Beisp:

1.) Oben: $(a+bx)' = b \Rightarrow x' \overset{a=0,\, b=1}{=} 1 \quad , \quad a' \overset{b=0}{=} 0$

$$\Rightarrow \underline{\underline{(a \cdot f(x))'}} = \underbrace{a'}_{0} \cdot f(x) + a f'(x) = \underline{\underline{a f'(x)}}$$

2.) $$(x^2)' = (\underset{\underset{f}{\uparrow}}{x} \cdot \underset{\underset{g}{\uparrow}}{x})' \underset{\underset{(ii)}{\uparrow}}{=} x' \cdot x + x \cdot x' = 1 \cdot x + x \cdot 1 = 2x$$

$$(x^3)' = (\underset{\underset{f}{\uparrow}}{x^2} \cdot \underset{\underset{g}{\uparrow}}{x}) = \underbrace{(x^2)'}_{2x} \cdot x + x^2 \cdot \underbrace{x'}_{1} = 3x^2$$

usw

$$\Rightarrow \underline{\underline{(x^n)' = n\, x^{n-1}}} \qquad \underline{\underline{\forall n \in \mathbb{N}}}$$

Dasselbe für alle $n \in \mathbb{Z}$: Übungen.

3.) $P(x) := \sum_{k=0}^{n} c_k x^k$ (<u>Polynom</u>)

$$\Rightarrow P'(x) \underset{\underset{(i)}{\uparrow}}{=} \sum_{k=0}^{n} (c_k x^k)' \underset{\underset{1.)}{\uparrow}}{=} \sum_{k=0}^{n} c_k k x^{k-1}$$

Kein Beitrag für $k=0$

$$\Rightarrow \underline{\underline{\left(\sum_{k=0}^{n} c_k x^k \right)' = \sum_{k=1}^{n} c_k \cdot k \cdot x^{k-1}}}$$

4.) $f(x) := \sum_{k=0}^{\infty} c_k x^k$ (<u>Potenzreihe</u>)

$\Rightarrow$ $f'(x) = \sum_{k=1}^{\infty} c_k \, k \, x^{k-1}$ („<u>man darf gliedweise differenzieren</u>")

$\uparrow$
wie in 3.)

[genauer: Analysis]

<u>Wichtigstes Beisp:</u> $\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$\Rightarrow$ $\exp'(x) = \sum_{k=1}^{\infty} \underbrace{\frac{1}{k!} \, k}_{\frac{k}{1\cdot 2 \cdot \ldots \cdot (k-1)\cdot k} = \frac{1}{(k-1)!}} \cdot x^{k-1} \overset{j := k-1}{=} \sum_{j=0}^{\infty} \frac{1}{j!} x^j = \exp(x)$

$$\boxed{\exp'(x) = \exp(x)} \quad \forall x \in \mathbb{R}$$

$\Leftrightarrow$ $\underline{\underline{(e^x)' = e^x}}$ $\forall x \in \mathbb{R}$

8.3 Mittelbare Funktionen

Eine Fkt. $h(x)$ der Form

$$\underline{\underline{h(x) = f(g(x))}}$$

heisst mittelbare Fkt. ($\hat{=}$ Verkettung/Verknüpfung von f und g).

Andere Schreibweise: $\underline{\underline{h = f \circ g}}$ bzw $h(x) = (f \circ g)(x)$

Beisp:

- $f(x) := x^2$, $g(x) := a + bx \Rightarrow f(g(x)) = (g(x))^2 = (a+bx)^2$

- $f(x) := \sin x$, $g(x) := x^2 \Rightarrow f(g(x)) = \sin(x^2)$

  $g(f(x)) = (\sin x)^2 = \sin^2 x$

- $g(x) := f^{-1}(x) \Rightarrow f(g(x)) = f(f^{-1}(x)) = x$

  $g(f(x)) = f^{-1}(f(x)) = x$

- $f(x) := e^x$, $g(x) := \cos x \Rightarrow f(g(x)) = e^{\cos x} = \sum_{k=0}^{\infty} \frac{1}{k!} (\cos x)^k$

