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# Stochastic aspects of climatic transitions—response to a periodic forcing

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ABSTRACT

The time-dependent properties of the Fokker–Planck equation corresponding to a zero-dimensional climate model, showing bistable behavior and subject to a weak external periodic forcing are analyzed. Conditions under which the response is amplified are found analytically. In this way the possibility of transitions between climatic states is established. The results are illustrated by the 100,000-yr periodicity of the eccentricity of the earth's orbit, in connection with glaciation cycles.

## 1. Introduction

In a previous paper (Nicolis and Nicolis, 1981, hereafter referred to as I), a nonlinear theory of climatic fluctuations has been developed. The starting point was to incorporate in the climate dynamics the effect of random imbalances between the various transport and radiative mechanisms. The usual, deterministic rate equations (such as the equation of energy conservation) were thus replaced by stochastic differential equations. Under the assumption of a Gaussian white noise, the latter were equivalent to a Fokker–Planck equation for the underlying probability density.

In I, the steady-state solutions of the Fokker–Planck equation have been analyzed in detail for a simple zero-dimensional model involving two stable climatic states separated by an unstable one. It was shown that the basic properties of the probability distribution are monitored by a quantity which was called the *climatic potential*, playing in the theory a role analogous to that of free energy in thermodynamics. The minima of this potential give the positions of the stable climatic states. Under certain conditions on the parameter values the depth of the minima could become equal, and as a result the stable states be equally dominant. This situation was referred to as the *climatic coexistence*.

The time-dependent behavior of the fluctuations turned out to be much more involved. Still, some results were obtained in paper I using the ideas of Kramers' theory of passage over a potential barrier, and confirmed by numerical simulations. The most striking of these results concerns the characteristic passage time between the two stable climatic states, which turned out to be

$$\tau \sim \exp\left(\frac{2}{q^2}\Delta U\right) \tag{1.1}$$

Here $q^2$ is the variance of the fluctuations and $\Delta U$ the height of the barrier—essentially the difference of the values of the climatic potential between the unstable and one of the stable states. The point is that if, as usually, fluctuations are small with respect to the magnitude of the barrier, $\tau$ is a long time scale of the order of $10^3$ years or more. Such scales are absent from the deterministic energy balance equations, which typically predict relaxation times of the order of the year.

The purpose of the present paper is to examine the consequences of the existence of a long time scale, eq. (1.1), in climate dynamics. It is well known that the glaciation cycles, which are certainly the most dramatic episodes of the quaternary era, have a dominant periodicity of 100,000 yrs. This time scale coincides with the period of



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variation of the eccentricity of the earth's orbit (Berger, 1978). Despite many efforts, however, the response of simple energy-balance models to a weak external signal having this periodicity turned out to be very weak and hence incapable of triggering a major climatic change. It is the purpose of our work to show that the situation may be completely different when the coupling between the external forcing and the internal fluctuations of the climatic system is considered explicitly.

In Section 2 we introduce a simple zero-dimensional climate model having two stable states separated by an unstable one, and summarize the properties of the deterministic response to a periodic variation of incoming solar energy. In Section 3 the stochastic description is set up. Sections 4 and 5 are devoted respectively, to the analytic and numerical results of the response to the periodic variation. We show that the response is considerably amplified when a matching between the characteristic time scale of fluctuations (eq. (1.1)) and the periodicity of the incoming solar energy occurs. A preliminary account of this result has been given in a recent communication by the author (Nicolis, 1980a).

## 2. A simple zero-dimensional model subject to a periodic forcing

In much of this paper we shall be concerned with a set of climatic variables $\bar{x}$ which in the absence of fluctuations, obey to a closed equation of evolution of the form:

$$\frac{d\bar{x}}{dt} = f(\bar{x}, \lambda, t) \equiv f_0(\bar{x}, \lambda) + \varepsilon f_1(\bar{x}, \lambda, t) \qquad (2.1)$$

Here $f$ is an appropriate nonlinear rate function, and $\lambda$ stands for a set of characteristic parameters such as albedo, emissivity and so forth. This function is decomposed into a part $f_0$ corresponding to an autonomous evolution, and to a time-dependent part $f_1$ describing the effect of some external forcing proportional to $\varepsilon$. As in paper I, of special interest for our work are cases where the steady-state solutions of the system in the absence of the above time-dependent forcing,

$$f_0(\bar{x}_s, \lambda) = 0 \qquad (2.2)$$

are multiple and see their stability properties change as the parameters $\lambda$ take different values.

