General Relativity: Exercises 2 -Solutions

May 25, 2011

Homework 1: Foucault Pendulum

Equations of parallel transport of vector V are

$$\dot{V}^{\mu} + \Gamma^{\mu}_{\kappa\sigma}\dot{x}^{\kappa}V^{\sigma} = 0, \tag{1}$$

where dot means differentiation according to t. Taking Christoffel symbols for 2-speheres and choosing parametrization $t = \phi$ we obtain system of coupled differential equations

$$\begin{aligned} \dot{V}^{\theta} - \sin\theta_0 \cos\theta_0 V^{\phi} &= 0, & (2) \\ \dot{V}^{\phi} + \cot\theta_0 V^{\theta} &= 0. & (3) \end{aligned}$$

We can solve this system by differentiating first equation according to ϕ and inserting second equation into it. We obtain differential equation of second order

$$\ddot{V}^{\theta} + \cos^2\theta_0 V^{\theta} = 0. \tag{4}$$

Solution to this differential equation is given by

$$V^{\theta} = A\cos(\phi\cos\theta_0) + B\sin(\phi\cos\theta_0), \tag{5}$$

where A and B are integration constants which we can determine from condition $V = (V^{\phi}, V^{\theta}) = (0, 1)$ at $\phi = 0$. We obtain $A = 1$ and $B = 0$. With knowledge of V^{θ} we can find (3) to be

$$\dot{V}^{\phi} = -\cot\theta_0 V^{\theta} = -\cot\theta_0 \cos(\phi\cos\theta_0), \tag{6}$$

which can be easily integrated and thus we obtain final solution

$$\begin{aligned} V^{\theta} &= \cos(\phi\cos\theta_0), & (7) \\ V^{\phi} &= -\frac{\sin(\phi\cos\theta_0)}{\sin\theta_0}. & (8) \end{aligned}$$

To find angle by which is vector rotated we need

$$\cos\alpha = \frac{V.V'}{|V|.|V'|} \tag{9}$$

where V is $V = V(\phi = 0)$ and V' is after parallel transport, i.e. $V' = V(\phi = 2\pi)$ From there we can see that after parallel tranport around full circle, i.e. from $\phi = 0$ to $\phi = 2\pi$, vector V will be rotated by angle

$$\alpha = 2\pi\cos\theta_0, \tag{10}$$

what is exactly angle by which Foucault pendulum will be rotated by one day. This is because motion of Foucault pendulum is inertial, i.e. is parallely transported. In one day it is paralely transported around full circle (because of rotation of Earth surface).

Homework 2: Twin Paradoxes

a) Call clock at Earth surface Clock 1 and one onboard of satellite Clock 2. For clock on Earth surface holds

$$d\tau_1^2 = dt^2, \tag{11}$$

For Earth observer Clock 2 are moving so space-time interval which Clock 2 pass (in respect to Earth observer) is

$$c^2 d\tau_2^2 = c^2 dt^2 - d\vec{x}^2 = \left[1 - \frac{1}{c^2}\left(\frac{d\vec{x}}{dt}\right)^2\right] c^2 dt^2 = \left(1 - \frac{\vec{v}^2}{c^2}\right) c^2 dt^2. \tag{12}$$

Comparing these two equations we obtain

$$d\tau_1 = \frac{d\tau_2}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} \approx \left(1 + \frac{\vec{v}^2}{2c^2}\right) d\tau_2, \tag{13}$$

after linearization. Orbital velocity velocity can be found from

$$\frac{GM}{r^2} = \frac{v^2}{r} \quad \Longrightarrow \quad v = \sqrt{\frac{GM}{r}} \approx 3,877 \text{ m/s}. \tag{14}$$

So, difference in flow of proper time for these two obserers is

$$\Delta\tau_1 \approx 1.0000000000835 \ \Delta\tau_2. \tag{15}$$

So, in one day there will be accumulated time difference

$$\Delta\tau_{SR} = \Delta\tau_1 - \Delta\tau_2 = 7.21 \ \mu s. \tag{16}$$

Note: In order to give rigorous answer to this problem we should count in also motion of observer 1. Observer on Earth surface makes rotation around full circle in 1 day. Highest velocity would be for observer at Earth Equator, where he would be effectively rotating with velocity of 40,000 km per day, i.e. $v_1 = 460$ m/s, what is about 10% of orbital velocity of satellite. Relation for dilatation of time is quadratic in velocity, so counting in this rotation of observer on Earth surface would cause correction about 1%, what is effect which we can neglect.

---

b) Similar effect works in GR,only here it is caused by gravitational field. From lecture on Newtonian limit You know that

$$g_{00} = -1 - \frac{2\phi}{c^2}. \tag{17}$$

Denote $g_{00}(1)$ to be metric coefficients at the place of observer on Earth, i.e.

$$g_{00}(1) = -1 - \frac{2\phi_1}{c^2}. \tag{18}$$

Then proper time at the place of Clock 1 will be

$$d\tau_1^2 = g_{00}(1) dt^2, \tag{19}$$

and at Clock 2

$$d\tau_2^2 = g_{00}(2) dt^2. \tag{20}$$

From where we obtain relation between proper times

$$d\tau_1 = \frac{\sqrt{g_{00}(1)}}{\sqrt{g_{00}(2)}} d\tau_2, \tag{21}$$

which after linearization of square roots is up to order $O(\phi)$

$$d\tau_1 = \left[1 + \frac{1}{c^2}(\phi_1 - \phi_2) + O(\phi^2)\right] d\tau_2 \approx \left[1 - \frac{G_N M_\oplus}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right] d\tau_2, \tag{22}$$

where r_1 is distance of observer 1 to center of Earth and r_2 distance of observer 2. Numerically

$$\Delta\tau_1 = 0.99999999894\Delta\tau_2. \tag{23}$$

Their difference accumulated during one day is

$$\Delta\tau_{GR} = \Delta\tau_1 - \Delta\tau_2 = -45.737\ \mu s. \tag{24}$$

c) You can see that total accumulated time difference due to GR and SR in one day will be

$$\Delta\tau = \Delta\tau_{GR} + \Delta\tau_{SR} = -38.527\ \mu s. \tag{25}$$

Interesting thing is that these two effect of dilation of time in some sense acts in "opposite ways", i.e. SR time-dilation cause satellite clock ticking slower than those on Earth surface and GR time-dilation cause satellite clock tick faster.