

Tutorial sheet 9

Discussion topic: Langevin model of Brownian motion

17. Characteristic functional of a stochastic process

In the lecture, the characteristic functional associated with a stochastic process $Y_X(t)$ has been defined as

$$G_Y[k(t)] \equiv \left\langle \exp \left[i \int k(t) Y(t) dt \right] \right\rangle,$$

with $k(t)$ a test function.

i. Characteristic functional and moments

Expand the exponential in power series of k and express the characteristic functional in terms of the n -time moments. How would you write the moment $\langle Y(t_1)Y(t_2) \cdots Y(t_n) \rangle$ as function of $G_Y[k(t)]$?

ii. Gaussian process

Give the characteristic functional associated with a Gaussian process.

18. Vibrating string

Consider a weightless elastic strings, whose extremities are fixed at points $x = 0$ and $x = L$ along some x -axis. Let $y(x)$ denote the displacement of the string transverse to this axis—for the sake of simplicity, we can assume that this displacement is one-dimensional—at position x . For small displacements, one can show that the elastic energy associated with a given profile $y(x)$ reads

$$E[y(x)] = \int_0^L \frac{1}{2} k \left[\frac{dy(x)}{dx} \right]^2 dx, \quad (1)$$

with k a positive constant.

When the string undergoes thermal fluctuations, induced by its environment at temperature T , $y(x)$ becomes a random function (of position, instead of time), where one expects that the probability for a given $y(x)$ should be proportional to $e^{-\beta E[y(x)]}$ with $\beta = 1/k_B T$. Here, we wish to consider a discretized version of the problem and view $y(x)$ as the realization of a stochastic function $Y(x)$.

i. Let $n \in \mathbb{N}$. Consider n points $0 < x_1 < x_2 < \cdots < x_n < L$ and let y_j be the displacement of the string at point x_j . Write down the energy of the string, assuming that it is straight between two successive points x_j, x_{j+1} .

Hint: For the sake of brevity, one can introduce the notations $x_0 = 0, x_{n+1} = L, y_0 = y_{n+1} = 0$.

ii. We introduce the n -point probability density

$$p_n(x_1, y_1; \dots; x_n, y_n) = \sqrt{\frac{2\pi L}{k\beta}} \prod_{j=0}^n \sqrt{\frac{k\beta}{2\pi(x_{j+1} - x_j)}} \exp \left[-\frac{k\beta}{2} \frac{(y_{j+1} - y_j)^2}{x_{j+1} - x_j} \right],$$

which for large n , agrees with the anticipated factor $e^{-\beta E[y(x)]}$ (are you convinced of that?).

Show that the various p_n satisfy the 4 properties given in the lecture. Write down the single-point and two-point averages $\langle Y(x_1) \rangle$ and $\langle Y(x_1)Y(x_2) \rangle$, as well as the autocorrelation function. Which properties does the process possess?

19. Examples of Markov processes

i. Wiener process

The stochastic process defined by the “initial condition” $p_{Y,1}(t=0, y) = \delta(y)$ for $y \in \mathbb{R}$ and the transition probability ($0 < t_1 < t_2$)

$$p_{Y,1|1}(t_2, y_2 | t_1, y_1) = \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \exp \left[-\frac{(y_2 - y_1)^2}{2(t_2 - t_1)} \right]$$

is called *Wiener process*.

Check that this transition probability obeys the Chapman–Kolmogorov equation, and that the probability density at time $t > 0$ is given by

$$p_{Y,1}(t, y) = \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t}.$$

Remark: Note that the above single-time probability density is solution of the diffusion equation

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2 f}{\partial y^2}$$

with diffusion coefficient $D = \frac{1}{2}$.

ii. Ornstein–Uhlenbeck process

The so-called *Ornstein–Uhlenbeck process* is defined by the time-independent single-time probability density

$$p_{Y,1}(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

and the transition probability

$$\mathcal{T}_{Y;\tau}(y_2 | y_1) = \frac{1}{\sqrt{2\pi(1 - e^{-2\tau})}} \exp \left[-\frac{(y_2 - y_1 e^{-\tau})^2}{2(1 - e^{-2\tau})} \right].$$

Check that this transition probability fulfills the Chapman–Kolmogorov equation, so that the Ornstein–Uhlenbeck process is Markovian. Show that the process is also Gaussian [*hint*: exercise 17.ii], stationary, and that its autocorrelation function is $\kappa(\tau) = e^{-\tau}$.

Merry Christmas and a happy New Year!