Tutorial sheet 7

Discussion topic: Relativistic Boltzmann kinetic equation

14. Electrical conductivity in a magnetic field[1]

We consider the problem of electric conduction in a metal subject to an electric field $\vec{\mathscr{E}}$ and a magnetic field $\vec{\mathscr{B}}$. The conduction electrons (mass m, charge $-e$) form a non-relativistic, highly degenerate ideal Fermi gas obeying the kinetic Lorentz equation (cf. Tutorial sheet 6) in the presence of an external force $\vec{F}(\vec{r})$, which is here simply the Lorentz force. We assume that the various densities are uniform and stationary: $f(t,\vec{r},\vec{p}) = f(\vec{p})$, where $\vec{p}$ denotes the linear momentum, and $n(t,\vec{r}) = n$, which leads to simplifications in the left-hand side of the Lorentz equation. In addition, we assume that the equilibrium distribution $f_{\rm eq.}$ is a function of energy only: $f^{(0)}(\vec{p}) = f^{(0)}(\varepsilon \equiv \vec{p}^2/2m_{\rm e})$.

i. We first take $\vec{\mathscr{B}} = \vec{0}$. Calculate $\delta f \equiv f - f^{(0)}$ and show that the electric current density $\vec{J}_{\rm el.}$ is given in the relaxation time approximation by

$$\vec{J}_{\rm el.} = -e^2 \int \tau_{\rm r}(|\vec{p}|)(\vec{v}\cdot\vec{\mathscr{E}})\,\vec{v}\,\frac{{\rm d}f^{(0)}}{{\rm d}\varepsilon}\,{\rm d}^3\vec{p}.$$

If $f^{(0)}$ is the Fermi distribution at $T = 0$

$$f^{(0)}(\varepsilon) = \frac{2}{(2\pi\hbar)^3}\Theta(\varepsilon_{\rm F} - \varepsilon)$$

with $\varepsilon_{\rm F}$ the Fermi energy, show that the electrical conductivity $\sigma_{\rm el.}$ is given by

$$\sigma_{\rm el.} = \frac{n\,e^2}{m}\tau_{\rm F}$$

where $\tau_{\rm F} \equiv \tau_{\rm r}(p_{\rm F})$, with $p_{\rm F}$ the Fermi momentum.

ii. We now take $\vec{\mathscr{B}} \neq \vec{0}$. How is the electrical conductivity modified if the applied $\vec{\mathscr{B}}$ field is parallel to $\vec{\mathscr{E}}$?

Consider then the case where the electric field is in the xy-plane and the magnetic field along the z-axis, $\vec{\mathscr{B}} = \mathscr{B}\,\vec{\rm e}_z$ with $\mathscr{B} > 0$. Show that the Lorentz equation in the relaxation time approximation becomes

$$-e\vec{v}\cdot\vec{\mathscr{E}}\,\frac{{\rm d}f^{(0)}}{{\rm d}\varepsilon} - e\big(\vec{v}\times\vec{\mathscr{B}}\big)\cdot\vec{\nabla}_{\vec{p}}\,\delta f = -\frac{\delta f}{\tau_r(|\vec{p}|)}.$$

We look for a solution of the form

$$\delta f = -\vec{v}\cdot\vec{C}\,\frac{{\rm d}f^{(0)}}{{\rm d}\varepsilon}$$

where $\vec{C}$ is a vector, function of $\vec{\mathscr{E}}$ and $\vec{\mathscr{B}}$ but independent of $\vec{v}$, to be determined. What should $\vec{C}$ be when $\vec{\mathscr{B}} = \vec{0}$? when $\vec{\mathscr{E}} = \vec{0}$? For the latter case, estimate first the average magnetic force on the electrons.

iii. Show that $\vec{C}$ satisfies the equation

$$-e\vec{\mathscr{E}} + \vec{\omega}\times\vec{C} = \frac{\vec{C}}{\tau_r(|\vec{p}|)},$$

with $\vec{\omega} = \omega\,\vec{\rm e}_z$, where $\omega \equiv e\mathscr{B}/m_{\rm e}$ is the Larmor frequency. Justify that $\vec{C}$ is necessarily of the form

[1] This exercise was shamelessly stolen from the book *Equilibrium and non-equilibrium statistical thermodynamics* by M. Le Bellac *et al.*

$\vec{C} = \alpha\vec{\mathscr{E}} + \delta\vec{\mathscr{B}} + \gamma\vec{\mathscr{B}} \times \vec{\mathscr{E}}$, where α, δ, γ are real numbers. Find the expression for $\vec{C}$ and show that

$$\delta f = \frac{e\tau_{\rm r}}{1+\omega^2\tau_{\rm r}^2}\big(\vec{\mathscr{E}} + \tau_{\rm r}\vec{\omega}\times\vec{\mathscr{E}}\big)\cdot\vec{v}\,\frac{{\rm d}f^{(0)}}{{\rm d}\varepsilon}.$$

iv. Calculate the electric current and the components $(\sigma_{\rm el.})_{ij}$, $i, j = x, y$ of the electrical conductivity tensor. Verify that $(\sigma_{\rm el.})_{xy} = -(\sigma_{\rm el.})_{yx}$ and comment on this relation.