Tutorial sheet 8

The exercise marked with a star is homework.

Discussion topic: Fokker–Planck equation

21. Brownian motion in the viscous limit

Consider the process of Brownian motion in the viscous limit, described by the stochastic differential equation

$$M\gamma \frac{\mathrm{d}x(t)}{\mathrm{d}t} = F_{\mathrm{L}}(t) \tag{1}$$

where $F_L(t)$ is a stochastic force with vanishing expectation value $\langle F_{\mathrm{L}}(t)\rangle = 0$ and memoryless auto-correlation function $\langle F_{\mathrm{L}}(t)F_{\mathrm{L}}(t')\rangle = 2DM^2\gamma^2\delta(t-t')$.

i. Show that Eq. (1) can be obtained from the Langevin equation introduced in the lecture in the combined limit $M \to 0$, $\gamma \to \infty$ with the product $M\gamma$ held constant.

ii. Show that the Fokker–Planck equation for the probability distribution $f(t,x) = p_1(t,x)$ associated with Eq. (1) is

$$\frac{\partial f(t,x)}{\partial t} = D\frac{\partial^2 f(t,x)}{\partial x^2}.$$

Why is this result satisfactory?

iii. Calculate the average displacement $\langle x(t) - x(t_0)\rangle$ and its variance $\langle [x(t) - x(t_0)]^2\rangle$ for $t > t_0$ and comment on the similarities and differences between the results obtained for the simplified model (1) and the Langevin model discussed in the lecture.

Remark: Equation (1) is actually the description of Brownian motion proposed by A. Einstein and M. Smoluchowski (1872–1917) in 1905–06, before the introduction of the Langevin model in 1908.

*22. Fokker–Planck equation as approximation to the master equation

In exercise **20.**, it was found that the evolution of the probability density $p_{Y,1}(\tau, y)$ of an homogeneous Markov process $Y(t)$ is governed by the so-called *master equation*

$$\frac{\partial p_{Y,1}(\tau,y)}{\partial \tau} = \int \left[\Gamma(y\,|\,y')\,p_{Y,1}(\tau,y') - \Gamma(y'\,|\,y)\,p_{Y,1}(\tau,y)\right]\mathrm{d}y',$$

where $\Gamma(y\,|\,y')$ denotes the transition rate from y' to y. In many situations, only small jumps $w \equiv y - y'$ occur. Rewriting $W(y',w) \equiv \Gamma(y'+w\,|\,y')$, this means that $W(y',w)$ is a sharply peaked function of w, while it varies slowly with y'. Assuming that $p_{Y,1}(\tau,y)$ also varies slowly with y, show that a Taylor expansion of the master equation up to second order yields the Fokker–Planck equation

$$\frac{\partial p_{Y,1}(\tau,y)}{\partial \tau} = -\frac{\partial}{\partial y}\left[M_1(y)p_{Y,1}(\tau,y)\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[M_2(y)p_{Y,1}(\tau,y)\right] \quad \text{with} \quad M_n(y) \equiv \int w^n\,W(y,w)\,\mathrm{d}w.$$