More specifically, suppose that $\bar{x}$ denotes the average surface temperature. The rate function $f$ in eq. (2.1) is then the difference between the solar influx $Q(1 - a(\bar{x}))$ [$a$ being the albedo] and the infra-red cooling rate, $\varepsilon_B \sigma \bar{x}^4$, [$\varepsilon_B$ being the emissivity and $\sigma$ the Stefan constant]. Equation (2.1) becomes:

$$\frac{d\bar{x}}{dt} = \frac{1}{C}[Q(1 - a(\bar{x})) - \varepsilon_B \sigma \bar{x}^4] \qquad (2.3)$$

where $C$ is the thermal inertia coefficient.

In the majority of climate models $Q$ is taken to be constant. On the other hand, it is known that the solar output displays very pronounced variability at different time scales. One example is the sunspot cycle which despite an inherent noise, shows an approximate 11-year periodicity. Another example more significant for our purposes is the slight change in the mean annual influx arising from the variation of the eccentricity of the earth's orbit. Hereafter we are interested in the effect of such time-dependent forcing, in the presence of fluctuations. To simplify the analysis as much as possible we describe the above-mentioned nearly periodic variation in the form

$$Q = Q_0(1 + \varepsilon \sin \omega t) \qquad (2.4)$$

The unperturbed solar constant divided by 4 is taken to be $Q_0 = 340$ W m$^{-2}$.

For temperature values $T$ near the present-day climate, $a(\bar{x})$ is usually taken to be a roughly linear function of its argument (Cess, 1976; Nicolis, 1980b). On the other hand, for very low $\bar{x}$, $a$ must tend to the albedo of ice, $a_{ice}$ whereas for high $\bar{x}$, $a$ should also saturate to some value, $a_{hot}$ descriptive of an ice-free earth. The simplest representation taking these features into account is the zero-dimensional (0-d) piecewise linear model proposed by Crafoord and Källén (1978). Analytically, we write:

$$\begin{aligned} &1 - a(\bar{x}) = 1 - a_{ice} = \gamma_1, && \bar{x} < T_1 \\ &1 - a(\bar{x}) = 1 - \alpha + \beta\bar{x} = \gamma_0 + \beta\bar{x}, && T_1 < \bar{x} < T_2 \\ &1 - a(\bar{x}) = 1 - a_{hot} = \gamma_2, && \bar{x} > T_2 \end{aligned} \qquad (2.5)$$

Using the explicit dependence of the albedo on $T$ as given by eqs. (2.5) in eq. (2.3) we see that in the absence of periodic forcing and for appropriate values of the parameters $\gamma_0$, $\gamma_1$, $\gamma_2$ and $\beta$ the system may admit *three* steady-state solutions. One of them, denoted hereafter by $T_+$, corresponds to the present-day climate and is asymptotically stable, provided the parameters $\gamma_0$ and $\beta$ are chosen in such a way that the planetary albedo is 0.30 and the emissivity is $\varepsilon_B = 0.61$. The second solution, denoted by $T_-$, corresponds to a deep-freeze climate and is also asymptotically stable. A third solution $T_0$ lies between $T_+$ and $T_-$ and is unstable.

Before we analyze the stochastic properties of the system defined by eqs. (2.3) to (2.5) we briefly review the main features of the deterministic response. We first write the energy balance equation in the form:

$$\frac{d\bar{x}}{dt} = \frac{1}{C}\{Q_0(1 - a(\bar{x})) - \varepsilon_B \sigma \bar{x}^4\}$$
$$+ \frac{1}{C} Q_0 \varepsilon (1 - a(\bar{x})) \sin \omega t = -U_0'(\bar{x})$$
$$+ \frac{1}{C} \varepsilon (1 - a(\bar{x}) \sin \omega t \tag{2.6}$$

where $U_0'$ denotes the derivative of the climatic potential introduced in paper I with respect to its argument:

$$U_0(x) = -\int f_0(x,\lambda)\,dx \tag{2.7}$$

As a rule, $\varepsilon$ is small. Hence, to a good approximation one may linearize the above equation around the stable states $T_+$ and $T_-$. Setting

$$\bar{x}_\pm = T_\pm + \delta T_\pm \tag{2.8}$$

we obtain:

$$\frac{d\delta T_\pm}{dt} = -U_0''(T_\pm)\,\delta T_\pm + \frac{1}{C} Q_0\, \varepsilon (1 - a - T_\pm)) \sin \omega t \tag{2.9}$$

