=$ $D_{1}^{1} \otimes D_{1}^{2} \otimes D_{2} \otimes D_{2}, \operatorname{dim} \mathcal{R}=6=\operatorname{dim} D_{1}^{1}+\operatorname{dim} D_{1}^{1}+\operatorname{dim} D_{2}+\operatorname{dim} D_{2}=1_{1}+2+2$
Definition: A representation $D$ is called decomposable if it can be decomposed further into a direct sum $D(g)=D_{1}(g) \oplus D_{2}(g) \oplus \ldots \oplus D_{k}(g)$, with $k>1$. The basis can be chosen such that the matrix has the shape

$$
D(g)=\left(\begin{array}{cccc}
D_{1}(g) & 0 & \ldots & 0 \\
0 & D_{2}(g) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & D_{k}(g)
\end{array}\right)
$$

Otherwise the representation is called indecomposable.
A reducible representation that can be decomposed into a direct sum of irreducible representations is called fully reducible.

## Remarks:

- the one-dimensional representations are always irreducible
- an irreducible representation is always indecomposable
- however, there are reducible representations that are indecomposable: Assume that a reducible representation has a a proper invariant subspace $V_{1}$ in $V$, i.e. $D(g)$ maps every vector of $V_{1}$ into $V_{1}$. Choose a basis such that the first basis vectors are within $V_{1}$, then the representation matrices have the form

$$
D(g)=\left(\begin{array}{cc}
D_{1}(g) & H(g) \\
0 & D_{2}(g)
\end{array}\right)
$$

Only if $H(g)=0$ the representation is decomposable.

Theorem: Every unitary representation $D$ is fully reducible.
Proof: If $D$ is irreducible, there is nothing to prove. Hence assume that $V_{1}$ is a proper invariant subspace and

$$
W=V_{1}^{\perp} \equiv\left\{w \in V \mid\left(w, V_{1}\right)=0\right\}
$$

its unitary-orthogonal complement, such that for all $v \in V: v=v_{1}+w$. Due to the unitarity of $D(g)$ :

$$
\forall g \in \mathcal{G}:\left(D(g) w, v_{1}\right)=\left(w, D^{\dagger}(g) v_{1}\right)=\left(w, D\left(g^{-1}\right) v_{1}\right)=0
$$

since $D\left(g^{-1}\right) v_{1} \in V_{1}$. Hence $W$ is also an invariant subspace. The argument can be repeated until $V$ is decomposed into irredubile parts.

Definition: Let $g \mapsto D(g)$ be a representation acting on a complex vector space, then the complex conjugate representation $g \mapsto D^{*}(g)$ is also a representation with

$$
D^{*}(e)=\mathbb{1}^{*}=\mathbb{1}, \quad D^{*}\left(g_{1} g_{2}\right)=\left(D\left(g_{1}\right) D\left(g_{2}\right)\right)^{*}=D\left(g_{1}\right)^{*} D\left(g_{2}\right)^{*}
$$

Definition: A representation is called real if it is equivalent to a representation with real matrices. A representation is called pseudoreal if $D(g)$ and $D^{*}(g)$ are equivalent.

## Remarks:

- the group $\mathrm{SU}(2)$ is pseudoreal: it has the representation

$$
U=\left(\begin{array}{cc}
a & b \\
-b^{*} & a^{*}
\end{array}\right), \quad|a|^{2}+|b|^{2}=1
$$

and is equivalent to

$$
\sigma_{2} U \sigma_{2}=U^{*}
$$

However, the representation is not real.

### 4.3 Direct sum and tensor product of representations

Definition: Let $D$ be a $n$-dimensional representation and $D^{\prime}$ a $m$-dimensional representation, with $\vec{x} \mapsto D(g) \vec{x}, \vec{y} \mapsto D(g) \vec{y}^{\prime}$, and the the direct sum of the vector spaces is

$$
\vec{s}=\vec{x} \oplus \vec{y} \in V \oplus V^{\prime},
$$

Then the direct sum representation of $D$ and $D^{\prime}$ is

$$
\begin{equation*}
\vec{s} \mapsto[D(g) \vec{x}] \oplus\left[D^{\prime}(g) \vec{y}\right] \equiv\left[D \oplus D^{\prime}\right](g) \vec{s}, \quad s_{i}+n j=x_{i}+y_{j} \mapsto D(g) x_{i} \oplus D^{\prime}(g) y_{j} . \tag{4.6}
\end{equation*}
$$

The representation $D \otimes D^{\prime}$ has dimension $n+m$.

Definition: Let $D$ be a $n$-dimensional representation and $D^{\prime}$ a $m$-dimensional representation, with $\vec{x} \mapsto D(g) \vec{x}, \vec{y} \mapsto D(y) \vec{y}^{\prime}$, and the tensor product of the vector spaces is

$$
t=\vec{x} \otimes \vec{y} \in V \otimes V^{\prime},
$$

i.e. a tensor of rank 2: $t_{i j}=x_{i} y_{j}$.

Then the tensor product representation of $D$ and $D^{\prime}$ is

$$
\begin{equation*}
t \mapsto[D(g) \vec{x}] \otimes\left[D^{\prime}(g) \vec{y}\right] \equiv\left[D \otimes D^{\prime}\right](g), \quad t_{i j} \mapsto \sum_{a, b} D(g)_{i a} D(g)_{b j} t_{a b} \tag{4.7}
\end{equation*}
$$

The representation $D \otimes D^{\prime}$ has dimension $n \cdot m$.

### 4.4 Characters and the Lemmas of Schur

### 4.4.1 Characters and Character Tables

Definition: A map $f: \mathcal{G} \rightarrow \mathbb{C}$ is called a class function of the group $\mathcal{G}$ if $f(g)=f\left(a g a^{-1}\right)$ for all $a \in \mathcal{G}$.