<u>Kettenregel</u>:

$$\underline{\underline{\left[f(g(x))\right]' = \underbrace{f'(g(x))}_{\text{„äussere Abl."}} \cdot \underbrace{g'(x)}_{\text{„nachdifferenzieren"/„innere Abl."}}}}$$

Bew: [genauer: Analysis]

$$\left[f(g(x))\right]' = \lim_{\tilde{x}\to x} \frac{f(g(\tilde{x}))-f(g(x))}{\tilde{x}-x} \cdot \underbrace{\frac{g(\tilde{x})-g(x)}{g(\tilde{x})-g(x)}}_{=1}$$

$$= \lim_{\tilde{x}\to x} \underbrace{\frac{f(g(\tilde{x}))-f(g(x))}{\underbrace{g(\tilde{x})}_{=:\tilde{g}}-\underbrace{g(x)}_{=:g}}}_{\to \lim\limits_{\tilde{g}\to g} \frac{f(\tilde{g})-f(g)}{\tilde{g}-g} = f'(g) = f'(g(x))} \cdot \underbrace{\frac{g(\tilde{x})-g(x)}{\tilde{x}-x}}_{\to g'(x)} = f'(g(x))\cdot g'(x)$$

Noch einfacher: $\dfrac{df(g(x))}{dx} = \dfrac{df(g(x))}{dg(x)}\,\dfrac{dg(x)}{dx}$ oder $\dfrac{df}{dx} = \dfrac{df}{dg}\cdot\dfrac{dg}{dx}$

Beisp:

1.) $\underline{\underline{(\sin(2x))'}} \overset{f(x)=\sin x,\; g(x)=2x}{=} f'(g(x)) \cdot g'(x) = \cos(2x)\cdot 2 = \underline{\underline{2\cos(2x)}}$

2.) Analog: $\underline{\underline{(f(bx))'}} \overset{g(x)=bx}{=} f'(bx)\cdot \overbrace{g'(x)}^{b} = \underline{\underline{b\cdot f'(bx)}}$

3.) $f(x) := x^n, \quad g(x) := a+bx$

$\Rightarrow f(g(x)) = (a+bx)^n$

$\Rightarrow \underline{\underline{\frac{d\,(a+bx)^n}{dx}}} = \underline{\underline{\underbrace{n(a+bx)^{n-1}}_{f'(g(x))} \cdot \underbrace{b}_{g'(x)}}}$

4.) $g(x) := f^{-1}(x) \Rightarrow f(g(x)) = f(f^{-1}(x)) = x$

$$\Rightarrow \underbrace{\Big(\underbrace{f(f^{-1}(x))}_{x}\Big)'}_{x' = 1} = f'(f^{-1}(x)) \cdot (f^{-1}(x))'$$

$$\Rightarrow (f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

5.) $f(x) := \exp(x) \Rightarrow f^{-1}(x) = \ln(x) \quad (x \in \mathbb{R}^+)$, $f'(x) = \exp(x)$

$$\Rightarrow (\ln x)' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\underbrace{\exp(\ln(x))}_{=x}} = \frac{1}{x}$$

$$\underline{\underline{\frac{d \ln(x)}{dx} = \frac{1}{x}}} \quad \forall x \in \mathbb{R}^+$$

8.4 Höhere Ableitungen

$$\frac{d^2 f(x)}{dx^2} = \frac{d^2}{dx^2} f(x) := \frac{d}{dx}\left(\frac{d}{dx} f(x)\right) \quad \text{usw}:$$

$$\frac{d^n f(x)}{dx^n} := \frac{d}{dx}\left(\frac{d^{n-1} f(x)}{dx^{n-1}}\right) \quad \text{„n-te Ableitung"}$$

Andere Schreibweisen:

$$f'(x),\ f''(x),\ f'''(x),\ f^{(4)}(x), \ldots, f^{(n)}(x) \quad (n \in \mathbb{N}_0)$$

bzw.