In the limit of long times the response around the present-day climate predicted by eq. (2.9) is easily seen to be of the form:

$$\delta T_+(t) = \frac{1}{C} \frac{1 - a_+}{\omega^2 + (U_0''(T_+))^2}$$
$$Q_0 \frac{\varepsilon U_0''(T_+)}{\cos \theta} \sin(\omega t + \theta) \tag{2.10a}$$

where the signal-response phase shift is given by

$$tg\theta = -\frac{\omega}{U_0''(T_+)} \tag{2.10b}$$

From this expression we see that if $\varepsilon$ is small the amplitude of the response is negligible. For instance, for the model considered in this Section with the usually accepted values for $C$ and $\beta$ and for $\varepsilon = 0.001$, which is the estimated change of solar influx arising from the eccentricity variation of the earth's orbit (Imbrie and Imbrie, 1980) one finds an upper bound for $\delta T_+$ of the order of 0.1 °K. Moreover if $\omega \ll U_{0+}''$ (that is, if the periodicity of the forcing is very long), the phase shift practically vanishes and the amplitude of the response is independent of the thermal inertia coefficient $C$. As we see later, these conclusions change radically when fluctuations are taken into account.

## 3. Stochastic description

As discussed in the Introduction, the deterministic description must often be extended to take into account the fluctuations, associated with random imbalances between the various transport and radiative mechanisms involved in the rate function $f(\bar{x}, \lambda, t)$. We denote their effect by a random force $F(t)$ and assume the latter to be $x$-independent and define a *white noise* (Wax, 1954):

$$\langle F(t) \rangle = 0$$
$$\langle F(t)\, F(t') \rangle = q^2\, \delta(t - t') \tag{3.1}$$

Here $\langle\ \rangle$ denotes the expectation operator over the ensemble of possible realizations.

Equation (2.1) is now to be replaced by the stochastic differential equation

$$\frac{dx}{dt} = f(x, \lambda, t) + F(t) \tag{3.2}$$

As is well known (see e.g. Arnold, 1973) eqs. (3.1) and (3.2) are equivalent to the following Fokker–Planck equation with nonlinear friction coefficient and constant diffusion coefficient:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x} f(x,\lambda t) P(x,t) + \frac{q^2}{2} \frac{\partial^2 P(x,t)}{\partial x^2} \tag{3.3}$$

where $P(x,t)$ is the probability density for having the value $x$ of the state variable at time $t$.

As shown in paper I, in the absence of time-dependent forcing,

$$f = f_0(x, \lambda)$$

eq. (3.3) has a stationary solution in the form of a two humped distribution (see Fig. 1)

$$P_s(x) = Z_0^{-1} \exp\left[-\frac{2}{q^2} U_0(x)\right] \tag{3.4}$$

$Z_0$ being the normalization factor. In the presence of forcing the above procedure is no longer applicable. Nevertheless, if the external periodicity is very long with respect to the characteristic relaxation time $(U''_{0\pm})^{-1}$, one expects that a *quasi-steady state regime* will be established in which the system would adapt at each moment to the instantaneous state of the external environment. Specifically, let us define a time-dependent potential

$$U(x, t) = -\int_x f(x, \lambda, t)\, dx \tag{3.5}$$

and the associated probability distribution

$$P_s(x, t) = Z^{-1}(t) \exp\left[-\frac{2}{q^2} U(x, t)\right] \tag{3.6}$$

This function cancels identically the right-hand side of eq. (3.3), but in the left-hand side it gives terms proportional to $\partial U/\partial t$ or, according to eqs. (2.1) and (2.4), terms of the order of the frequency $\omega$ of the external forcing. If $\omega$ is small, it is sensible to assume that on the time scale of interest, the maxima of $P$ will have relaxed to the values $x_\pm$ given by the deterministic description of Section 2, which are the minima of $U$. Moreover, because of the smallness of the deterministic response $\delta T_\pm$ (see eq. 2.10a), we may assume that these extrema remain fixed, and are essentially identical to the values $T_\pm$ which correspond to the steady-state solution in the absence of the forcing (see Fig. 1). In short, we expect that the exact probability $P$ will have properties similar to $P_s(x,t)$ as far as the location of the most probable states is concerned.