## Remarks:

- if follows from the definition that a class function is constant on each conjugacy class
- the trace and determinant of a matrix group are class functions

Definition: A character of a representation $g \rightarrow D(g)$ is the class function

$$
\begin{equation*}
\chi_{D}(g)=\operatorname{tr}(D(g)) \tag{4.8}
\end{equation*}
$$

Theorem: Characters have the following properties:

1. Equivalent representations have the same character.
2. The dimension of the representation is $\operatorname{dim} D=\chi_{D}(e)$.
3. For unitary representations it holds that $\chi_{D}\left(g^{-1}\right)=\chi_{D}^{*}(g)$.
4. For $D=D_{1} \oplus D_{2} \oplus \ldots \oplus D_{k}$ it holds that $\chi_{D}=\chi_{D_{1}}+\chi_{D_{2}}+\ldots \chi_{D_{k}}$
5. It holds that $\chi_{D_{1} \otimes D_{2}}=\chi_{D_{1}} \chi_{D_{2}}$

## Proof:

1. Due to the cyclicity of the trace, it does not change for a similarity transformation
2. For a $n$-dimensional representation: $\chi_{D}(e)=\operatorname{tr} D(e)=\operatorname{tr} \mathbb{1}_{n \times n}=n$
3. This follows from unitarity: $\operatorname{tr} D^{-1}(g)=\operatorname{tr} D^{\dagger}(g)$. For representations where $D(g)$ is similar to $D\left(g^{-1}\right)$, the character is real: $\chi_{D}(g)=\chi_{D}\left(g^{-1}\right)=\chi_{D}^{*}(g)$
4. This is evident in the basis where $D$ is block-diagonal.
5. 

$$
\chi_{D_{1} \otimes D_{2}}=\operatorname{tr}\left(D_{1} \otimes D_{2}\right)=\sum_{i}\left(D_{1}\right)_{i i} \operatorname{tr} D_{2}=\operatorname{tr} D_{1} \operatorname{tr} D_{2}=\chi\left(D_{1}\right) \chi\left(D_{2}\right)
$$

## Example:

1. character table of the dihedral group $\mathcal{D}_{3}$, including the reducible representation $D_{2} \otimes D_{2}$ with $c=\cos (2 \pi / 3), s=\sin (2 \pi / 3)$

$$
\left(D_{2} \otimes D_{2}\right)\left(c_{3}\right)=\left(\begin{array}{cccc}
c^{2} & -c s & -s c & s^{2}  \tag{4.9}\\
c s & c^{2} & -s^{2} & -s c \\
s c & -s^{2} & c^{2} & -c s \\
s^{2} & s c & c s & c^{2}
\end{array}\right) \quad\left(D_{2} \otimes D_{2}\right)(\sigma)=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

| $\mathcal{D}_{3}$ | $\chi_{D_{1}^{1}}$ | $\chi_{D_{1}^{2}}$ | $\chi_{D_{2}}$ | $\chi_{D_{3}}$ | $\chi_{D_{2} \otimes D_{2}}$ | $\mathcal{R}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $K_{e}$ | 1 | 1 | 2 | 3 | 4 | 6 |
| $K_{c_{3}}$ | 1 | 1 | -1 | 0 | 1 | 0 |
| $K_{\sigma}$ | 1 | -1 | 0 | 1 | 0 | 0 |

where $\mathcal{D}_{3}$ has three conjugacy classes and three irreducible representations $\chi_{D_{1}^{1}}, \chi_{D_{1}^{2}}, \chi_{D_{2}}$ and all others being reducible, e.g. $D_{2} \otimes D_{2}=D_{2} \oplus D_{1}^{1} \oplus D_{1}^{2}$.

### 4.4.2 Motivation: The benzene ring

The benzene ring has as its symmetry group the Dihedral group $\mathcal{D}_{6}$. The energy of this molecule is given by pairs of nearest neighbors

$$
H=\epsilon\left(\begin{array}{llllll}
0 & 1 & 0 & 0 & 0 & 1  \tag{4.10}\\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0
\end{array}\right)
$$

and is proportional to the discrete second derivative on the ring.
The six-dimensional representation of $\mathcal{D}_{6}$ is spanned by the same 6 basis vectors $\vec{e}_{k}$ as the Hamiltonian.
and is given by the generators $c_{6} \in \operatorname{cal} D_{6}$ which is an elementary rotation by $2 \pi / 6$ and by $\sigma_{d} \in \mathcal{D}_{6}$ which is a reflection with two fixed corners of the hexagon:

$$
D\left(c_{6}\right)=\left(\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0  \tag{4.11}\\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right), \quad D\left(\sigma_{d}\right)=\left(\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{array}\right) .
$$

Since this representation is reducible, we can determine a new basis in which the matrices are
block-diagonal corresponding to invariant subspaces:

$$
\begin{array}{ll}
\vec{f}_{1}=\frac{1}{\sqrt{6}} \sum_{i=1}^{6} \cos (2 \pi k) \vec{e}_{k} & \\
\overrightarrow{f_{2}}=\frac{1}{\sqrt{6}} \sum_{i=1}^{6} \cos (\pi k) \vec{e}_{k} & \\
\overrightarrow{f_{3}}=\frac{1}{\sqrt{3}} \sum_{i=1}^{6} \cos \left(\frac{2}{3} \pi k\right) \vec{e}_{k}, & \vec{f}_{4}=\frac{1}{\sqrt{3}} \sum_{i=1}^{6} \sin \left(\frac{2}{3} \pi k\right) \vec{e}_{k} \\
\overrightarrow{f_{5}}=\frac{1}{\sqrt{6}} \sum_{i=1}^{6} \cos \left(\frac{1}{3} \pi k\right) \vec{e}_{k} & \vec{f}_{6}=\frac{1}{\sqrt{6}} \sum_{i=1}^{6} \sin \left(\frac{1}{3} \pi k\right) \vec{e}_{k} \tag{4.15}
\end{array}
$$

