Tutorial sheet 7

The exercises marked with a star are homework.

Discussion topics:

– Langevin model of Brownian motion
– Markov processes; Fokker–Planck equation

18. Vibrating string

Consider a weightless elastic string, whose extremities are fixed at points $x = 0$ and $x = L$ along the x-axis. Let $y(x)$ denote the displacement of the string transverse to this axis—for the sake of simplicity, we can assume that this displacement is one-dimensional—at position x. For small displacements, one can show that the elastic energy associated with a given profile $y(x)$ reads

$$E[y(x)] = \int_0^L \frac{k}{2}\left[\frac{\mathrm{d}y(x)}{\mathrm{d}x}\right]^2 \mathrm{d}x, \tag{1}$$

with k a positive constant.

When the string undergoes thermal fluctuations, induced by its environment at temperature T, $y(x)$ becomes a random function (of position, instead of time), where one expects that the probability for a given $y(x)$ should be proportional to $\mathrm{e}^{-\beta E[y(x)]}$ with $\beta = 1/k_BT$. Here, we wish to consider a discretized version of the problem and view $y(x)$ as the realization of a stochastic function $Y(x)$.

i. Let $n \in \mathbb{N}$. Consider n points $0 < x_1 < x_2 < \cdots < x_n < L$ and let y_j be the displacement of the string at point x_j. Write down the energy of the string, assuming that it is straight between two successive points x_j, x_{j+1}.

Hint: For the sake of brevity, one can introduce the notations $x_0 = 0$, $x_{n+1} = L$, $y_0 = y_{n+1} = 0$.

ii. We introduce the n-point probability density

$$p_n(x_1, y_1; \ldots; x_n, y_n) = \sqrt{\frac{2\pi L}{k\beta}} \prod_{j=0}^{n} \sqrt{\frac{k\beta}{2\pi(x_{j+1} - x_j)}} \exp\left[-\frac{k\beta}{2}\frac{(y_{j+1} - y_j)^2}{x_{j+1} - x_j}\right],$$

which for large n, agrees with the anticipated factor $\mathrm{e}^{-\beta E[y(x)]}$ (are you convinced of that?).

Show that the various p_n satisfy the 4 properties of n-point densities given in the lecture. Write down the single-point and two-point averages $\langle Y(x_1)\rangle$ and $\langle Y(x_1)Y(x_2)\rangle$, as well as the autocorrelation function. Which properties does the process possess?

*19. Another view of the Fokker–Planck equation in one dimension

Consider an arbitrary one-dimensional Markovian process $X(t)$, taking its values in a real interval $[a, b]$, and such that the corresponding first two coefficients $\mathcal{M}_1(t, x)$, $\mathcal{M}_2(t, x)$ in the Kramers–Moyal expansion are actually independent of time.

i. Stationary solutions

Recall the form of the Fokker–Planck equation. Assuming that there is no flow of probability across the boundaries $x = a$ and $x = b$ (“reflecting boundary conditions”), write down the differential equation obeyed by the stationary solution $p^{\text{st.}}_{X,1}(x)$ to the Fokker–Planck equation. Show that

$$p^{\text{st.}}_{X,1}(x) = \frac{C}{\mathcal{M}_2(x)} \exp\left[2\int_a^x \frac{\mathcal{M}_1(x')}{\mathcal{M}_2(x')}\,\mathrm{d}x'\right], \tag{2}$$

where C is a constant which need not be computed. Why is this solution unique?

ii. Transforming the Fokker–Planck equation

Assume now that $\mathcal{M}_2$ is actually constant. Let $V(x) \equiv \frac{1}{2}[\mathcal{M}_1(x)]^2 + \frac{\mathcal{M}_2}{2}\frac{\mathrm{d}\mathcal{M}_1(x)}{\mathrm{d}x}$.

Perform the change of unknown function $p_{X,1}(t,x) = [p^{\mathrm{st.}}_{X,1}(x)]^{1/2}\psi(t,x)$ in the Fokker–Planck equation, where $p^{\mathrm{st.}}_{X,1}(x)$ is the stationary solution (2), and deduce the equation obeyed by $\psi(t,x)$. What do you recognize?

In the new language you just found, to which known problem is that of the Fokker–Planck equation for the Langevin model [$\mathcal{M}_1(x) = \gamma x$, $\mathcal{M}_2 = D$, $x \in \mathbb{R}$] equivalent?