$$\dot{f}(t),\ \ddot{f}(t),\ \dddot{f}(t),\ f^{(n)}(t)$$

↑ ab $n=4$

Beisp:

1.) $\sin'(x) = \cos(x)$

$\sin''(x) = \cos'(x) = -\sin(x)$

$\sin'''(x) = -\sin'(x) = -\cos(x)$

$\sin^{(4)}(x) = -\cos'(x) = \sin(x)$ usw.

$$\sin^{(n+4)}(x) = \sin^{(n)}(x) \quad \forall\, n \in \mathbb{N}_0$$

ebenso für $\cos(x)$

2.) $f(t) := \ln t \Rightarrow \dot{f}(t) = \frac{1}{t}\,,\ \ddot{f}(t) = \frac{d}{dt}\left(\frac{1}{t}\right) = -\frac{1}{t^2}$

8.5 Ableitung vektorwertiger Funktionen

Vektorwertige Fkt'en aus Kap. 4:

$$\vec{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix} = \sum_{k=1}^{n} f_k(t)\, \vec{e}_k \quad , \; n \in \mathbb{N}$$

Def:

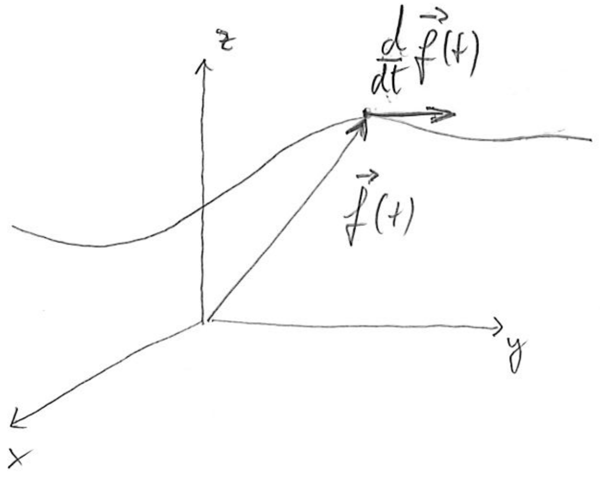
$$\frac{d}{dt} \vec{f}(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t} \underbrace{\left(\vec{f}(t+\Delta t) - \vec{f}(t) \right)}_{\sum_{k=1}^{n} \left(f_k(t+\Delta t) - f_k(t) \right) \vec{e}_k} =$$

$$= \lim_{\Delta t \to 0} \sum_{k=1}^{n} \underbrace{\frac{f_k(t+\Delta t) - f_k(t)}{\Delta t}}_{\to \dot{f}_k(t)} \vec{e}_k \quad \Leftrightarrow$$

$$\frac{d}{dt} \vec{f}(t) = \begin{pmatrix} \dot{f}_1(t) \\ \vdots \\ \dot{f}_n(t) \end{pmatrix} = \sum_{k=1}^{n} \dot{f}_k(t)\, \vec{e}_k$$

(„komponentenweise")

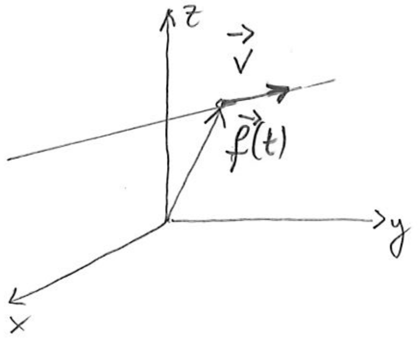
Veranschaulichung im $\mathbb{R}^3$:



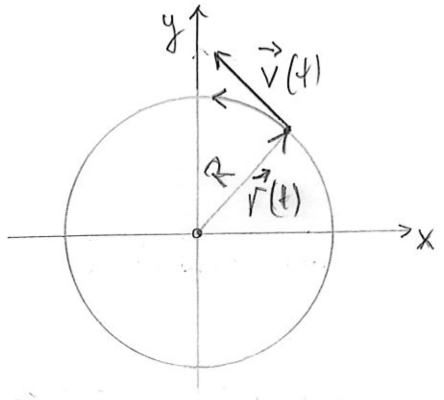
z.B. $\vec{f}(t)$ „Ort" $\Rightarrow$ $\frac{d}{dt}\vec{f}(t)$ „Geschw."