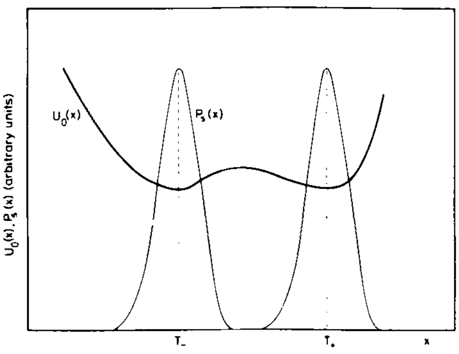


*Fig. 1.* A typical two-humped probability distribution in the case of coexisting climatic states.

On the other hand, as shown in paper I, in addition to the evolution of the extrema, there is a slow interpeak relaxation process associated with the adjustment of the probability mass around the extrema. As its rate may be comparable to the above estimated rate of change of $P_s(x,t)$, it is essential to incorporate it into the description. We do this by applying, as in paper I, Kramers' theory of diffusion over a potential barrier (Wax, 1954). Actually because of the time-dependence of the friction coefficient in eq. (3.3) we need a generalization of Kramers' theory, and this is most conveniently carried out using a recent reformulation of this theory due to Gardiner (1980).

Let $M(x,t)$, denote the total probability mass from zero up to some value of $x$ of the state variable:

$$M(x, t) = \int_0^x P(x', t)\, dx' \tag{3.7}$$

Of particular interest are the values of this quantity associated with the domains of attraction of the two maxima of $P$:

$$N_-(t) = M(T_0, t)$$
$$N_+(t) = 1 - N_-(t) = 1 - M(T_0, t) \tag{3.8}$$

We also introduce the corresponding expressions for the quasi-stationary distribution $P_s(x,t)$:

$$n_-(t) = 1 - n_+(t) = \int_0^{T_0} P_s(x', t)\, dx' \tag{3.9}$$

We now formulate the main

*Assumption*: In the whole range of values of $x$, the $x$-dependence of $P(x,t)$ is taken to be proportional to $P_s(x,t)$. To ensure proper normalization this implies that:

$$P(x,t) \cong N_-(t)\frac{1}{n_-(t)}P_s(x,t) \qquad x < T_0$$

$$P(x,t) \cong N_0(t)\frac{1}{n_0(t)}P_s(x,t) \cong 0 \qquad x \cong T_0$$

$$P(x,t) \cong N_+(t)\frac{1}{n_+(t)}P_s(x,t) \qquad x > T_0 \qquad (3.10)$$

The problem of solving the Fokker–Planck equation amounts now to finding an equation for either of the two *weight functions* $N_-(t)$ or $N_+(t)$, which reflect the relative importance of the two stable states. To this end we differentiate eq. (3.7) with respect to time and substitute the time derivative of $P$ from the Fokker–Planck equation:

$$\dot{M} \equiv \frac{dM}{dt} = \int_0^x \frac{\partial}{\partial x}\left[U'(x,t)\,P(x,t) + \frac{q^2}{2}\frac{\partial P(x,t)}{\partial t}\right]dx$$

$$= U'(x,t)\,P(x,t) + \frac{q^2}{2}\frac{\partial P(x;t)}{\partial x} \qquad (3.11)$$

where we assumed that the probability flux is zero at the boundary $x = 0$ (see paper I for a discussion of this point). Taking into account the explicit form of $P_s(x,t)$, eq. (3.6), we write eq. (3.11) in the equivalent form

$$\dot{M} = \frac{q^2}{2}P_s(x,t)\frac{\partial}{\partial x}\frac{P(x,t)}{P_s(x,t)} \qquad (3.12)$$

from which we get

$$\int_{T_-}^{T_0} dx\,\frac{\dot{M}(x,t)}{P(x,t)} = \frac{q^2}{2}\left[\frac{P(T_0,t)}{P_s(T_0,t)} - \frac{P(T_-,t)}{P_s(T_-,t)}\right] \qquad (3.13)$$

According to eq. (3.6) and Fig. 1, the function $P_s(x,t)$ is sharply peaked on the two minima $T_-$ and $T_+$ of the potential $U$ and is practically vanishing at $T_0$. Conversely, $P_s^{-1}(x,t)$ presents a very sharp maximum at $x = T_0$. Thus, for all practical purposes only the value of $\dot{M}(T_0,t)$ matters in eq. (3.13). Now, from definition (3.7) and our main Assumption,

$$M(T_0,t) = N_-(t)\frac{1}{n_-(t)}\int_0^{T_0} P_s(x',t)\,dx' = N_-(t) \qquad (3.14)$$

