CHAPTER VII
Symmetries in classical field theory

VII.1 Basics of classical field theory

Consider independent scalar fields \( \varphi_a(x) \), \( a = 1, \ldots, N \), defined at each point \( x \) of a \( D \)-dimensional space-time \( M_D \) on which a metric with signature \((-+, +, +, \cdots, +)\) is defined — the obvious case, which we consider in the following, being that of the 4-dimensional Minkowski\(^{(af)}\) space \( M_4 \) of special relativity. As we shall illustrate on a few examples, the fields \( \varphi_a(x) \) may be real- or complex-valued, and may represent the components in a given reference frame of a vector or tensor field.

VII.1.1 Definitions

The dynamical behavior of the fields is described by a *Lagrange density*

\[
\mathcal{L} = \{ \varphi_a(x) \}, \{ \partial_\mu \varphi_a(x) \}.
\]

(VII.1a)

which is a functional of the fields and of their derivatives \( \partial_\mu \varphi_a(x) \) with \( \mu \in \{0, 1, 2, 3\} \), where \( \partial_\mu \) denotes the partial derivative with respect to the coordinate \( x^\mu \) of the space-time point \( x \) in a given reference frame.

Integrating the Lagrange density over the spatial coordinates yields the *Lagrangian*

\[
\mathcal{L} = \int \mathcal{L} \{ \varphi_a(x) \}, \{ \partial_\mu \varphi_a(x) \} \, d^3 \mathbf{r}.
\]

(VII.1b)

In turn, integrating the Lagrangian over a time interval \([t_1, t_2]\) gives the *action*

\[
S = \int_{t_1}^{t_2} L \, dt = \int \mathcal{L} \{ \varphi_a(x) \}, \{ \partial_\mu \varphi_a(x) \} \, d^4x
\]

(VII.1c)

with \( d^4x \) the 4-volume element, \( d^4x = dx^0 \, dx^1 \, dx^2 \, dx^3 \).

Remarks:

* If the Lagrange density of a system of fields is invariant under the transformations of the Lorentz or Poincaré groups, then this also holds for the resulting action, since \( d^4x \) is Lorentz-invariant.

* The Lagrange density for a given physical system is not unique. Anticipating on next section, two Lagrange densities \( \mathcal{L} \) and \( \mathcal{L}' \) that only differ by a 4-divergence \( \partial_\mu M^\mu(x) \), with \( M \) a 4-vector field, are equivalent: their integrals give respective actions \( S, S' \), which lead to the same equations of motion.

\(^{(af)}\)H. Minkowski, 1864–1909
VII.1.2 Euler–Lagrange equations

According to the principle of the stationary action, or Hamilton’s principle, the equations of motion obeyed by a system of fields can be obtained by demanding that the action $S$ should be stationary when the field configuration is varied, assuming that the fields do not vary at the boundaries of the space-time region over which the action is computed.

Considering a (small) variation $\delta \varphi_a$ of each field $\varphi_a$, which results in a variation $\delta \partial_\mu \varphi_a = \partial_\mu \delta \varphi_a$ of its partial derivatives, the ensuing variation of the action (VII.1c) reads

$$\delta S = \int \sum_a \left[ \frac{\partial L}{\partial \varphi_a} \delta \varphi_a + \frac{\partial L}{\partial (\partial_\mu \varphi_a)} \delta \partial_\mu \varphi_a \right] d^4x,$$

where the second term involves a non-written sum over the Lorentz indices $\mu$. When integrating that term by partial integration, the contribution from the integrated term is zero, thanks to the assumption that $\delta \varphi_a$ vanishes on the boundaries of the integration region. One thus obtains

$$\delta S = \int \sum_a \left[ \frac{\partial L}{\partial \varphi_a} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \varphi_a)} \right] \delta \varphi_a d^4x.$$

The stationarity of the action is realized when $\delta S = 0$ for arbitrary variations $\delta \varphi_a$, i.e. when each field $\varphi_a$ and its derivatives obey the Euler–Lagrange equations

$$\frac{\partial L}{\partial \varphi_a}[\{\varphi_a\}, \{\partial_\mu \varphi_a\}] = \partial_\mu \left[ \frac{\partial L}{\partial (\partial_\mu \varphi_a)} \right].$$

(VII.2)

These represent the equations of motion of the fields.