The trivial representation is spanned by $\vec{f}_{1}$, and there is another one-dimensional irrep which is spanned by $\vec{f}_{2}$ : under elementary rotation $c_{6}$ :

$$
\begin{equation*}
\overrightarrow{f_{2}} \mapsto \frac{1}{\sqrt{6}} \sum_{i=1}^{6} \cos (\pi k) \vec{e}_{k+1}=\frac{1}{\sqrt{6}} \sum_{i=1}^{6} \cos (\pi k-\pi) \vec{e}_{k}=-\vec{f}_{2} \tag{4.16}
\end{equation*}
$$

The vectors $\vec{f}_{3}$ and $\vec{f}_{4}$ span a 2-dimensional representation $D_{2}^{1}$ : and the vectors $\vec{f}_{5}$ and $\vec{f}_{6}$ span another, inequivalent 2-dimensional representation $D_{2}^{2}$ :

$$
\begin{array}{rlr}
D_{2}^{1}\left(c_{6}\right)=\left(\begin{array}{cc}
\cos (2 \pi / 3) & -\sin (2 \pi / 3) \\
\sin (2 \pi / 3) & \cos (2 \pi / 3)
\end{array}\right), & D_{2}^{1}\left(\sigma_{d}\right)=\left(\begin{array}{cc}
\cos (2 \pi / 3) & -\sin (2 \pi / 3) \\
-\sin (2 \pi / 3) & -\cos (2 \pi / 3)
\end{array}\right) \\
D_{2}^{2}\left(c_{6}\right)=\left(\begin{array}{cc}
\cos (\pi / 3) & -\sin (\pi / 3) \\
\sin (\pi / 3) & \cos (\pi / 3)
\end{array}\right), & D_{2}^{2}\left(\sigma_{d}\right)=\left(\begin{array}{cc}
\cos (\pi / 3) & -\sin (\pi / 3) \\
-\sin (\pi / 3) & -\cos (\pi / 3)
\end{array}\right) \tag{4.18}
\end{array}
$$

They are inequivalent because the characters are different, e.g.

$$
\begin{equation*}
\chi_{D_{2}^{1}}\left(c_{6}\right)=2 \cos (2 \pi / 3)=-1 \quad \text { versus } \quad \chi_{D_{2}^{2}}\left(c_{6}\right)=2 \cos (\pi / 3)=1 \tag{4.20}
\end{equation*}
$$

Hence the reduction of the six-dimensional representation $D$ is of the Dihedral group $\mathcal{D}_{6}$ is

$$
\begin{equation*}
D=D_{1}^{1} \oplus D_{1}^{2} \oplus D_{2}^{1} \oplus D_{2}^{2} \tag{4.21}
\end{equation*}
$$

### 4.4.3 Lemma of Schur

Theorem: (Lemma of Schur): Let $D_{1}, D_{2}$ be irreducible representations of a group $\mathcal{G}$ in vector spaces $V_{1}, V_{2}$ of dimension $n_{1}, n_{2}$. Let $H: V_{1} \rightarrow V_{2}$ be a linear map such that

$$
\begin{equation*}
\forall g \in \mathcal{G}: H D_{1}(g)=D_{2}(g) H \tag{4.22}
\end{equation*}
$$

Then either

1. $H=0$, or
2. $n_{1}=n_{2}, H$ is invertible and for all $g \in \mathcal{G}: D_{2}(g)=H D_{1}(g) H^{-1}$.

## Proof:

1. $H\left(V_{1}\right)=\{\overrightarrow{0}\}$
2. $H\left(V_{1}\right)=V_{2}$, then $H$ is surjective, also either $\operatorname{Ker} H=V_{1}$ (which is the first case) or $\operatorname{Ker} H=\{\overrightarrow{0}\}$, then $H$ is bijective and hence invertible. It follows that $D_{1}$ and $D_{2}$ are equivalent representations.

The Lemma of Schur has important consequences in physics, in particular with systems with Hamiltonians that are invariant under a group of symmetry transformations $\mathcal{G}$.
Theorem: Let $D$ be an irreducible representation on a vector space $V$. Let the linear operator $H$ acting also on $V$ commute with all representation matrices:

$$
\begin{equation*}
\forall g \in \mathcal{G}: H D(g)=D(g) H \tag{4.23}
\end{equation*}
$$

Then $H$ is a multiple of the identity: $H=\lambda \mathbb{1}$.
Proof: Coming Soon

Theorem: Every irreducible representation of an Abelian group is one-dimensional
Proof: $\quad$ Let $D$ be a irreducible representation. Then for all $g, h \in \mathcal{G}: D(g) D(h)=D(h) D(g)$. Due to the previous theorem, with $H=D(h), D(h)=\lambda \mathbb{1}$. Hence all representation matrices are diagonal, and only one-dimensional representations can be indecomposable, hence irreducible.

Theorem: Let $D$ be a representation on $V$ and let $D$ be fully reducible into pairwise inequivalent irreducible representations:

$$
D=D_{1} \oplus D_{2} \oplus \ldots \oplus D_{k}
$$

Let the linear operator $H$ on $V$ commute with every $D(g)$ :

$$
\forall g \in \mathcal{G}:[H, D(g)]=0
$$

Then the $n_{i}$-dimensional invariant subspaces $V_{i}$ belonging to the irreducible representations $D_{i}$ are eigenspaces of $H: H_{i}=\lambda_{i} \mathbb{1}_{n_{i} \times n_{i}}$
Proof: Let $D$ be a, irreducible representation. Then for all $g, h \in \mathcal{G}: D(g) D(h)=D(h) D(g)$. Due to the previous theorem, with $H=D(h), D(h)=\lambda \mathbb{1}$. Hence all representation matrices are diagonal, and only one-dimensional representations can be indecomposable, hence irreducible.

