General Relativity: Exercises 1

Till: May 16, 2011

Problem 1

Standard way to compute Christoffel symbols is from definition

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right). \tag{1}$$

There exist easier way to do it. Take functional

$$L = g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} = g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau}, \tag{2}$$

and show that Lagrange-Euler equations

$$\frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^{\lambda}} = \frac{\partial L}{\partial x^{\lambda}}, \tag{3}$$

for this functional will be

$$\frac{d^2 x^{\lambda}}{d\tau^2} + \frac{1}{2} g^{\lambda\sigma} \left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right) \frac{d\dot{x}^{\mu}}{d\tau} \frac{d\dot{x}^{\nu}}{d\tau} = 0. \tag{4}$$

Actually this is equation of geodetic

$$\frac{d^2 x^{\lambda}}{d\tau^2} + \Gamma^{\lambda}_{\mu\nu} \frac{d\dot{x}^{\mu}}{d\tau} \frac{d\dot{x}^{\nu}}{d\tau} = 0. \tag{5}$$

So, from this equation You can read-off all Christoffel symbols.

Problem 2

Remember previous homework-i.e. calculation Christoffel symbols for 2-sphere- and how much time did it consume. With help of this variational principle do it fast.

Homework 1

Show that covariant derivative of vector

$$\nabla_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \Gamma^{\nu}_{\mu\lambda} V^{\lambda}, \tag{6}$$

transform under coordinate change

$$x^{\mu} \longmapsto x'^{\mu}, \tag{7}$$

like tensor.

Homework 2

One of properties of derivative is that it respect Leibniz rule. Show that covariant derivative do so

$$\nabla_{\lambda}(A^{\mu}_{\nu} B^{\kappa}) = (\nabla_{\lambda} A^{\mu}_{\nu}) B^{\kappa} + A^{\mu}_{\nu} (\nabla_{\lambda} B^{\kappa}). \tag{8}$$

Homework 3

Imagine infinite cylinder and find metric on it. What is the scalar curvature of surface of this cylinder. How do You find answer to this question without explicit calculation?