Example: The simplest possible example is that of a single real scalar field $\varphi$ with the Lagrange density

$$L[\varphi, \partial_\mu \varphi] = -\frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2} m^2 \varphi^2.$$

(VII.3)

Straightforward calculations, where one should not forget the identity $\partial^\mu \varphi = \eta^{\mu\nu} \partial_\nu \varphi$, yield

$$\frac{\partial L}{\partial \varphi} = -m^2 \varphi \quad \text{and} \quad \frac{\partial L}{\partial (\partial_\mu \varphi)} = -\partial^\mu \varphi,$$

so that the Euler–Lagrange equation gives $-m^2 \varphi = -\partial_\mu \partial^\mu \varphi$, that is, restoring the $x$-dependence of the field

$$\partial_\mu \partial^\mu \varphi(x) - m^2 \varphi(x) = 0$$

(VII.4)

which is known as the Klein–Gordon equation.

Remark: In analogy to the construction in analytical mechanics, one associates to each field $\varphi_a$ governed by the Lagrange density $L$ a conjugate momentum

$$\pi_a = \frac{\partial L}{\partial (\partial_0 \varphi_a)}.$$

(VII.5)

With its help, one defines the Hamilton density

$$\mathcal{H}[(\pi_a), \{\varphi_a\}] = \sum_a \pi_a \partial_0 \varphi_a - L,$$

(VII.6)

whose integral over the spatial coordinates gives the Hamilton function of the system.

For the real scalar field of the previous example, one at once finds $\pi = -\partial^0 \varphi = \partial_0 \varphi$, which yields $\mathcal{H} = \frac{1}{2} (\partial_0 \varphi)^2 + \frac{i}{2} (\nabla \varphi)^2 + \frac{1}{2} m^2 \varphi^2.$

\(^{(ag)}\text{O. Klein, 1894–1977} \quad ^{(ah)}\text{W. Gordon, 1893–1939}\)
VII.2 Symmetries in field theories and conserved quantities

An important feature of classical field theory lies in its relation to symmetries. On the one hand, implementing a symmetry into a given theory is rather simple, as we shall see in the following section. On the other hand, if a continuous symmetry is present, then there a exist corresponding conserved quantity (or set of conserved quantities), as we shall now see.

VII.2.1 Symmetries in a field theory

Using the Lagrange formalism, one can easily study the behavior of a system of classical fields under various types of transformations. For physical applications, two big classes of such transformations are of interest, which may possibly be applied simultaneously.

On the one hand, one can consider space-time transformations, affecting the “position” of the field system. These are generically of the form

\[ x \rightarrow x' = f(x, \{\theta_c\}) , \]  

where \( f \) denotes a sufficiently regular function depending on real parameters \( \{\theta_c\} \). Introducing Minkowski coordinates, the transformation reads

\[ x^\mu \rightarrow x'^\mu = f^\mu(\{x^\nu\}, \{\theta_c\}) \]  

for \( \mu = 0, 1, 2, 3 \) (or up to \( D - 1 \) in the more general case of a \( D \)-dimensional space-time \( \mathbb{M}_D \)).

On the other hand, one may also consider “internal” transformations of the fields, of the form

\[ \varphi_a(x) \rightarrow \varphi_a'(x) = g_a(\varphi_b(x), \{\Lambda_c(x)\}) , \]  

where \( g_a \) is some function depending on real parameters \( \{\Lambda_c(x)\} \). An important possibility, corresponding to so-called gauge transformations, is that the latter parameters indeed depend on the space-time position of the fields. In case of space- and time-independent parameters, one also talks of global transformations.

A group of transformations is said to be a symmetry of a physical system described by fields if the associated action (VII.1c) is invariant under the transformations.

VII.2.2 Noether theorem

Let us now show that every continuous group of symmetry transformations of the action \( S \) of a system leads to the existence of a continuous quantity, which constitutes the Noether theorem.

