General Relativity: Exercises 3 -Solutions

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Homework 1: Ideal fluid

First, it is important to stress difference between "classical" velocity $\mathbf{v}$ and four-velocity U^μ

$$\mathbf{v} = \frac{d\mathbf{x}}{dt}, \qquad U^\mu = \frac{dx^\mu}{d\tau}, \tag{1}$$

so You can find relations

$$U^0 = \frac{dt}{d\tau} = \gamma \qquad U^i = \frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} = \gamma v^i, \tag{2}$$

where $\gamma = (1+v^2)^{-\frac{1}{2}}$. Then components of

$$T^{\mu\nu} = p\eta^{\mu\nu} + (p+\rho)U^\mu U^\nu, \tag{3}$$

expressed in terms of $\mathbf{v}$ will be

$$\begin{aligned} T^{00} &= -p + (p+\rho)\gamma^2, && (4)\\ T^{0i} &= (p+\rho)\gamma^2 v^i, && (5)\\ T^{ij} &= p\delta^{ij} + (p+\rho)\gamma^2 v^i v^j. && (6) \end{aligned}$$

Conservation law of energy-momentum tensor

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0, \tag{7}$$

can be split to four equations

$$\begin{aligned} \frac{\partial T^{0\nu}}{\partial x^\nu} &= \frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0i}}{\partial x^i}. && (8)\\ \frac{\partial T^{i\nu}}{\partial x^\nu} &= \frac{\partial T^{i0}}{\partial t} + \frac{\partial T^{ij}}{\partial x^j}. && (9) \end{aligned}$$

Explicitly this is

$$\begin{aligned} \frac{\partial T^{0\nu}}{\partial x^\nu} &= -\frac{\partial p}{\partial t} + \left(\frac{\partial p}{\partial t} + \frac{\partial \rho}{\partial t}\right)\gamma^2 + (p+\rho)\frac{\partial \gamma^2}{\partial t} + \left(\frac{\partial p}{\partial x^j} + \frac{\partial \rho}{\partial x^j}\right)\gamma^2 v^j + (p+\rho)\frac{\partial \gamma^2 v^j}{\partial x^j}, && (10)\\ \frac{\partial T^{i\nu}}{\partial x^\nu} &= \left(\frac{\partial p}{\partial t} + \frac{\partial \rho}{\partial t}\right)\gamma^2 v^i + (p+\rho)\frac{\partial \gamma^2 v^i}{\partial t} + \frac{\partial p}{\partial x^i} + \left(\frac{\partial p}{\partial x^j} + \frac{\partial \rho}{\partial x^j}\right)\gamma^2 v^i v^j + (p+\rho)\frac{\partial \gamma^2 v^i v^j}{\partial x^j}. && (11) \end{aligned}$$

Multiply first equation by v^i and substract it from second equation to obtain

$$\left(v^i\frac{\partial p}{\partial t} + \frac{\partial p}{\partial x^i}\right) + (p+\rho)\gamma^2\left[\frac{\partial v^i}{\partial t} + v^j\frac{\partial v^i}{\partial x^j}\right] = 0, \tag{12}$$

what can be expressed as

$$\frac{\partial v^i}{\partial t} + v^j\frac{\partial v^i}{\partial x^j} = -\frac{1}{(p+\rho)\gamma^2}\left(v^i\frac{\partial p}{\partial t} + \frac{\partial p}{\partial x^i}\right), \tag{13}$$

or in vector notation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1-v^2}{(p+\rho)}\left(\mathbf{v}\frac{\partial p}{\partial t} + \nabla p\right). \tag{14}$$

Homework 2: Lie derivative

a) First find commutator of vector fields

$$[\epsilon, V] = \epsilon^\mu \partial_\mu V^\nu \partial_\nu - V^\mu \partial_\mu \epsilon^\nu \partial_\nu = \epsilon^\mu (\partial_\mu V^\nu) \partial_\nu - V^\mu (\partial_\mu \epsilon^\nu) \partial_\nu + \underbrace{\epsilon^\mu V^\mu (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)}_{0}, \tag{15}$$

where last bracket vanishes because partial derivatives commute. From there follows

$$[\epsilon, V] = (\epsilon^\mu \partial_\mu V^\nu - V^\mu \partial_\mu \epsilon^\nu) \partial_\nu, \tag{16}$$

and from lecture notes You know that Lie derivative acts on vector (See review from 25th of May) as

$$\Delta_\epsilon V^\mu = \epsilon^\mu \partial_\mu V^\nu - V^\mu \partial_\mu \epsilon^\nu. \tag{17}$$

You can see that this is exactly what You wanted to prove, because $\Delta_\epsilon V^\mu$ is μ-component of Lie derivative. So, in coordinate basis ∂_μ we obtained exactly equation (16).

b) We have metric

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2, \tag{18}$$

Lie derivative acts on metric as

$$\Delta_\epsilon g_{\mu\nu} = \epsilon^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} + g_{\mu\alpha} \frac{\partial \epsilon^\alpha}{\partial x^\nu} + g_{\alpha\nu} \frac{\partial \epsilon^\alpha}{\partial x^\mu}, \tag{19}$$

where ϵ is some vector field which can be expressed in (θ, ϕ) coordinates as

$$\epsilon = u\frac{\partial}{\partial\theta} + v\frac{\partial}{\partial\phi}. \tag{20}$$

Then we have three independent components of equation (19). Explicitly

$$\Delta_\epsilon g_{\theta\theta} = 2\frac{\partial u}{\partial \theta}, \tag{21}$$

$$\Delta_\epsilon g_{\theta\phi} = \frac{\partial u}{\partial \phi} + \sin^2\theta \frac{\partial v}{\partial \theta}, \tag{22}$$

$$\Delta_\epsilon g_{\phi\phi} = 2\sin\theta\cos\theta u + 2\sin^2\theta\frac{\partial v}{\partial\phi}. \tag{23}$$

Lie derivative tells us how some tensor changes under change of coordinates. This means that to find vector field ϵ which leaves metric invariant is equivalent to finding such vector field ϵ for which Lie derivative of metric vanishes. This means that all left-hand sides of previous equations are zero and we obtain system of differential equations

$$\frac{\partial u}{\partial \theta} = 0, \qquad \frac{\partial v}{\partial\theta}\sin^2\theta + \frac{\partial u}{\partial\phi} = 0, \qquad u + \frac{\partial v}{\partial \phi}\tan\theta = 0. \tag{24}$$

From first of these equations we obtain

$$u = u(\phi), \tag{25}$$

then second equation can be integrated according to θ

$$v = u'(\phi)\cot\theta + C, \tag{26}$$

and we can insert this result to third equation to obtain

$$u''(\phi) = -u(\phi), \tag{27}$$

which solution is

$$u(\phi) = A\cos\phi + B\sin\phi, \tag{28}$$

and thus

$$v(\phi,\theta) = -A\sin\phi\cot\theta + B\cos\phi\cot\theta + C. \tag{29}$$

So, most general vector field which leaves metric of two-sphere invariant is

$$\epsilon = (A\cos\phi + B\sin\phi)\,\frac{\partial}{\partial\theta} + (-A\sin\phi\cot\theta + B\cos\phi\cot\theta + C)\,\frac{\partial}{\partial\phi} \tag{30}$$

or we can rewrite it as

$$\epsilon = AX + BY + CZ, \tag{31}$$

i.e. as a linear combination of three linearly independent vector fields

$$X = \cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}, \tag{32}$$

$$Y = \sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}, \tag{33}$$

$$Z = \frac{\partial}{\partial\phi}. \tag{34}$$