*20. Master equation for Markov processes

The purpose of this exercise is to derive a linear integrodifferential equation—equivalent to the Chapman–Kolmogorov equation—for the transition probability and the single-time density of an (almost) arbitrary *homogeneous* Markov process $Y(t)$, i.e. a process for which the probability transition $p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)$ only depends on the time difference $\tau \equiv t_2 - t_1$. In analogy with stationary processes, the latter will be denoted by $\mathcal{T}_{Y;\tau}(y_2 \,|\, y_1)$.

We assume that for time differences τ much smaller than some time scale τ_{c}, the transition probability is of the form

$$\mathcal{T}_{Y;\tau}(y_2 \,|\, y_1) = [1 - \gamma(y_1)\tau]\,\delta(y_2 - y_1) + \Gamma(y_2 \,|\, y_1)\tau + o(\tau), \tag{3a}$$

where $o(\tau)$ denotes a term which is much smaller than τ in the limit $\tau \to 0$. The nonnegative quantity $\Gamma(y_2 \,|\, y_1)$ is the transition rate from y_1 to y_2, and $\gamma(y_1)$ is its integral over y_2

$$\gamma(y_1) = \int \Gamma(y_2 \,|\, y_1)\,\mathrm{d}y_2. \tag{3b}$$

i. Compute the integral of the transition probability $\mathcal{T}_{Y;\tau}(y_2 \,|\, y_1)$ over final states y_2.

ii. Master equation

Starting from the Chapman–Kolmogorov equation

$$\mathcal{T}_{Y;\tau+\tau'}(y_3 \,|\, y_1) = \int \mathcal{T}_{Y;\tau'}(y_3 \,|\, y_2)\,\mathcal{T}_{Y;\tau}(y_2 \,|\, y_1)\,\mathrm{d}y_2,$$

and assuming that $\tau' \ll \tau_{\mathrm{c}}$—note that no assumption on τ is needed—, show that after leaving aside a negligible term, one obtains

$$\mathcal{T}_{Y;\tau+\tau'}(y_3 \,|\, y_1) = \big[1 - \gamma(y_3)\,\tau'\big]\,\mathcal{T}_{Y;\tau}(y_3 \,|\, y_1) + \tau'\!\int \Gamma(y_3 \,|\, y_2)\,\mathcal{T}_{Y;\tau}(y_2 \,|\, y_1)\,\mathrm{d}y_2,$$

Check that this leads in the limit $\tau' \to 0$ to the integrodifferential equation

$$\frac{\partial \mathcal{T}_{Y;\tau}(y_3 \,|\, y_1)}{\partial \tau} = -\gamma(y_3)\mathcal{T}_{Y;\tau}(y_3 \,|\, y_1) + \int \Gamma(y_3 \,|\, y_2)\,\mathcal{T}_{Y;\tau}(y_2 \,|\, y_1)\,\mathrm{d}y_2,$$

and eventually, after invoking Eq. (3b) and relabeling the variables, to the *master equation*

$$\frac{\partial \mathcal{T}_{Y;\tau}(y \,|\, y_0)}{\partial \tau} = \int \big[\Gamma(y \,|\, y')\,\mathcal{T}_{Y;\tau}(y' \,|\, y_0) - \Gamma(y' \,|\, y)\,\mathcal{T}_{Y;\tau}(y \,|\, y_0)\big]\,\mathrm{d}y'. \tag{4}$$

Note that this evolution equation has the structure of a balance equation, with a gain term, involving the rate $\Gamma(y \,|\, y')$, and a loss term depending on the rate $\Gamma(y' \,|\, y)$.

iii. Evolution equation for the single-time probability density

Starting from the consistency condition

$$p_{Y,1}(\tau, y) = \int \mathcal{T}_{Y;\tau}(y \,|\, y_0)\,p_{Y,1}(t{=}0, y_0)\,\mathrm{d}y_0, \tag{5}$$

show that the above master equation leads to

$$\frac{\partial p_{Y,1}(\tau,y)}{\partial\tau} = \int \big[\Gamma(y\,|\,y')\,\mathcal{T}_{Y;\tau}(y'\,|\,y_0) - \Gamma(y'\,|\,y)\,\mathcal{T}_{Y;\tau}(y\,|\,y_0)\big]\,p_{Y,1}(t\!=\!0,y_0)\,\mathrm{d}y_0\,\mathrm{d}y'.$$

Check that this leads to the evolution equation

$$\frac{\partial p_{Y,1}(\tau,y)}{\partial\tau} = \int \big[\Gamma(y\,|\,y')\,p_{Y,1}(\tau,y') - \Gamma(y'\,|\,y)\,p_{Y,1}(\tau,y)\big]\,\mathrm{d}y', \tag{6}$$

which is formally identical to the master equation for $\mathcal{T}_{Y;\tau}$. How can you interpret this equation?