VII.2.2a Result and discussion

More precisely, we shall find that if the integral of a Lagrange density \( \mathcal{L}[\varphi_a(x), \{\partial_\mu \varphi_a(x)\}] \) over a 4-volume \( \Omega \) — which will be further specified hereafter — is invariant under simultaneous infinitesimal transformations

\[ x^\mu \rightarrow x'^\mu = x^\mu + \delta x^\mu \quad \forall \mu = 0, 1, 2, 3 , \]  

\[ \varphi_a(x) \rightarrow \varphi_a'(x') = \varphi_a(x) + \delta \varphi_a(x) \quad \forall a \]  

of the space-time coordinates and the fields, then the Noether current with components

\[ J^\mu = \sum_a \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_a)} \delta \varphi_a + T^\mu_\text{can} \delta x^\nu \]  

obeys

\[ \int_\Omega \partial_\mu J^\mu(x) d^4x = 0 , \]  

\( \text{(ai) E. Noether, 1882–1935} \)
where the **canonical energy-momentum tensor** is defined by

\[
T^{\mu\nu}_{\text{can}} \equiv \eta^{\mu\nu} \mathcal{L} - \sum_a \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_a)} \partial^\nu \varphi_a.
\]  

(VII.12)

In the above, the 4-volume \( \Omega \) is delimited in the temporal direction by two constant-time hyperplanes \( t = t_1 \) and \( t = t_2 \), while it extends to infinity in the spatial directions, in agreement with Eqs. (VII.1b) and (VII.1c). That is, the boundary \( \partial \Omega \) of \( \Omega \) consists of the two hyperplanes \( t = t_1 \) and \( t = t_2 \) and of the hypersurface \( \partial \mathbb{R}^3 \) at spatial infinity for every \( t \in [t_1, t_2] \) — where one assumes that every field vanishes together with its derivatives. Accordingly, an integral over \( \partial \Omega \) actually reduces to integrals over the spatial coordinates \( \vec{r} \in \mathbb{R}^3 \) at \( t = t_1 \) and \( t = t_2 \). In particular, the divergence theorem reads for any 4-vector-field \( B^\mu \)

\[
\int_{\Omega} \partial_\mu B^\mu(x) \, d^4x = \oint_{\partial \Omega} B^\mu(x) \, d^3\sigma_\mu = \left[ \int_{\mathbb{R}^3} B^\mu(t, \vec{r}) \, d^3\vec{r} \right]_{t=t_1}^{t=t_2},
\]

where we used the fact that on the hyperplane \( t = t_1 \) resp. \( t = t_2 \), the outwards-oriented hypersurface element \( d^3\sigma_\mu \) is purely along the time direction with \( d^3\sigma_0 = d^3\vec{r} \) resp. \( -d^3\vec{r} \).

Applying the latter result to the relation (VII.11) obeyed by the Noether current (VII.10) yields

\[
\int_{\Omega} \partial_\mu J^\mu(x) \, d^4x = 0 = \left[ \int_{\mathbb{R}^3} J_0^\mu(t, \vec{r}) \, d^3\vec{r} \right]_{t=t_1}^{t=t_2},
\]

which shows that the Noether charge

\[
Q \equiv \int_{\mathbb{R}^3} J_0^\mu(t, \vec{r}) \, d^3\vec{r}
\]

(VII.13)

is a conserved quantity, a constant of motion.

**Remarks:**

* Using \( \eta^{00} = -1 \) and \( \partial^0 \varphi_a = -\partial_0 \varphi_a \), definition (VII.12) shows that the 00-component of the canonical energy-momentum tensor coincides with the Hamilton density (VII.6): \( T^{00}_{\text{can}} = \mathcal{H} \).

* Strictly speaking, \( J^\mu \) is a current density.

* Instead of the invariance of the action, one often finds in the literature the stronger requirement of having an invariant Lagrange density \( \mathcal{L} \) — which of course automatically leads to an invariant action. This stronger condition has the advantage that it leads to the “local conservation equation” \( \partial_\mu J^\mu(x) = 0 \) in lieu of Eq. (VII.11), yet it is not needed for the existence of the conserved Noether charge.

**VII.2.2 b Proof**

Let us first rewrite the expression of action of the fields

\[
S = \int_\Omega \mathcal{L} \left[ \{ \varphi_a(x) \}, \{ \partial_\mu \varphi_a(x) \} \right] \, d^4x
\]

(VII.14)

valid for the transformed fields and coordinates (VII.9). When transforming the latter, one simultaneously modifies the integration 4-volume: \( \Omega \to \Omega' \). Accordingly, the transformed action reads

\[
S' = \int_{\Omega'} \mathcal{L} \left[ \{ \varphi_a'(x') \}, \{ \partial_\mu \varphi_a'(x') \} \right] \, d^4x'.
\]