### 4.4.4 Orthogonality Relations

## Groups with a Mean

Before we can formulate the orthogonality relations, some remarks are in place: We will consider in the following groups on which one can define means: Virtually all groups in physics (finite groups, compact Lie groups) belong to that class.

Definition: Let $\mathcal{G}$ be a finite or compact group, and let $f: \mathcal{G} \rightarrow \mathbb{C}$ be a map on which we consider the mean:

$$
\begin{align*}
& \mathcal{M}(f)=\frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} f(g)  \tag{4.24}\\
& \mathcal{M}(f)=\int_{\mathcal{G}} d \mu(g) f(g), \quad \int_{\mathcal{G}} d \mu(g)=1 \tag{4.25}
\end{align*}
$$

The mean of $f$ on the group has the follow properties:

- linearity: $\mathcal{M}\left(\alpha f_{1}+\beta f_{2}\right)=\alpha \mathcal{M}\left(f_{1}\right)+\beta \mathcal{M}\left(f_{2}\right)$
- positivity: for $f \geq 0$ also $\mathcal{M}(f) \geq 0$
- norm: if for all $g \in \mathcal{G}: f(g)=1$, then $\mathcal{M}(f)=1$
- left- and right-invariance:

$$
\mathcal{M}\left(h \circ l_{a}\right)=\mathcal{M}\left(h \circ r_{a}\right)=\mathcal{M}(f) \quad \text { for } \quad f(g) \circ l_{a}=f(a g), \quad f(g) \circ r_{a}=f(g a)
$$

## Remarks:

- Groups for which a mean can be defined, have also a scalar product for these maps

$$
\begin{equation*}
f_{1}, f_{2}: \mathcal{G} \rightarrow \mathbb{C}: \quad\left(f_{1}, f_{2}\right)=\mathcal{M}\left(f_{1}^{*} \cdot f_{2}\right) \tag{4.26}
\end{equation*}
$$

- Every representation $D(g)$ of a group $\mathcal{G}$ on which a mean $\mathcal{M}$ can be defined, acting on a vector space with a scalar product $\langle.,$.$\rangle is equivalent to a unitary representation U(g)$.


## Orthogonality Theorem

Theorem: Let $D_{1}$ and $D_{2}$ be two irreducible representations of a group $\mathcal{G}$ with a mean, and $V_{1}$, $V_{2}$ be the corresponding invariant subspaces. Let $U: V_{1} \rightarrow V_{2}$ be a linear map, and define the matrix $H: V_{1} \rightarrow V_{2}$ with

$$
\begin{equation*}
H_{i j}=\mathcal{M}\left(\left(D_{2}(g) U D_{1}^{-1}(g)\right)_{i j}\right), \quad i=1, \ldots \operatorname{dim} V_{2}, \quad j=1, \ldots \operatorname{dim} V_{1} \tag{4.27}
\end{equation*}
$$

1. If $D_{1}$ and $D_{2}$ are inequivalent, then $\left(\chi_{D_{1}}, \chi_{D_{2}}\right)=0$.
2. If $D_{1}$ and $D_{2}$ are equivalent, then $\left(\chi_{D_{1}}, \chi_{D_{2}}\right)=1$.

## Proof:

1. Due to the first alternative of Schurs Lemma, we know that in the inequivalent case $H=0$, hence for all $i, j: \mathcal{M}\left(\left(D_{2}(g) U D_{1}^{-1}(g)\right)_{i j}\right)=0$. Now set $U$ such that only the entry at $(p, q)$ is 1 and all other entries are $0:(U)_{i j}=\delta_{p i} \delta_{q j}$. Then

$$
\begin{equation*}
\mathcal{M}\left(D_{2}(g)_{i p} D_{1}^{-1}(g)_{q j}\right)=0 \tag{4.28}
\end{equation*}
$$

and with $i=p, j=q$ :

$$
\begin{align*}
& \sum_{i, j} \mathcal{M}\left(D_{2}(g)_{i i} D_{1}^{-1}(g)_{j j}\right)=\mathcal{M}\left(\sum_{i, j} D_{2}(g)_{i i} D_{1}^{-1}(g)_{j j}\right) \\
= & \mathcal{M}\left(\operatorname{tr} D_{2}(g) \operatorname{tr} D_{1}^{-1}(g)\right)=\mathcal{M}\left(\chi_{D_{1}}^{*} \chi_{D_{2}}\right)=\left(\chi_{D_{1}}, \chi_{D_{2}}\right)=0 \tag{4.29}
\end{align*}
$$

2. Due to the second alternative of Schurs Lemma, in a basis where $D \equiv D_{1}=D_{2}$, for all $\tilde{g} \in \mathcal{G}: D(\tilde{g}) H=H D(\tilde{g})$ and hence $H=\lambda \mathbb{1}$, where $\lambda$ is

$$
\begin{equation*}
\lambda=\frac{\operatorname{tr} H}{\operatorname{dim} D}=\frac{\operatorname{tr}\left(D(g) U D^{-1}(g)\right)}{\operatorname{dim} D}=\frac{\operatorname{tr}(U)}{\operatorname{dim} D} \tag{4.30}
\end{equation*}
$$

with $U$ now a square matrix. Now set $(U)_{a b}=\delta_{a j} \delta_{b p}$, then $\operatorname{tr}(U)=\delta_{j p}$ and

$$
\begin{equation*}
\mathcal{M}\left(D(g)_{i j} D^{-1}(g)_{p q}\right)=\frac{1}{\operatorname{dim} D} \delta_{j p} \delta_{i q} \tag{4.31}
\end{equation*}
$$

Theorem: The characters of all irreducible representations are orthonormal elements in the linear space of complex-valued class functions, with a scalar product

$$
\begin{equation*}
\left(\chi_{m}, \chi_{n}\right)=\delta_{m n}, \quad m, n \in\{1, \ldots r\} \tag{4.32}
\end{equation*}
$$

