Tutorial sheet 10

Discussion topic: Fokker–Planck equation

20. Fokker–Planck equation as approximation to the master equation

One can show that the evolution of the probability density $p_{Y,1}(\tau, y)$ of an homogeneous¹ Markov process Y(t) is governed by the so-called *master equation*

$$\frac{\partial p_{Y,1}(\tau,y)}{\partial \tau} = \int \left[\Gamma(y \,|\, y') \, p_{Y,1}(\tau,y') - \Gamma(y' \,|\, y) \, p_{Y,1}(\tau,y) \right] \mathrm{d}y',$$

where $\Gamma(y | y')$ denotes the transition rate from y' to y. In many situations, only small jumps $w \equiv y - y'$ occur. Rewriting $W(y', w) \equiv \Gamma(y' + w | y')$, this means that W(y', w) is a sharply peaked function of w, while it varies slowly with y'. Assuming that $p_{Y,1}(\tau, y)$ also varies slowly with y, show that a Taylor expansion of the master equation up to second order yields the Fokker–Planck equation

$$\frac{\partial p_{Y,1}(\tau,y)}{\partial \tau} = -\frac{\partial}{\partial y} \left[M_1(y) p_{Y,1}(\tau,y) \right] + \frac{1}{2} \frac{\partial^2}{\partial y^2} \left[M_2(y) p_{Y,1}(\tau,y) \right] \quad \text{with} \quad M_n(y) \equiv \int w^n W(y,w) \, \mathrm{d}w.$$

21. Another view of the Fokker–Planck equation in one dimension

Consider an arbitrary one-dimensional Markovian process X(t), taking its values in a real interval [a, b], and such that the corresponding first two coefficients $M_1(t, x)$, $M_2(t, x)$ in the Kramers–Moyal expansion are actually independent of time.

i. Stationary solutions

Recall the form of the Fokker–Planck equation. Assuming that there is no flow of probability across the boundaries x = a and x = b ("reflecting boundary conditions"), write down the differential equation obeyed by the stationary solution $p_{X,1}^{\text{st.}}(x)$ to the Fokker–Planck equation. Show that

$$p_{X,1}^{\text{st.}}(x) = \frac{C}{M_2(x)} \exp\left[2\int_a^x \frac{M_1(x')}{M_2(x')} \,\mathrm{d}x'\right],\tag{1}$$

where C is a constant which need not be computed. Why is this solution unique?

ii. Transforming the Fokker–Planck equation

Assume now that M_2 is actually constant. Let $V(x) \equiv \frac{1}{2}[M_1(x)]^2 + \frac{M_2}{2}\frac{\mathrm{d}M_1(x)}{\mathrm{d}x}$.

Perform the change of unknown function $p_{X,1}(t,x) = [p_{X,1}^{\text{st.}}(x)]^{1/2} \psi(t,x)$ in the Fokker–Planck equation, where $p_{X,1}^{\text{st.}}(x)$ is the stationary solution (1), and deduce the equation obeyed by $\psi(t,x)$. What do you recognize?

In the new language you just found, to which known problem is that of the Fokker–Planck equation for the Ornstein–Uhlenbeck process $[M_1(x) = \gamma x, M_2 = D, x \in \mathbb{R}]$ equivalent?

¹... which means that the transition probability $p_{Y,1|1}(t_2, y_2|t_1, y_1)$ actually only depends on the time difference $t_2 - t_1$.