Tutorial sheet 6

The exercise marked with a star is homework.

16. A bizarre exercise with generating functions

In this exercise, you are to consider functions of a variable z, whose Taylor expansion involves graphic coefficients $(\bullet)$, $(\bullet\bullet)$, $(\bullet\bullet\bullet)$ and so on. One can multiply those coefficients using "normal" rules, such that each bullet $\bullet$ remains confined within its original subgraph, and that different subgraphs do not merge: for instance $(\bullet)^2 = (\bullet)(\bullet)$ is not the same as $(\bullet\bullet)$.

i. Let $f(z) \equiv (\bullet)\, z + (\bullet\bullet)\,\frac{z^2}{2!} + (\bullet\bullet\bullet)\,\frac{z^3}{3!} + (\bullet\bullet\bullet\bullet)\,\frac{z^4}{4!} + \cdots$.

Using the Taylor expansion of e^x for small x, compute $\exp[f(z)]$ to order $\mathcal{O}(z^4)$.

ii. Consider now graphs consisting of one, two, three, four, … bullets, that are now no longer enclosed in "connected groups": $\bullet$, $\bullet\,\bullet$, $\bullet\bullet\bullet$, $\bullet\bullet\bullet\bullet$, … . All bullets of a given (non-connected) graph are supposed to be distinguishable, i.e. can be designated with different labels, as e.g. (1,2,3) for the bullets of $\bullet\bullet\bullet$.

a) For each non-connected graph with $n = 2$, 3 or 4 bullets, find all possible ways of "decomposing" the graph by regrouping all its bullets in non-overlapping connected groups of $1 \leq m \leq n$ bullets. For $n = 1$, i.e. $\bullet$, there is a single possibility: $(\bullet)$.

Hint: Starting with $n = 3$, there might be different groupings with the same "topology", say for instance (for $n = 3$) with one pair and one single bullet: you may regroup these groupings — but do not forget their multiplicity, i.e. how many groupings have that topology.

b) Compare your "rewritings" of the disconnected graphs $\bullet\,\bullet$, $\bullet\bullet\bullet$, $\bullet\bullet\bullet\bullet$ in terms of connected subgraphs with the coefficients of z^2, z^3, z^4 of the function $\exp[f(z)]$ in question **i.** What do you notice?[1]

*17. Examples of Markov processes

The lecture introduced the so-called *Markov processes*, which are entirely determined by their single-time density $p_{Y,1}$ and their conditional probability density $p_{Y,1|1}$. The latter, which is referred to as *transition probability*, obeys the *Chapman–Kolmogorov equation*

$$p_{Y,1|1}(t_3, y_3 \,|\, t_1, y_1) = \int p_{Y,1|1}(t_3, y_3 \,|\, t_2, y_2)\, p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)\, \mathrm{d}y_2 \quad \text{for } t_1 < t_2 < t_3. \tag{1}$$

i. Wiener process

The stochastic process defined by the "initial condition" $p_{Y,1}(t\!=\!0, y) = \delta(y)$ for $y \in \mathbb{R}$ and the transition probability $(0 < t_1 < t_2)$

$$p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1) = \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \exp\left[-\frac{(y_2 - y_1)^2}{2(t_2 - t_1)}\right]$$

is called *Wiener process*.

Check that this transition probability obeys the Chapman–Kolmogorov equation, and that the probability density at time $t > 0$ is given by

$$p_{Y,1}(t, y) = \frac{1}{\sqrt{2\pi t}}\, e^{-y^2/2t}.$$

Remark: Note that the above single-time probability density is solution of the diffusion equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial y^2}$$

with diffusion coefficient $D = \frac{1}{2}$.

[1] You are free to go to graphs with 5 or 6 bullets if you cannot sleep at night!

ii. Ornstein–Uhlenbeck process

The so-called *Ornstein–Uhlenbeck process* is defined by the time-independent single-time probability density

$$p_{Y,1}(y) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-y^2/2}$$

and the transition probability ($\tau > 0$)

$$p_{Y,1|1}(t+\tau, y \,|\, t, y_0) = \frac{1}{\sqrt{2\pi(1-\mathrm{e}^{-2\tau})}}\,\exp\left[-\frac{(y-y_0\mathrm{e}^{-\tau})^2}{2(1-\mathrm{e}^{-2\tau})}\right].$$

a) Check that this transition probability fulfills the Chapman–Kolmogorov equation, so that the Ornstein–Uhlenbeck process is Markovian. Show that the process is also Gaussian, stationary, and that its autocorrelation function is $\kappa(\tau) = \mathrm{e}^{-\tau}$.

b) What is the large-τ limit of the transition probability? And its limit when τ goes to 0^+?

c) Viewing the above transition probability as a function of τ and y, can you find a partial differential equation, of which it is a (fundamental) solution?

Hint: Let yourself be inspired(?) by the remark at the end of question **i.**