Proof: The orthogonality was shown in the previous theorem. The character of an irrep $D$ has also norm 1 :

$$
\begin{equation*}
\mathcal{M}\left(\chi_{D}^{*} \chi_{D}\right)=1 \tag{4.33}
\end{equation*}
$$

## Reduction Formula

Theorem: (Reduction Formula): For a group $\mathcal{G}$ with a mean, the irreducible representation $D_{n}$ occurs in a representation

$$
\begin{equation*}
D=\bigoplus_{n=1}^{r} c_{n} D_{n} \tag{4.34}
\end{equation*}
$$

$D$ exactly $c_{n}=\left(\chi_{D}, \chi_{n}\right)$ times.
Proof: We just need to make use of the orthogonality:

$$
\begin{equation*}
\left(\chi_{D}, \chi_{n}\right)=\left(\bigoplus_{m=1}^{r} c_{m} \chi_{m}, \chi_{n}\right)=\bigoplus_{m=1}^{r} c_{m} \delta_{n m}=c_{n} \tag{4.35}
\end{equation*}
$$



Table 4.1: The character matrix. The rows and columns are normalized.

Theorem: Two irreducible representation are equivalent if and only if they have the same characters (without proof).
Proof:

### 4.5 Representation Theory of Finite Groups

### 4.5.1 Finding all Irreducible Representations

We want to find all irreducible representations of a finite group. In order to do so, we consider the regular representation, which has the character

$$
\chi_{\mathrm{reg}}(g)=\operatorname{tr}(\mathcal{R})(g)= \begin{cases}|\mathcal{G}| & \text { for } g=e  \tag{4.36}\\ 0 & \text { else }\end{cases}
$$

Theorem: (Burnside Theorem) Every irreducible representation $D_{n}$ of a finite group $\mathcal{G}$ is contained in the $|\mathcal{G}|$-dimensional regular representation $\mathcal{R}$ exactly $\operatorname{dim}\left(D_{n}\right)$ times:

$$
\begin{equation*}
\mathcal{R}=\bigoplus_{n=1}^{r} \operatorname{dim}\left(D_{n}\right) D_{n}, \quad|\mathcal{G}|=\sum_{n=1}^{r}\left(\operatorname{dim}\left(D_{n}\right)^{2}\right) \tag{4.37}
\end{equation*}
$$

Proof: We need to compute the coefficients $c_{n}$ from the orthogonality relations:

$$
\begin{equation*}
c_{n}=\left(\chi_{n}, \chi_{\mathrm{reg}}\right)=\frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \chi_{n}^{*}(g) \chi_{r e g}(g)=\frac{1}{\mathcal{G}} \chi_{n}^{*}(e) \chi_{\mathrm{reg}}(e) \tag{4.38}
\end{equation*}
$$

where the last equation holds since the mean is left-invariant. With $\chi_{n}(e)=\operatorname{dim}\left(D_{n}\right)$ and $\chi_{\mathrm{reg}}(e)=|\mathcal{G}|$ we obtain the result.

### 4.5.2 The character matrix

We have already had an example of a character table.
Definition: The character matrix is ar $\times k$ dimensional matrix that contains the normalized characters for each conjucacy class $K_{i}$ :

$$
\begin{equation*}
h_{i} \cdot \chi_{j}\left(K_{i}\right), \quad h_{i}\left(\frac{\left|K_{i}\right|}{|\mathcal{G}|}\right)^{1 / 2} \tag{4.39}
\end{equation*}
$$

In fact, the character matrix is a square matrix:
Theorem: There number of irreps $r$ equals the number of conjugacy classes $k$. Every class function is a linear combination of orthonormal characters.
Proof: $\quad$ First, we show that $r \leq k$. This is because the columns are unitary-orthogonal to each other, for $n, m \in\{1, \ldots r\}$ :

$$
\begin{equation*}
\sum_{i=1}^{k} h_{i} \chi_{m}^{*}\left(K_{i}\right) h_{i} \chi_{n}\left(K_{i}\right)=\frac{1}{|\mathcal{G}|} \sum_{i=1}^{k}\left|K_{i}\right| \chi_{m}^{*}\left(K_{i}\right) \chi_{n}\left(K_{i}\right)=\left(\chi_{m}, \chi_{n}\right)=\delta_{m, n} \tag{4.40}
\end{equation*}
$$

It is still possible that two distinct conjugacy classes have the same character. That this is not the case can be seen as follows, from

$$
\begin{equation*}
\chi_{n}\left(K_{i}\right)=\sum_{m} \chi_{m}\left(K_{i}\right)\left(\chi_{m}, \chi_{n}\right)=\sum_{K_{j}}\left(\frac{1}{|\mathcal{G}|} \sum_{m}\left|K_{j}\right| \chi_{m}\left(K_{i}\right) \chi_{m}^{*}\left(K_{j}\right)\right) \chi_{n}\left(K_{j}\right) . \tag{4.41}
\end{equation*}
$$

The only contribution from this sum is for $K_{i}=K_{j}$, hence we obtain the orthogonality relation for $i, j \in\{1, \ldots k\}$ :

$$
\begin{equation*}
\sum_{m=1}^{r} h_{i} \chi_{m}^{*}\left(K_{i}\right) h_{j} \chi_{m}\left(K_{j}\right)=\delta_{i, j} . \tag{4.42}
\end{equation*}
$$

### 4.5.3 Representations of the Symmetric Group

Definition: Consider the symmetric group $S_{n}$ acting on a $n$-dimensional vector space, i.e $\pi \in S_{n}$ acts on the vector

$$
\begin{equation*}
\vec{v}=\sum_{i=1}^{n} \alpha_{i} \vec{e}_{i} \mapsto L(\pi) \vec{v}=\sum_{i=1}^{n} \alpha_{i} \vec{e}_{\pi(i)} \tag{4.43}
\end{equation*}
$$

which defines the so-called permuation representation.