Introducing the boundary \( \partial \Omega \) of the 4-volume \( \Omega \) — which is also, up to terms of order \( \delta x^\mu \), the boundary of \( \Omega' \) —, one may decompose the integral over \( \Omega' \) into the sum of an integral over \( \Omega \) and of an integral over the layer of thickness \( \delta x^\mu \) that corresponds to the difference \( \Omega' \setminus \Omega \):
\[ S' = \int_{\Omega} \mathcal{L}'\left[\{\varphi'_a(x)\}, \{\partial_\mu \varphi_a(x)\}\right] d^4 x + \oint_{\partial \Omega} \mathcal{L}'\left[\{\varphi'_a(x)\}, \{\partial_\mu \varphi_a(x)\}\right] \delta x^\mu d^3 \sigma_\mu, \quad (\text{VII.15}) \]

where \( d^3 \sigma_\mu \) is a normal (hyper)surface 4-vector at each point of the boundary \( \partial \Omega \). Note that every space-time position is now expressed in terms of the “old” positions \( x \), which allows us to suppress it from now on for the sake of brevity. Accordingly, we introduce the variation of the field \( \varphi_a \) at a fixed point:

\[ \delta \varphi_a(x) \equiv \varphi'_a(x) - \varphi_a(x), \]

which will generally differ from \( \delta \varphi_a(x) \) if \( x' \) and \( x \) are different, i.e. if the space-time coordinates are transformed: to first order in the variation \( \delta x \), one has

\[ \delta \varphi_a = \varphi'_a(x' - \delta x) - \varphi_a(x) \approx \varphi'_a(x') - \delta x_\mu \partial^\mu \varphi_a(x) - \varphi_a(x) \]

i.e.

\[ \delta \varphi_a \approx \delta \varphi_a(x) - \delta x_\mu \partial^\mu \varphi_a(x). \quad (\text{VII.16}) \]

Coming back to Eq. (VII.15), one may write \( \varphi'_a(x) = \varphi_a(x) + \delta \varphi_a(x) \) in the arguments of the Lagrange density and expand to first order in the variations \( \delta x, \delta \varphi_a \), and \( \delta (\partial_\mu \varphi_a) = \partial_\mu (\delta \varphi_a) \):

\[ S' \approx \int_{\Omega} \left[ \mathcal{L}'\left[\{\varphi_a\}, \{\partial_\mu \varphi_a\}\right] + \sum_a \left( \frac{\partial \mathcal{L}}{\partial \varphi_a} \delta \varphi_a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_a)} \partial_\mu \delta \varphi_a \right) \right] d^4 x + \oint_{\partial \Omega} \mathcal{L}'\left[\{\varphi_a\}, \{\partial_\mu \varphi_a\}\right] \delta x^\mu d^3 \sigma_\mu. \]

Note that in the integral over the boundary \( \partial \Omega \), we have suppressed terms stemming from the expansion of the Lagrange density that yield contributions of at least second order in the variations. From now on, we shall also omit the argument \( \{\varphi_a\}, \{\partial_\mu \varphi_a\} \) of the Lagrange density and its (functional) derivatives, since it is the same in every term.

One may now subtract the action (VII.14) from the new expression of \( S' \), which is straightforward. Performing an integration by parts for the term in the 4-volume integral involving the partial derivatives \( \partial_\mu (\delta \varphi_a) \), one obtains

\[ \delta S \equiv S' - S = \int_{\Omega} \sum_a \left[ \frac{\partial \mathcal{L}}{\partial \varphi_a} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_a)} \right) \right] \delta \varphi_a d^4 x + \oint_{\partial \Omega} \left[ \sum_a \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_a)} \delta \varphi_a + \mathcal{L} \delta x^\mu \right] d^3 \sigma_\mu. \]

Thanks to the Euler–Lagrange equations (VII.2), the integrand of the 4-volume integral identically vanishes, since we assume that the field configuration \( \{\varphi_a(x)\} \) is a solution of the equations of motion. In turn, expressing \( \delta \varphi_a \) through \( \delta \varphi_a \) [Eq. (VII.16)] in the integral over \( \partial \Omega \) yields

\[ \delta S = \oint_{\partial \Omega} \left[ \sum_a \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_a)} (\delta \varphi_a - \delta x_\mu \partial^\nu \varphi_a) + \mathcal{L} \delta x^\mu \right] d^3 \sigma_\mu. \]