Consequently, eq. (3.13) becomes, after utilizing once again our main Assumption:

$$\dot{N}_-(t)\int_{T_-}^{T_0} P_s^{-1}(x',t)\,dx' = \frac{q^2}{2}\left[\frac{N_0(t)}{n_0(t)} - \frac{N_-(t)}{n_-(t)}\right] \qquad (3.15)$$

A similar procedure leads to an equation relating $\dot{N}_+$, $N_0$ and $N_+$. Summarizing:

$$\dot{N}_-(t) = \lambda_0(t)\,N_0(t) - \lambda_-\,N_-(t) \qquad (3.16a)$$

$$\dot{N}_+(t) = \lambda_0'\,N_0(t) - \lambda_+\,N_+(t) \qquad (3.16b)$$

where we have set

$$\lambda_0 = \frac{q^2}{2}\frac{1}{n_0}\left(\int_{T_-}^{T_0} P_s^{-1}(x')\,dx'\right)^{-1}$$

$$\lambda_- = \frac{q^2}{2}\frac{1}{n_-}\left(\int_{T_-}^{T_0} P_s^{-1}(x')\,dx'\right)^{-1}$$

$$\lambda_0' = \frac{q^2}{2}\frac{1}{n_0}\left(\int_{T_0}^{T_+} P_s^{-1}(x')\,dx'\right)^{-1}$$

$$\lambda_+ = \frac{q^2}{2}\frac{1}{n_+}\left(\int_{T_0}^{T_+} P_s^{-1}(x')\,dx'\right)^{-1} \qquad (3.17)$$

From now on the time dependence of $n$ and $P_s$ will not be indicated explicitly.

Summing the two relations (3.16) and taking into account that $N_+ + N_- = 1$, we obtain

$$N_0 = \frac{\lambda_- N_- + \lambda_+ N_+}{\lambda_0 + \lambda_0'} \qquad (3.18)$$

Substituting back into eq. (3.16a) we obtain a closed equation for the weight $N_-(t)$ of the probability function around $x = T_-$:

$$\dot{N}_- = r_+ - (r_- + r_+)\,N_- \qquad (3.19)$$

with

$$r_- = \frac{\lambda_0' \lambda_-}{\lambda_0 + \lambda_0'}$$

$$r_+ = \frac{\lambda_0 \lambda_+}{\lambda_0 + \lambda_0'} \tag{3.20}$$

Both $r_-$ and $r_+$ can easily be evaluated asymptotically, by expanding $P_s^{-1}$ in eq. (3.17) around its unique maximum at $T_0$ and by computing the integral using the steepest descent method (seel also paper I). One finds in this way:

$$r_- = \frac{1}{2\pi} |-U''(T_0)\, U''(T_-)|^{1/2} \exp\left[-\frac{2}{q^2}\Delta U_-\right]$$

$$r_+ = \frac{1}{2\pi} |-U''(T_0)\, U''(T_+)|^{1/2} \exp\left[-\frac{2}{q^2}\Delta U_+\right] \tag{3.21}$$

where

$$\Delta U_\pm = U(T_0, t) - U(T_\pm, t) \tag{3.22}$$

is the instantaneous value of the potential barrier separating the stable states from the unstable one.

## 4. Stochastic response to a weak periodic forcing

We now evaluate the solution $N_-(t)$ of eq. (3.19) —which will automatically give to us the $N_+(t)$ as well—in the case in which the system is submitted to the weak periodic forcing described in Section 2. This means that the potential $U(x,t)$ is to be split in a way similar to eq. (2.1)

$$U(x,t) = U_0(x) + \varepsilon U_1(x,t) \tag{4.1}$$

If $\varepsilon$ is a small quantity, the coefficients $r_-$ and $r_+$ appearing in eq. (3.18) could be linearized around their values corresponding to the absence of forcing:

$$r_\pm = r_{0\pm} + \varepsilon \rho_\pm \sin\omega t \tag{4.2}$$

This allows us to seek for solutions of the form:

