General Relativity: Exercises 4 -Solutions

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Homework 1: Gravitational radius

Trivial calculations using

$$a = \frac{2GM}{c^2}, \tag{1}$$

from where You will obtain:
a) Gravitational radius of supermassive black hole in centre of Milky Way

$$a_{MW} = 1.21 \times 10^7 \text{ km}. \tag{2}$$

b) Schwarzchild radius of electron

$$a_e = 1.353 \times 10^{-57} \text{ m} = 8.371 \times 10^{-23} l_P. \tag{3}$$

c) Moon

$$a_{☾} = 1.1 \times 10^{-4} \text{ m}. \tag{4}$$

Homework 2: Proper and "radar" distance in Schwarzchild solution

a) For observer at r_1 is "radar" distance between r_1 and r_2 given by (in $c = 1$ units)

$$l_R(r_1, r_2) = \tau(r_1, r_2) = t(r_1, r_2)\sqrt{g_{tt}(r_1)}, \tag{5}$$

where $\tau(r_1, r_2)$ is time of travel of photon between r_1 and r_2 as measured by observer 1 (i.e. his proper time), which can be obtained by finding coordinate time of given process and multiplying by metric coefficient at the place of observer 1 to adjust for gravitational dilation of time.
Because light travel along null geodesics we know that

$$0 = g_{tt}dt^2 + g_{rr}dr^2, \tag{6}$$

from where we can find coordinate time of travel of photon between r_1 and r_2

$$t(r_1, r_2) = \pm \int_{r_1}^{r_2} \sqrt{-\frac{g_{rr}}{g_{tt}}} dr = \pm \int_{r_1}^{r_2} \frac{dr}{1 - \frac{a}{r}} = \pm \left[r + a \ln\left(\frac{r}{a} - 1\right)\right]_{r_1}^{r_2}. \tag{7}$$

So, then "radar" distance for observer at r_1 will be

$$l_R(r_1, r_2) = \left[r + a \ln\left(\frac{r}{a} - 1\right)\right]_{r_1}^{r_2} \sqrt{1 - \frac{a}{r_1}}. \tag{8}$$

b)First find expression for proper distance in Schwarzchild solution

$$l_p(r_1, r_2) = \int_{r_1}^{r_2} \sqrt{g_{rr}} dr = \int_{r_1}^{r_2} \sqrt{\frac{1}{1 - \frac{a}{r}}} dr = \left[\sqrt{r(r-a)} + a \ln\left(\sqrt{\frac{r}{a} - 1} + \sqrt{\frac{r}{a}}\right)\right]_{r_1}^{r_2}, \tag{9}$$

and so proper distance between $r_1 = a$ and $r_2 = r' > a$ is given by

$$l_p(a, r' > a) = \sqrt{r'(r' - a)} + a \ln \left(\sqrt{\frac{r'}{a} - 1} + \sqrt{\frac{r'}{a}} \right) < \infty. \tag{10}$$

Radar distance between $r_1 = a$ and $r_2 = r' > a$ is obtained from equation (8) as

$$l_R(a, r') = \left[r + a \ln \left(\frac{r}{a} - 1 \right) \right]_a^{r'} \sqrt{1 - \frac{a}{r}} \bigg|_a . \tag{11}$$

We can see that expression in square brackets is infinite, but it is multiplied by expression which goes to zero. We can find limit of this expression and so we obtain

$$l_R(a, r') = 0. \tag{12}$$

c) So, we can conclude that in this situation radar distance is zero and proper distance is finite. Interesting thing is that if we would have opposite situation, i.e. observer at $r_1 > a$ and mirror at $r_2 = a$, then expression for radar distance would be infinite.