C.2.5 Markov processes

We now introduce a class of stochastic processes which are often encountered in physics—or, one should rather say, which are often used to model physical phenomena due to their simplicity, since they are entirely determined by the two densities $p_{Y,1}$ and $p_{Y,1|1}$.

C.2.5 a Markov property

A *Markov*[(61)] *process* is a stochastic process $Y(t)$ for which for all $n \in \mathbb{N}^*$ and arbitrary ordered times $t_1 < t_2 < \cdots < t_{n-1} < t_n < t_{n+1}$, the conditional probability densities obey the *Markov property*

$$p_{Y,1|n}(t_{n+1}, y_{n+1} \,|\, t_1, y_1; t_2, y_2; \ldots; t_{n-1}, y_{n-1}; t_n, y_n) = p_{Y,1|1}(t_{n+1}, y_{n+1} \,|\, t_n, y_n). \tag{C.22}$$

Viewing t_n as being "now", this property means that the (conditional) probability that the process takes a given value y_{n+1} in the future (at t_{n+1}) only depends on its present value y_n, not on the values it took in the past.

An even more drastically "memoryless" class of processes is that of *fully random processes*, for which the value taken at a given time is totally independent of the past values. For such a process, the conditional probability densities equal the joint probability densities—i.e. $p_{Y,n|m} = p_{Y,n}$ for all m, n—, and repeated applications of Bayes' theorem (C.13) show that the n-point density factorizes into the product of n single-time densities,

$$p_{Y,n}(t_1, y_1; \ldots; t_n, y_n) = p_{Y,1}(t_1, y_1) \cdots p_{Y,1}(t_n, y_n).$$

One can check that a Markov process is entirely determined by the single-time probability density $p_{Y,1}(t_1, y_1)$ and by the *transition probability* $p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)$, or equivalently by $p_{Y,1}(t_1, y_1)$ and the two-time density $p_{Y,2}(t_1, y_1; t_2, y_2)$.
For instance, the 3-time probability density can be rewritten as

$$\begin{aligned} p_{Y,3}(t_1, y_1; t_2, y_2; t_3, y_3) &= p_{Y,1|2}(t_3, y_3 \,|\, t_1, y_1; t_2, y_2)\, p_{Y,2}(t_1, y_1; t_2, y_2) \\ &= p_{Y,1|1}(t_3, y_3 \,|\, t_2, y_2)\, p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)\, p_{Y,1}(t_1, y_1), \end{aligned} \tag{C.23}$$

where we have used twice Bayes' theorem (C.13) and once the Markov property (C.22).

Remarks:

∗ The Markov property (C.22) characterizes the n-point densities for ordered times. The value for arbitrary t_1, t_2, ..., t_n follows from the necessary invariance [property (C.11b)] of $p_{Y,n}$ when two pairs (t_j, y_j) and (t_k, y_k) are exchanged.

∗ The single-time probability density $p_{Y,1}$ and the transition probability $p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)$ are not fully independent of each other, since they have to obey the obvious identity

$$p_{Y,1}(t_2, y_2) = \int p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)\, p_{Y,1}(t_1, y_1)\, \mathrm{d}y_1. \tag{C.24}$$

C.2.5 b Chapman–Kolmogorov equation

Integrating Eq. (C.23) over the intermediate value y_2 of the stochastic process, while taking into account the consistency condition (C.11c), gives

$$p_{Y,2}(t_1, y_1; t_3, y_3) = p_{Y,1}(t_1, y_1) \int p_{Y,1|1}(t_3, y_3 \,|\, t_2, y_2)\, p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)\, \mathrm{d}y_2,$$

where $t_1 < t_2 < t_3$.

[(61)] ... or *Markoff* in the older literature.

Dividing by $p_{Y,1}(t_1,y_1)$, one obtains the *Chapman*[bn]*–Kolmogorov*[bo] *equation*

$$p_{Y,1|1}(t_3,y_3\,|\,t_1,y_1) = \int p_{Y,1|1}(t_3,y_3\,|\,t_2,y_2)\, p_{Y,1|1}(t_2,y_2\,|\,t_1,y_1)\,\mathrm{d}y_2 \quad \text{for } t_1<t_2<t_3, \tag{C.25}$$

which gives a relation—a nonlinear integral-functional equation—fulfilled by the transition probability of a Markov process.

Reciprocally, two arbitrary nonnegative functions $p_{Y,1}(t_1,y_1)$, $p_{Y,1|1}(t_2,y_2\,|\,t_1,y_1)$ obeying the two identities (C.24) and (C.25) entirely define a unique Markov process.

Remark: The Chapman–Kolmogorov equation follows quite obviously when invoking the Markov property in the more generic relation (C.15), which holds for every stochastic process. In contrast to the latter, Eq. (C.25) is closed, i.e. does not depend on another function.

C.2.5 c Examples of Markov processes

Wiener process

The stochastic process defined by the "initial condition" $p_{Y,1}(t=0,y) = \delta(y)$ for $y \in \mathbb{R}$ and the Gaussian transition probability $(0<t_1<t_2)$

$$p_{Y,1|1}(t_2,y_2\,|\,t_1,y_1) = \frac{1}{\sqrt{2\pi(t_2-t_1)}}\exp\left[-\frac{(y_2-y_1)^2}{2(t_2-t_1)}\right] \tag{C.26a}$$

is called *Wiener*[bp] *process*.