$$N_-(t) = \hat{N}_- \sin(\omega t + \phi) + N_{0-} \tag{4.3}$$

where $N_{0-}$ is the stationary solution in the absence of the forcing

$$N_{0-} = \frac{r_{0+}}{r_{0-} + r_{0+}} \tag{4.4}$$

Substituting eqs. (4.2)–(4.4) into eq. (3.19) and keeping only linear terms we find straightforwardly the amplitude of the response:

$$\hat{N}_- = \frac{1}{\left[1 + \left(\dfrac{\omega}{r_{0+} + r_{0-}}\right)^2\right]^{1/2}} \varepsilon \frac{\rho_+ - N_{0-}(\rho_+ + \rho_-)}{r_{0+} + r_{0-}} \tag{4.5}$$

and its phase shift:

$$\phi = -\mathrm{arc}tg \frac{\omega}{r_{0+} + r_{0-}} \tag{4.6}$$

Thus, the characteristics of the stochastic response are monitored by the quantity $\omega/(r_{0+} + r_{0-})$, which is the ratio of the two characteristic times of interest in this problem: The period of the external forcing, and the characteristic time of interpeak relaxation in the absence of forcing (cf. eq. (3.19)). This is in agreement with the qualitative arguments advanced at the beginning of Section 3.

Let us now evaluate more explicitly expressions (4.5) and (4.6). In the limit where the variance $q^2$ is small compared to the magnitude of the barrier $\Delta U_\pm$, the coefficients $\rho_\pm$ are given by

$$\rho_\pm = \frac{1}{\pi q^2} (-U_{00}'' U_{0\pm}'')^{1/2} \Delta U_{1\pm}$$

$$\exp\left[-\frac{2}{q^2}\Delta U_{0\pm}\right] \tag{4.7a}$$

where

$$U_{00}'' \equiv U_0''(T_0)$$

$$U_{0\pm}'' \equiv U_0''(T_\pm)$$

with (see eq. 4.1):

$$\begin{aligned} \Delta U_{0\pm} &= U_0(T_\pm) - U_0(T_0) \\ \Delta U_{1\pm} &= U_1(T_\pm) - U_1(T_0) \end{aligned} \tag{4.7b}$$

On the other hand from eqs. (3.21)

$$r_{0\pm} = \frac{1}{2\pi} (-U_{00}'' U_{0-}'')^{1/2} \exp\ -\frac{2}{q^2}\Delta U_{0\pm}] \tag{4.8}$$

and from eq. (4.4)

$$N_{0-} = \frac{(U''_{0+})^{1/2} \exp\left[-\frac{2}{q^2}\Delta U_{0+}\right]}{(U''_{0-})^{1/2} \exp\left[-\frac{2}{q^2}\Delta U_{0-}\right] + (U''_{0+})^{1/2} \exp\left[-\frac{2}{q^2}\Delta U_{0+}\right]} \tag{4.9}$$

Using these relations it is easy to see that if the values of the time-independent part of the climatic potential $U$, at the two stable states $T_+$ and $T_-$ are not equal, the amplitude $N_-$ of the response behaves as

$$\hat{N}_- \sim \exp\left[-\frac{2}{q^2}\,|U_{0+} - U_{0-}|\right] \tag{4.10}$$

and is therefore negligibly small if the variance $q^2$ is small, as it is expected to be. Therefore the only case where we may have a significant response is when

$$U_{0+} \sim U_{0-} \tag{4.11}$$

In paper I we referred to this situation as the *climatic coexistence* case. Expression (4.5) now becomes

$$\hat{N}_- = \frac{2}{q^2} \frac{1}{\left[1 + \frac{\omega}{r_{0+} + r_{0-}}^2\right]^{1/2}} \varepsilon\left\{\frac{(U''_{0-}\,U''_{0+})^{1/2}\,(\Delta U_{1+} - \Delta U_{1-})}{|(U''_{0-})^{1/2} + (U''_{0+})^{1/2}|^2}\right\} \tag{4.12}$$

One can easily check that $(U''_{0\pm})^{1/2}$ are typically of order unity and from eqs. (4.7b), the quantity inside the curly brackets turns out to be about $2 \times 10^3$ yr$^{-1}$ K$^2$. For a forcing amplitude of 0.001 corresponding to the eccentricity variation of the earth's orbit (Imbrie and Imbrie, 1980) we find therefore that the stochastic response $\hat{N}_-$ is crucially dependent on the magnitude of the $q$-dependent factors

$$\frac{2}{q^2}\left[1 + \left(\frac{\omega}{r_{0+} + r_{0-}}\right)^2\right]^{-1/2}$$

Notice the highly singular dependence of this part on $q$. As a matter of fact, we have two competing factors: $2/q^2$ which increases if $q$ is becoming small, and the inverse square root which, in view of eq. (4.8), decreases for a fixed $\omega$ if $q$ becomes small.