After some straightforward rewriting, this becomes

\[ \delta S = \oint_{\partial \Omega} \left[ \sum_a \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_a)} \delta \varphi_a + \left( \eta^{\mu \nu} \mathcal{L} - \sum_a \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_a)} \partial^\nu \varphi_a \right) \delta x_\nu \right] d^3 \sigma_\mu. \quad (\text{VII.17}) \]

Using the definitions of the Noether current (VII.10) and the canonical energy-momentum tensor (VII.12), this becomes

\[ \delta S = \oint_{\partial \Omega} J^\mu d^3 \sigma_\mu. \]

The action is stationary for any infinitesimal transformation of the fields and space-time coordinates if its variation \( \delta S \) vanishes, which invoking the divergence theorem precisely gives the announced result (VII.11) — which concludes the proof.
VII.2.3 Examples

VII.2.3 a Energy and momentum of a field system

Considering a Lagrange density which is invariant under space-time translations

\[ x^\mu \to x'^\mu = x^\mu + a^\mu \]  \hspace{1cm} (VII.18)

with \( a^\mu \in \mathbb{R} \) for every \( \mu \in \{0,1,2,3\} \), one finds by setting \( \delta \varphi = 0 \) and \( \delta x_\nu = a_\nu \) in the expression (VII.10) of the Noether current that the latter becomes proportional to the canonical energy-momentum tensor. Accordingly, the four components

\[ P^\nu = \int_{\mathbb{R}^3} T^\nu_{\text{can}}(t, \vec{r}) \, d^3\vec{r} \quad \text{for} \quad \nu = 0, 1, 2, 3, \]

are conserved quantities, which corresponds to the energy \( (P^0) \) and the momentum of the field system, and form a four-vector.

VII.2.3 b Complex scalar field

Let \( \varphi(x) \) denote a complex scalar field, with the Lagrange density

\[ \mathcal{L}[\varphi, \varphi^*, \partial_\mu \varphi, \partial_\mu \varphi^*] = -\left( \partial_\mu \varphi \right) \left( \partial^\mu \varphi^* \right) - m^2 \varphi \varphi^*, \]  \hspace{1cm} (VII.19)

where \( \varphi \) and \( \varphi^* \) are to be considered as independent fields.

The Euler–Lagrange equations

\[ \frac{\partial \mathcal{L}}{\partial \varphi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial \left( \partial_\mu \varphi \right)} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \varphi^*} = \partial_\mu \frac{\partial \mathcal{L}}{\partial \left( \partial_\mu \varphi^* \right)} \]

are easily computed and lead to respective Klein–Gordon equations

\[ \partial_\mu \partial^\mu \varphi^*(x) - m^2 \varphi^*(x) = 0 \quad \text{and} \quad \partial_\mu \partial^\mu \varphi(x) - m^2 \varphi(x) = 0, \]  \hspace{1cm} (VII.20)

which are naturally equivalent if \( m \in \mathbb{R} \).

One sees at once that the Lagrange density (VII.19) is invariant under the “global U(1) transformation”

\[ \begin{align*}
\varphi(x) &\to \varphi'(x) = e^{-i\Lambda} \varphi(x) \\
\varphi^*(x) &\to \varphi^*(x) = e^{i\Lambda} \varphi(x)
\end{align*} \]  \hspace{1cm} (VII.21)

where \( \Lambda \in \mathbb{R} \) is invariant of the space-time position. In the case of an infinitesimally small \( \Lambda \), these transformations become

\[ \begin{align*}
\varphi'(x) &\simeq \varphi(x) - i\Lambda \varphi(x) \\
\varphi^*(x) &\simeq \varphi^*(x) + i\Lambda \varphi^*(x)
\end{align*} \]

which correspond to infinitesimal variations \( \delta \varphi = -i\Lambda \varphi, \delta \varphi^* = i\Lambda \varphi^* \) of the independent fields. Inserting the latter in the expression (VII.10) of the Noether current yields

\[ J^\mu = i \left( \varphi^* \partial^\mu \varphi - \varphi \partial^\mu \varphi^* \right), \]  \hspace{1cm} (VII.22)

where we have divided by \( \Lambda \). One easily checks with the equations of motion (VII.20) that this current, whose spatial components are reminiscent of the probability current of non-relativistic quantum mechanics, obeys the local conservation equation \( \partial_\mu J^\mu = 0 \).