## Remarks:

- an example for $\pi \in S_{4}$ : e.g.

$$
\pi=(1,2)(34) \mapsto L(\pi)=\left(\begin{array}{cccc}
0 & 1 & 0 & 0  \tag{4.44}\\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{array}\right), \quad \pi=(1,4,3)(2) \mapsto L(\pi)=\left(\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{array}\right)
$$

- the trace of $L(\pi)$ is the number of cycles of length 1 , hence the character is $\chi_{L}(\pi)=\nu_{1}$
- for $\left|K_{\nu}\right|$ the number of elements in the conjugacy class for cycle $\nu$, the squared norm is

$$
\begin{equation*}
\left(\chi_{L}, \chi_{L}\right)=\frac{1}{\left|S_{n}\right|} \sum_{\nu}\left|K_{\nu}\right| \nu_{1}^{2}=\frac{1}{n!} \sum_{\nu} P(n, \nu) \nu_{1}^{2}=2, \tag{4.45}
\end{equation*}
$$

hence the representation is reducible

## Young Tableaux and Young Diagrams

The irreducible representations can be labelled by Young diagrams. First we combine the cycle types $\nu$ of a permutation $\pi \in S_{n}$ into an integer partition of $n$ :

$$
\begin{array}{llllllllll}
\lambda_{1} & = & \nu_{1} & + & \nu_{2} & + & \nu_{3} & + & \cdots & + \\
\lambda_{2} & = & & \nu_{2} & + & \nu_{3} & + & \cdots & + & \nu_{n} \\
\lambda_{3} & = & & & & \nu_{3} & + & \cdots & + & \nu_{n} \\
\vdots & & & & & & & & & \vdots \\
\lambda_{n} & = & & & & & & & & \nu_{n}
\end{array}
$$

Since $\sum_{i} \nu_{i}=n$ (every number in $\{1, \ldots n\}$ is exactly in one cycle), also $\sum_{i} \lambda_{i}=n$. With $\lambda=\left[\lambda_{1}, \ldots, \lambda_{n}\right]$ with $\lambda_{i} \geq \lambda_{i+1}$, the ordered parts, we write $\lambda \vdash n$ to denote that the parts parititions $n$.
Definition: With every partition of the $P(n)$ partitions, Young diagram $T_{\lambda}$ can be associated, which is a set of boxes that left-aligned such that the number of boxes in the row below is not larger than that of a row.

Definition: A Young tableau is a map that assign the number $\{1, \ldots n\}$ into the boxes of a Young diagram $T_{\lambda}$. A standard Young tableau is a Young tableau such that the number are sorted into the boxes such that to the right and to the bottom the number decrease.

## Remarks:

- the symmetric group acts on the set of Young tableaux of the shape $\lambda$.
- every Young tableau defines an operators that is obtained by symmetrizing in all rows, and anti-symmetrizing in all columns, which define the Young symmetrizer $Y$

$$
\begin{equation*}
S=\prod_{i=1}^{n} S_{i}, \quad A=\prod_{i=1}^{n} A_{i}, \quad Y=A S, \quad Y^{2}=k Y \tag{4.46}
\end{equation*}
$$

with $k$ some normalization constant.

- the Young symmetrizers $Y$ act on the
- howevever, the set of vectors obtained from $Y$ are linear dependent. It turns out that a basis for the irrep $\lambda$ is obtained by restricting the set of $Y$ to the standard Young tableaux

Theorem: (Hook formula): The dimension of an irreducible representation $\lambda \vdash n$ of $S_{n}$ is given by the number of standard Young tableaux, and it can be computed by the formula

$$
\begin{equation*}
d^{\lambda}=\frac{n!}{\prod_{i, j} h_{i j}^{\lambda}} \quad h_{i j}^{\lambda}=\lambda_{i}-j+\lambda_{j}^{\prime}-i+1 \tag{4.47}
\end{equation*}
$$

with $h_{i j}^{\lambda}$ the hooks lengths.
We will see explictly how the irreps are computed via Young Diagrams for $S_{3}$

## Irreducible representations of $S_{3}$

[coming soon]

The dimensions obtained from the number of standard Young tableaux is:

$$
\begin{equation*}
d^{[3,0,0]}=\frac{3!}{3 \cdot 2}=1, \quad d^{[2,1,0]}=\frac{3!}{3}=2, \quad d^{[0,0,1]}=\frac{3!}{3 \cdot 2}=1, \tag{4.48}
\end{equation*}
$$

and with the theorem of Burnside:

$$
\begin{equation*}
\sum_{\lambda \vdash n}\left(d^{\lambda}\right)^{2}=1^{2}+2^{2}+1^{2}=6=3!=\left|S_{3}\right| \tag{4.50}
\end{equation*}
$$

## Irreducible representations of $S_{4}$

[coming soon]
The dimensions obtained from the number of standard Young tableaux is:

$$
\begin{align*}
& d^{[4,0,0,0]}=\frac{4!}{4 \cdot 3 \cdot 2}=1, \quad d^{[3,1,0,0]}=\frac{4!}{4 \cdot 2}=3, \quad d^{[2,2,0,0]}=\frac{4!}{3 \cdot 2 \cdot 2}=2,  \tag{4.51}\\
& d^{[2,1,1,0]}=\frac{4!}{4 \cdot 2}=3, \quad d^{[0,0,0,1]}=\frac{4!}{4 \cdot 3 \cdot 2}=1, \tag{4.52}
\end{align*}
$$

and with the theorem of Burnside:

$$
\begin{equation*}
\sum_{\lambda \vdash n}\left(d^{\lambda}\right)^{2}=1^{2}+3^{2}+2^{2}+3^{2}+1^{2}=24=4!=\left|S_{4}\right| \tag{4.53}
\end{equation*}
$$