One easily checks that the transition probability (C.26a) satisfies the Chapman–Kolmogorov equation (C.25), and that the probability density at time $t>0$ is given by

$$p_{Y,1}(t,y) = \frac{1}{\sqrt{2\pi t}}\,\mathrm{e}^{-y^2/2t}. \tag{C.26b}$$

The Wiener process is obviously not a stationary process, since for instance the second moment $\langle [Y(t)]^2\rangle = t$ depends on time.

Remark: The single-time probability density (C.26b) is solution of the diffusion equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial y^2} \tag{C.27}$$

with diffusion coefficient $D=\frac{1}{2}$.

Poisson process

Consider now the integer-valued stochastic process $Y(t)$ defined by the Poisson-distributed [cf. Eq. (B.10)] transition probability $(0\le t_1\le t_2)$

$$p_{Y,1|1}(t_2,n_2\,|\,t_1,n_1) = \frac{(t_2-t_1)^{n_2-n_1}}{(n_2-n_1)!}\,\mathrm{e}^{-(t_2-t_1)} \quad \text{for } n_2\ge n_1 \tag{C.28}$$

and 0 otherwise, and by the single-time probability density $p_{Y,1}(t=0,n) = \delta_{n,0}$. That is, a realization $y(t)$ is a succession of unit steps taking place at arbitrary instants, whose number between two given times t_1, t_2 obeys a Poisson distribution with parameter t_2-t_1.

$Y(t)$ is a non-stationary Markov process, called *Poisson process*.

Remark: In both Wiener and Poisson processes, the probability density of the *increment* (y_2-y_1 resp. n_2-n_1) between two successive instants t_1, t_2 only depends on the time difference t_2-t_1, not on t_1 (or t_2) alone. Such increments are called *stationary*. Since in addition successive increments are independent, both processes are instances of *Lévy*[bq] *processes*.

[bn] S. Chapman, 1888–1970 [bo] A. N. Kolmogorov, 1903–1987 [bp] N. Wiener, 1894–1964 [bq] P. Lévy, 1886–1971

C.2.5 d Stationary Markov processes

An interesting case in physics is that of stationary Markov processes. For such processes, the transition probability $p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)$ only depends on the time difference $\tau \equiv t_2 - t_1$, which is hereafter reflected in the use of the special notation

$$\mathcal{T}_{Y;\tau}(y_2 \,|\, y_1) \equiv p_{Y,1|1}(t_1+\tau, y_2 \,|\, t_1, y_1). \tag{C.29}$$

Using this notation, the Chapman–Kolmogorov equation (C.25) takes the form (both τ and τ' are taken to be nonnegative)

$$\boxed{\mathcal{T}_{Y;\tau+\tau'}(y_3 \,|\, y_1) = \int \mathcal{T}_{Y;\tau'}(y_3 \,|\, y_2)\, \mathcal{T}_{Y;\tau}(y_2 \,|\, y_1)\, \mathrm{d}y_2.} \tag{C.30}$$

If a Markov process is also stationary, the single-time probability density $p_{Y,1}(y)$ does not depend on time. Invoking a setup in which the probability density would first be time-dependent, i.e. in which the stochastic process Y is not (yet) stationary, $p_{Y,1}$ characterizes the large-time "equilibrium" distribution, reached after a sufficiently large τ, irrespective of the "initial" distribution $y(t)$ at some time $t = t_0$. Taking as initial condition $p_{Y,1}(t{=}t_0, y) = \delta(y - y_0)$, where y_0 is arbitrary, one finds

$$p_{Y,1}(y) = \lim_{\tau\to+\infty} \mathcal{T}_{Y;\tau}(y \,|\, y_0).$$

This follows from the identities

$$p_{Y,1}(t_0+\tau, y) = \int p_{Y,2}(t_0+\tau, y; t_0, y')\,\mathrm{d}y' = \int p_{Y,1|1}(t_0+\tau, y \,|\, t_0, y')\, p_{Y,1}(t_0, y')\,\mathrm{d}y'$$
$$= \int \mathcal{T}_{Y;\tau}(y \,|\, y')\, p_{Y,1}(t_0, y')\,\mathrm{d}y',$$

which with the assumed initial distribution $p_{Y,1}(t_0, y')$ gives the result. □

Ornstein–Uhlenbeck process

An example of stationary Markov process is the *Ornstein*[(br)]*–Uhlenbeck*[(bs)] *process* [42] defined by the (time-independent) single-time probability density

$$p_{Y,1}(y) = \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-y^2/2} \tag{C.31a}$$

and the transition probability

$$\mathcal{T}_{Y;\tau}(y_2 \,|\, y_1) = \frac{1}{\sqrt{2\pi(1-\mathrm{e}^{-2\tau})}} \exp\left[-\frac{(y_2 - y_1\mathrm{e}^{-\tau})^2}{2(1-\mathrm{e}^{-2\tau})}\right]. \tag{C.31b}$$

One can show that the Ornstein–Uhlenbeck process is also Gaussian and that its autocorrelation function is $\kappa(\tau) = \mathrm{e}^{-\tau}$.

The Doob's[(bt)] theorem actually states that the Ornstein–Uhlenbeck process is, up to trivial scalings or translations of the time argument, the only process which is Markovian, Gaussian and stationary.

C.2.5 e Master equation for a Markov process

Exercise 25!!!

[(br)] L. Ornstein, 1880–1941 [(bs)] G. Uhlenbeck, 1900–1988 [(bt)] J. L. Doob, 1910–2004