For usual values of $q^2$ and $\Delta U_\pm(r_{0+} + r_{0-})$ is a very small quantity. Therefore, if $\omega$ is of the order of 1 (such as the frequency associated with the 11- or 22-yr solar cycle), the inverse square root factor would be exceedingly small and the stochastic response to this type of forcing would be negligible.

The situation is completely different if $\omega$ and $(r_{0+} + r_{0-})$, in other words, the two inverse characteristic times of the problem, are of the same order of magnitude. With the value of $q$ for which this equality is achieved one finds that the amplitude $\hat{N}_-$ is of the order of 0.1, which is quite appreciable compared to the steady-state value $N_{0-} \sim 0.5$ one would obtain in the absence of forcing when the two states $T_+$ and $T_-$ are equally dominant. Everything happens as if the barrier that has to be overcome for a transition between $T_+$ and $T_-$ say (reminiscent of a glaciation), becomes significantly smaller for certain time intervals. The situation is represented in curve (a) of Figs. 2 and 3. It can be easily shown that the quasi-steady state of the probability distribution $P_s(x,t)$ is hardly affected by the forcing. The behavior of $P(x,t)$ is therefore entirely dominated by the two weight factors $N_-$ and $N_+$ (see eqs. (3.10)).

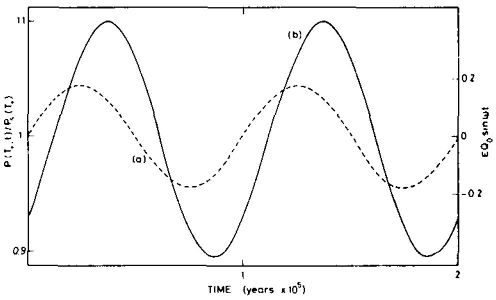


*Fig. 2.* Curve (a): Time dependence of the periodic forcing with frequency $\omega = 2\pi/10^5$ yr$^{-1}$ and an amplitude $\varepsilon = 0.001$ simulating the variation of the eccentricity of the earth's orbit. Curve (b): Time evolution of the probability of the stable state $P(T_+,t)$ divided by its value in the absence of forcing $P_s(T_+) \sim P_s(T_-)$, in the presence of the forcing represented in curve (a). Here and in Fig. 2 the time scale is normalized in such a way that $C = 1$.

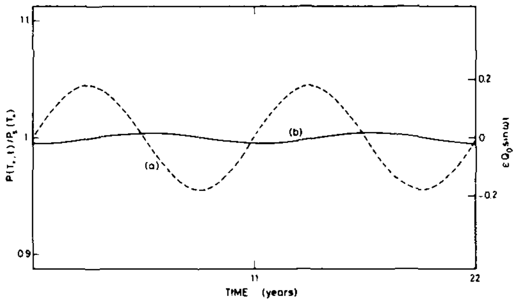


*Fig. 3.* Curve (a): Time dependence of the periodic forcing with frequency $\omega = 2\pi/11$ yr$^{-1}$ and an amplitude $\varepsilon = 0.001$, simulating a possible variation of the solar influx with the sunspot cycle. Curve (b): Time evolution of the probability of the stable state $P(T_+,t)$ divided by its value in the absence of forcing $P_s(T_+) \sim P_s(T_-)$, in the presence of the forcing represented in curve (a).

Similar conclusions have been reached by Benzi *et al.* (1981) on the basis of computer simulations. They refer to this phenomenon as *stochastic resonance.* As we see however from eq. (4.12) the system does not exhibit a resonance in the usual sense of the term, but rather the ability to amplify the response to a low frequency forcing under certain conditions.

## 5. Numerical results—concluding remarks

For the model described by eqs. (2.3) to (2.5), the time-dependent Fokker–Planck equation, eq. (3.3), was integrated numerically using a method developed by Chang and Cooper (1970). First, the steady-state probability distribution in the absence of forcing was obtained. And next, the forcing was added and the long time behavior of the probability was determined. The following parameter values have been adopted: $a_{\rm ice} = 0.82$, $a_{\rm hot} = 0.25$, $\beta = 0.0075$. These values correspond to the coexistence case $U_{0+} \sim U_{0-}$. In the absence of forcing the height of the potential barrier separating the unstable state from either of the two stable states is found to be $\Delta U_{0\pm} \sim 213$ yr$^{-1}$ K$^2$.