### 4.6 Representation of Lie Groups

### 4.6.1 Irredubile Representations of $\mathrm{U}(1)$

For the unitary group $U(1)$, every group element is a separate conjugacy class. Hence all functions are class functions. Since $U(1)$ is Abelian, $l l$ irreducible representations are one-dimensional, and there is a basis where the irreps are unitary. Hence every irrep has the form

$$
\begin{equation*}
D^{e^{i} \theta}=e^{i h(\theta)}, \quad h(\theta) \in \mathbb{R} \tag{4.54}
\end{equation*}
$$

The homomorphism property of a representation implies

$$
\begin{equation*}
e^{i h(0)}=e^{i h(2 \pi)}=1, \quad e^{i h\left(\theta_{1}+\theta_{2}\right)}=e^{i h\left(\theta_{1}\right)+h\left(\theta_{2}\right)} \tag{4.55}
\end{equation*}
$$

hence $h(\theta)$ is a linear function with $h(\theta+2 \pi)=h($ theta $)+2 \pi n$, and the irreps are

$$
\begin{equation*}
D_{n}: e^{i \theta} \mapsto D_{n}\left(e^{i \theta}\right)=e^{i n \theta}, \quad n \in \mathbb{Z} \tag{4.56}
\end{equation*}
$$

The characters of the infinite irreps are

$$
\begin{equation*}
\chi_{n}(\theta)=e^{i n \theta}, \quad\left(\chi_{m}, \chi_{m}\right)=\mathcal{M}\left(\chi_{m}^{*} \chi_{n}\right)=\frac{1}{2 \pi} \int d \theta e^{-i m \theta} e^{i n \theta}=\delta_{m n} \tag{4.57}
\end{equation*}
$$

Hence the situation is similar to what we found for the representations of finite groups:
Theorem: Every class function $f(\theta)$ is a linear combination of the unitary-orthogonal characters:

$$
\begin{equation*}
f(\theta)=\sum_{n} c_{n} \chi_{n}(\theta), \quad c_{n}=\frac{1}{2 \pi} \int e^{-i n \theta} f(\theta)=\left(\chi_{n}, f\right) \tag{4.58}
\end{equation*}
$$

### 4.6.2 Irredubile Representations of $\mathrm{SU}(2)$

For a non-Abelian Lie group, not all irreps are one-dimensional. For $U \in S U(2)$, The characters of the trivial irrep and the defining irrep are:

$$
\begin{equation*}
\chi_{1}(U)=1 \quad \chi_{2}(U)=2 \cos (\theta)=e^{i \theta}+e^{-i \theta} \tag{4.59}
\end{equation*}
$$

The scalar products can be computed via the so-called reduced Haar measure:

$$
\begin{array}{cl}
\int d \mu_{r e d} \cos ^{2}(\theta)=\frac{1}{4} & \int d \mu_{\mathrm{red}} \cos (\theta)=0
\end{array} d \mu_{\mathrm{red}}=\frac{2}{\pi} \sin ^{2}(\theta) d \theta
$$

In general,

$$
\begin{equation*}
\int d \mu_{r e d} \cos ^{2 p}(\theta)=\frac{1}{4^{p}} \frac{(2 p)!}{p!(p+1)!} \tag{4.62}
\end{equation*}
$$

A three-dimensional representation can be obtained as follows: consider

$$
\begin{equation*}
D=D_{2} \otimes D_{2} \quad \text { with } \quad \chi_{D}(\theta)=\chi_{2}(\theta) \chi_{2}(\theta)=4 \cos ^{2}(\theta) \tag{4.63}
\end{equation*}
$$

Since $\left(\chi_{1}, \chi_{D}\right)=1$ and $\left(\chi_{2}, \chi_{D}\right)=0$, the trivial representation is present once, and the twodimensional representation is not contained. Hence

$$
\begin{equation*}
D_{2} \otimes D_{2}=D_{1} \oplus D_{3}, \quad \chi_{3}(\theta)=e^{2 i \theta}+1+e^{-2 i \theta}, \quad\left(\chi_{3}, \chi_{3}\right)=1 \tag{4.64}
\end{equation*}
$$

hence $D_{3}$ is irreducible with $\operatorname{dim}\left(D_{3}\right)=\chi_{3}(e)=3$.
Theorem: The three-dimensional representation $D_{3}$ of $\mathrm{SU}(2)$ is the matrix group $\mathrm{SO}(3)$.

Proof: $\quad D_{2} \otimes D_{2}$ acts on the tensor vector space $\mathbb{C}^{2} \otimes \mathbb{C}^{2}$ as follows:

$$
\left(\begin{array}{l}
x_{1} y_{1}  \tag{4.65}\\
x_{1} y_{2} \\
x_{2} y_{1} \\
x_{2} y_{2}
\end{array}\right) \mapsto\left(\begin{array}{cccc}
a^{2} & a b & b a & b^{2} \\
-a \bar{b} & a \bar{a} & -b \bar{b} & b \bar{a} \\
-\bar{b} a & -\bar{b} b & \bar{a} a & \bar{a} b \\
\bar{b}^{2} & -\bar{b} \bar{a} & -\bar{a} \bar{b} & \bar{a}^{2}
\end{array}\right) \cdot\left(\begin{array}{l}
x_{1} y_{1} \\
x_{1} y_{2} \\
x_{2} y_{1} \\
x_{2} y_{2}
\end{array}\right)
$$

The following basis transformation into a anti-symmetric singlet and symmetric triplet decouples the tensor representation in its irreducible subspaces:

$$
\begin{equation*}
Y_{0,0}=\frac{1}{2}\left(x_{1} y_{2}-x_{2} y_{1}\right), \quad\left(Y_{1,1}, Y_{1,0}, Y_{1,-1}\right)=\left(x_{1}, y_{1}, \frac{1}{\sqrt{2}\left(x_{1} y_{2}+x_{2} y_{1}\right)}, x_{2} y_{2}\right) \tag{4.66}
\end{equation*}
$$