Beisp: [genau wie in Kap. 4]

1.) $\vec{f}(t) := \underset{\uparrow \text{ Startpkt.}}{\vec{r}_0} + t\,\vec{v} = \begin{pmatrix} r_{01} + t v_1 \\ r_{02} + t v_2 \\ r_{03} + t v_3 \end{pmatrix}$ (Ort)



$$\frac{d}{dt}\vec{f}(t) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{v}$$ (konstante Geschw.)



2.) $\underset{\substack{\uparrow \\ \text{„Ort"}}}{\vec{r}}(t) := \begin{pmatrix} R\cos(\omega t) \\ R\sin(\omega t) \end{pmatrix}$, $R, \omega > 0$

$$\Rightarrow \underline{\underline{|\vec{r}(t)|}} = \sqrt{R^2\cos^2(\omega t) + R^2\sin^2(\omega t)} = \sqrt{R^2} \underset{\substack{\uparrow \\ R>0}}{=} \underline{\underline{R}}$$

$$\Rightarrow \vec{v}(t) := \frac{d}{dt}\vec{r}(t)$$ („Geschw.")

$$= \begin{pmatrix} R\,(-\sin(\omega t)\cdot\omega) \\ R\,(\cos(\omega t)\cdot\omega) \end{pmatrix} = R\omega \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \end{pmatrix}$$

$$\Rightarrow \underline{\underline{|\vec{v}(t)|}} = \sqrt{R^2\omega^2\sin^2(\omega t) + R^2\omega^2\cos^2(\omega t)} = \underline{\underline{R\omega}} = \omega\cdot|\vec{r}(t)| \quad (\omega \geq 0)$$

$$\Rightarrow \quad \vec{r}(t)\cdot\vec{v}(t) = R\begin{pmatrix}\cos(\omega t)\\ \sin(\omega t)\end{pmatrix}\cdot R\omega\begin{pmatrix}-\sin(\omega t)\\ \cos(\omega t)\end{pmatrix}$$

$$= R^2\omega\Big(\cos(\omega t)(-\sin(\omega t)) + \sin(\omega t)\cos(\omega t)\Big) = 0$$

$$\Rightarrow \quad \underline{\vec{v}(t)\perp\vec{r}(t)} \ \text{(senkrecht)} \ \forall t \ !$$

$$\Rightarrow \quad \underline{\vec{a}(t)} := \frac{d}{dt}\vec{v}(t) = \frac{d^2}{dt^2}\vec{r}(t) \qquad (\text{„Beschl."})$$

$$= R\omega\begin{pmatrix}-\cos(\omega t)\cdot\omega\\ -\sin(\omega t)\cdot\omega\end{pmatrix} = \underline{-\omega^2\vec{r}(t)} \qquad \Rightarrow \underline{\vec{a}(t)\perp\vec{v}(t)}$$

$$\Rightarrow \underline{|\vec{a}(t)|} = \omega^2|\vec{r}(t)| = \underline{\omega^2 R} = \omega|\vec{v}(t)| = \underline{|\vec{v}(t)|^2/R}$$

<u>Genauso im $\mathbb{R}^3$:</u> $\quad \vec{r}(t) := \begin{pmatrix}R\cos(\omega t)\\ R\sin(\omega t)\\ 0\end{pmatrix}$

$$\Rightarrow \quad \vec{v}(t)\perp\vec{r}(t)$$

$$\vec{a}(t) = -\omega^2\vec{r}(t) \perp \vec{v}(t) \qquad \text{usw.}$$

Vgl. EP 1 (Th. Huser)