Curve (b) of Fig. 2 gives the main result, in the case of a long periodicity simulating the 100,000-yr variation of eccentricity. We start with a steady-state solution in the absence of forcing such that $P_s(T_+) \sim P_s(T_-)$, and choose the variance $q^2$ such that $\omega = r_{0+} + r_{0-}$ (see eq. 4.5). This yields $q^2/(2\Delta U_{0\pm}) \sim 0.12$. The presence of forcing introduces then a rather dramatic variation of $P(T_+,t)$ of the order of 20%. This reflects the fact that the passage over the barrier becomes easier during certain time intervals. Note also that there is a considerable time lag between forcing (curve (a)) and response (curve (b)), in quantitative agreement with eq. (4.6):

$$\phi \cong -45°$$

That is, the maximum of the response at $T_+$ lags behind the forcing by about 12,500 yrs.

According to the analytic treatment, a measure of the importance of the response is also the total probability for remaining at temperatures higher than the unstable $T_0$, denoted by $N_+(t)$ (see eq. (3.7) and (3.8)). It is thus of interest to consider the numerically computed

$$\sum_{T \geqslant T_0} (P(T,t))$$

reduced by its value in the absence of the forcing, as a function of time. One finds that the amplitude of the response is also of about 20%, in agreement with Fig. 2 and with the analytic prediction, eq. (4.12).

Curve (b) of Fig. 3 gives the stochastic response to an 11-yr periodicity simulating a possible variation of the solar influx with the sunspot cycle. We see that the variation is now practically negligible, as expected from the analysis of the preceding Section. In addition, if one considers as before, the total probability $\sum_{T \geqslant T_0} P(T,t)$ the amplitude of the response is so small that its value is certainly within the numerical error. This is again in complete agreement with the analytical expression (4.12).

The pronounced difference between the two responses can be understood as follows: In the presence of a long periodicity the system is given enough time to perceive the lowering of the potential barrier that occurs periodically, and perform more easily a transition between the two climatic states. In contrast, for a short periodicity the system is unable to adjust to the instantaneous external conditions in view of the large value of the characteristic passage time, eq. (1.1).

In summary, in this paper we performed a stochastic analysis of a simple 0-d energy balance model showing bistable behavior, in the presence of a periodic forcing. The amplitude of the forcing was so small that the deterministic response was negligible. Yet in the presence of fluctuations, the amplitude of the response could change dramatically, depending on two basic quantities: (i) the properties of the climatic potential and (ii) a characteristic time scale related to the variance of fluctuations. Under certain conditions the passage over the potential barrier is facilitated and the shape of the probability distribution changes periodically, favoring one of the stable states during certain time intervals. An attempt was made to relate these results to the 100,000-yr periodicity in glaciation cycles. We have been able to work out a comprehensive analytical theory of these phenomena, which is in complete agreement with the numerical simulations.

The work we reported can be extended in many directions. It would be interesting to consider the effect of fluctuations that couple to the system in a multiplicative way through such parameters as $Q$ and $\varepsilon_B$. Similarly, we can relax the hypothesis of purely periodic variation of the solar influx and analyze the effect of a random forcing around some mean periodicity. Finally, we could use more sophisticated climate models taking spatial effects into account. This latter extension is particularly interesting in view of the local character of the fluctuations.

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## СТОХАСТИЧЕСКИЕ АСПЕКТЫ КЛИМАТИЧЕСКИХ ПЕРЕХОДОВ— ОТКЛИК НА ПЕРИОДИЧЕСКОЕ ВОЗДЕЙСТВИЕ

Анализируются зависящие от времени свойства уравнения фоккера-Планка, соответствующего нульмерной модели климата, показывающего поведение с двумя устойчивыми состояниями при слабом периодическом внешнем воздействии. Аналитически найдены условия, при которых отклик системы усиливается. Таким путем устанавливается возможность переходов между различными климатическими состояниями. Результаты иллюстрируются периодичностью в 100,000 лет экспентричности земной орбиты в связи с циклами оледенения.