Tutorial sheet 8

Discussion topic: Stochastic processes

15. Electrical conductivity in a magnetic field: Hall effect

This exercise is a sequel to exercise 14 from the previous tutorial sheet. We again consider the problem of electric conduction in a metal subject to an electric field $\vec{\mathscr{E}}$ and a magnetic field $\vec{\mathscr{B}}$. The conductor is a rectangular parallelepiped, with its sides along the coordinate axes. Let L denote the length of the side parallel to the x-axis, l the length of the side along the y-axis, and d the length of the side along the z-axis.

i. Drude–Lorentz model

To model the effect of collisions in a simple way, one introduces an "average equation of motion" for the conduction electrons—i.e., an evolution equation for their average velocity $\langle\vec{v}\rangle$

$$\frac{\mathrm{d}\langle\vec{v}\rangle}{\mathrm{d}t} = -\frac{\langle\vec{v}\rangle}{\tau_\mathrm{r}} - \frac{e}{m}\big(\vec{\mathscr{E}} + \langle\vec{v}\rangle\times\vec{\mathscr{B}}\big).$$

Give a physical interpretation for this equation. Check that in the stationary regime one has

$$\langle v_x\rangle = -\frac{e\tau_\mathrm{r}}{m}\,\mathscr{E}_x - \omega\tau_\mathrm{r}\langle v_y\rangle\,, \qquad \langle v_y\rangle = -\frac{e\tau_\mathrm{r}}{m}\,\mathscr{E}_y + \omega\tau_\mathrm{r}\langle v_x\rangle\,,$$

with ω the Larmor frequency defined in exercise 14. Show that if one takes $\tau_\mathrm{r} = \tau_\mathrm{F}$, one recovers the same expression for the conductivity tensor as in exercise 14.

ii. Calculate in terms of $\mathscr{E}_x$ the value $\mathscr{E}_\mathrm{H}$ of $\mathscr{E}_y$ which cancels $(J_{\mathrm{el.}})_y$. Verify that the transport of electrons in that situation is the same as in the case $\vec{\mathscr{B}} = \vec{0}$, in other words $(J_{\mathrm{el.}})_x = \sigma_{\mathrm{el.}}\mathscr{E}_x$. The field intensity $\mathscr{E}_\mathrm{H}$ is called *Hall field*, and the *Hall resistance* is defined by

$$R_\mathrm{H} \equiv \frac{V_\mathrm{H}}{I}$$

where V_H is the *Hall voltage*, $V_\mathrm{H}/l = \mathscr{E}_\mathrm{H}$, and I the total electric current along the x direction. Show that R_H is given by

$$R_\mathrm{H} = \frac{\mathscr{B}}{nde},$$

with n the density of conduction electrons. By noting that R_H is independent of the relaxation time, find its expression using an elementary argument.

16. Bulk viscosity

In a hydrodynamic model, the (conserved) stress-energy tensor for a fluid obeying the Navier–Stokes equation can be written as

$$T_{ij}(t,\vec{r}) = \mathcal{P}(t,\vec{r})\delta_{ij} + \pi_{ij}^{\mathrm{shear}}(t,\vec{r}) + \Pi(t,\vec{r})\delta_{ij},$$

where $\mathcal{P}$ is the equilibrium pressure, $\pi_{ij}^{\mathrm{shear}}$ the traceless shear stress tensor—which depends on the first derivatives of the flow velocity $\vec{\mathsf{v}}(t,\vec{r})$—, and Π the bulk pressure. The latter is of the form $\Pi = \zeta\vec{\nabla}\cdot\vec{\mathsf{v}}$, with ζ the bulk viscosity. Thus, ζ measures the size of the deviation [in compressible ($\vec{\nabla}\cdot\vec{\mathsf{v}}\neq 0$) flows] of the trace $\sum_i T_{ii}$ of the stress-energy tensor from the equilibrium value $3\mathcal{P}$.

Microscopically, the stress-energy tensor for a gas of non-interacting particles is related to the phase space density $f(t,\vec{r},\vec{p})$ through

$$T_{ij}(t,\vec{r}) = \int p_i p_j f(t,\vec{r},\vec{p})\,\frac{\mathrm{d}^3\vec{p}}{E_{\vec{p}}},$$

where $E_{\vec{p}}$ denotes the energy of a particle with momentum $\vec{p}$. Similarly, the energy density e—which is related to the equilibrium pressure over the equation of state of the fluid—is given by

$$e(t,\vec{r}) = \int E_{\vec{p}}\, f(t,\vec{r},\vec{p})\, \mathrm{d}^3\vec{p}.$$

i. Gas of massless particles

For a noninteracting gas of massless particles, the equation of state reads $e = 3\mathcal{P}$.[1] Using the known relation for $E_{\vec{p}}$ as a function of momentum, check that, irrespective of the expression for f, the trace of the stress-energy tensor simply equals e. What does this mean for the bulk viscosity ζ?

ii. Nonrelativistic ideal gas

The internal energy U and pressure $\mathcal{P}$ of a nonrelativistic ideal gas occupying a volume $\mathcal{V}$ obey the relation $U = \frac{3}{2}\mathcal{P}\mathcal{V}$.[1] Using the nonrelativistic relation for $E_{\vec{p}}$ and appropriate approximations, compute the trace of the stress-energy tensor and express it as a (simple) function of the energy density e and the particle number density n defined as the integral over momenta of the phase space density. What does this again give for the bulk viscosity ζ?

Remark: Since no knowledge of the phase space density f was actually needed in the above derivations, the systems need not be equilibrated for the result(s) to hold.

[1]Check your favorite course on equilibrium statistical physics and thermostatics!