In this basis, the transformation is as follows:

$$
\left(\begin{array}{c}
Y_{0,0}  \tag{4.67}\\
Y_{1,1} \\
Y_{1,0} \\
Y_{1,-1}
\end{array}\right) \mapsto\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & a^{2} & \sqrt{2} a b & b^{2} \\
0 & -\sqrt{2} a \bar{b} & a \bar{a}-b \bar{b} & \sqrt{2} b \bar{a} \\
0 & \bar{b}^{2} & -\sqrt{2} \bar{a} \bar{a} \bar{b} & \bar{a}^{2}
\end{array}\right)\left(\begin{array}{c}
Y_{0,0} \\
Y_{1,1} \\
Y_{1,0} \\
Y_{1,-1}
\end{array}\right)
$$

. With the additional modification

$$
\begin{equation*}
\left(X_{1}, X_{2}, X_{3}\right)=\left(\frac{1}{\sqrt{(2)}}\left(Y_{1,1}-Y_{1,-1}\right), \frac{1}{i \sqrt{(2)}}\left(Y_{1,1}-Y_{1,-1}\right), Y_{1,0}\right) \tag{4.68}
\end{equation*}
$$

we can identify $\mathrm{SO}(3)$ transformation as follow:

$$
\begin{equation*}
\vec{X} \mapsto(\vec{e}, \vec{X}) \vec{e}+\vec{e} \times \vec{X} \sin (2 \theta)-\vec{e} \times(\vec{e} \times \vec{X}) \cos (2 \theta)=R(2 \theta, \vec{e}) \vec{X} \tag{4.69}
\end{equation*}
$$

where $2 \theta$ is the angle that rotates three-dimensional vectors $\vec{X}$ around the axis $\vec{e}$.

## Remarks:

- Note that the 3-dim irrep $U \mapsto R(U) \in \mathrm{SO}(3)$ is not faithful since $R(U)=R(-U)$. We have found this result already in Sec. 2.5.2.


### 4.6.3 Irreducible Representations of SU(3)

Every $\mathrm{SU}(3)$ matrix is similar (in the same conjugacy class) to a diagonal matrix

$$
U=\left(\begin{array}{ccc}
e^{i \theta_{1}} & 0 & 0  \tag{4.70}\\
0 & e^{i \theta_{2}} & 0 \\
0 & 0 & e^{-i\left(\theta_{1}+\theta_{2}\right)}
\end{array}\right)
$$

with $\theta_{1}, \theta_{2} \in[0,2 \pi]$. The defining representation denoted by 3 is 3 -dimensional and

$$
\begin{align*}
\chi_{3}\left(\theta_{1}, \theta_{2}\right) & =e^{i \theta_{1}}+e^{i \theta_{2}}+e^{-i\left(\theta_{1}+\theta_{2}\right)}  \tag{4.71}\\
\chi_{3}^{*} \chi_{3} & =3+2 \cos \left(\theta_{1}-\theta_{2}\right)+2 \cos \left(2 \theta_{1}-\theta_{2}\right)+2 \cos \left(\theta_{1}+2 \theta_{2}\right)  \tag{4.72}\\
d \mu_{\text {red }} & =\frac{8}{3 \pi^{2}} \sin ^{2}\left(\frac{\theta_{1}-\theta_{2}}{2}\right) \sin ^{2}\left(\frac{2 \theta_{1}+\theta_{2}}{2}\right) \sin ^{2}\left(\frac{\theta_{1}+2 \theta_{2}}{2}\right) d \theta_{1} d \theta_{2}  \tag{4.73}\\
\left(\chi_{3}, \chi_{3}\right) & =\int d \mu_{\text {red }}\left(\theta_{1}, \theta_{2}\right)\left|\chi_{3}\left(\theta_{1}, \theta_{2}\right)\right|^{2}=1 \tag{4.74}
\end{align*}
$$

hence it is irreducible.

## Remarks:

- The complex conjugate representation $U \mapsto U^{*}$ is denoted by $\overline{3}$ is not equivalent to the representation 3 , since $\chi_{\overline{3}}=\chi_{3}^{*} \neq c h i_{3}$.
- It is also irreducible since $\left|\chi_{\overline{3}}\right|^{2}=\left|\chi_{3}\right|^{2}$.
- The tensor representation $3 \otimes \overline{3}$ is reducible: The trivial representation occurs once since $\left(\chi_{3 \otimes \overline{3}}, \chi_{1}\right)=1$. The other irreducible representation is the 8 -dimensional adjoint representation:

$$
\begin{equation*}
\left(\chi_{8}, \chi_{8}\right)=\left(\chi_{3}^{2}, \chi_{3}^{2}\right)-2\left(\chi_{3}, \chi_{3}\right)+\left(\chi_{1}, \chi_{1}\right) \quad \text { with } \quad\left(\chi_{3}^{2}, \chi_{3}^{2}\right)=2 \tag{4.75}
\end{equation*}
$$

Hence $3 \otimes 3=1 \oplus 8$.

- Likewise, $3 \otimes 3=6 \oplus \overline{3}$ and $3 \otimes 3 \otimes 3=10 \oplus 8 \oplus 8 \oplus 1$


## Chapter 5

## Applications of Representation Theory in Physics

### 5.1 Quantum Chromodynamics

In QCD, as gauge theory with $\mathrm{SU}(3)$ gauge group, the quarks transform under the fundamental irrep 3 , the anti-quarks under the complex conjugate irrep $\overline{3}$, the gluons under the adjoint representation 8 , the mesons and baryons under the singlet representation.

The multiplet structure of hadrons is obtained from the representation theory of the flavor group $U_{L}(3) \times U_{R}(3)$. The irreducible representations of

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[^0]:    ${ }^{1}$ Integrable systems (those that can be solved from the initial conditions) have so many symmetries, that there is no space for dynamics.

