Tutorial sheet 12

The exercises marked with a star are homework.

Discussion topic: Boltzmann equation: relaxation time approximation and its application to the computation of transport coefficients

31. Boltzmann gas in a harmonic trap

Consider a system of N neutral particles of mass m and let $\bar{\mathsf{f}}(t,\vec{r},\vec{p})$ denote its phase-space density.

i. Balance equation

Let $g(\vec{r},\vec{p})$ denote a dynamical quantity.

a) Denoting by $\mathcal{I}_{\rm coll.}$ the collision integral of the Boltzmann transport equation, show that the latter leads to the generic balance equation

$$\frac{\mathrm{d}\langle g\rangle}{\mathrm{d}t} - \big\langle \vec{v}\cdot\vec{\nabla}_{\vec{r}}\, g\big\rangle - \big\langle \vec{F}\cdot\vec{\nabla}_{\vec{p}}\, g\big\rangle = \big\langle g\,\bar{\mathsf{f}}^{-1}\mathcal{I}_{\rm coll.}\big\rangle, \tag{1}$$

where the angular brackets denote an average over position and momenta:

$$\langle g\rangle \equiv \frac{1}{N}\int g(\vec{r},\vec{p})\,\bar{\mathsf{f}}(t,\vec{r},\vec{p})\,\mathrm{d}^6\mathcal{V}.$$

Hint: You may use the fact that $\bar{\mathsf{f}}$ vanishes at "the edges" of phase space.

b) Explain why $\big\langle g\,\bar{\mathsf{f}}^{-1}\mathcal{I}_{\rm coll.}\big\rangle$ vanishes when g is of the form $g(\vec{r},\vec{p}) = a(\vec{r}) + \vec{b}(\vec{r})\cdot\vec{p} + c(\vec{r})\vec{p}^{\,2}$ with a, $\vec{b}$, c arbitrary functions of $\vec{r}$.

ii. Let the system be trapped in a harmonic potential, namely $\vec{F} = -\vec{\nabla}V$ with $V(\vec{r}) = \frac{1}{2}m\omega_0^2\vec{r}^{\,2}$.

a) Derive using Eq. (1) the coupled system of equations

$$\frac{\mathrm{d}\langle\vec{r}^{\,2}\rangle}{\mathrm{d}t} = 2\big\langle\vec{r}\cdot\vec{v}\big\rangle \quad , \quad \frac{\mathrm{d}\langle\vec{r}\cdot\vec{v}\rangle}{\mathrm{d}t} = \big\langle\vec{v}^{\,2}\big\rangle - \omega_0^2\big\langle\vec{r}^{\,2}\big\rangle \quad , \quad \frac{\mathrm{d}\langle\vec{v}^{\,2}\rangle}{\mathrm{d}t} = -2\omega_0^2\big\langle\vec{r}\cdot\vec{v}\big\rangle. \tag{2}$$

b) Find and discuss the solutions of the system (2) evolving in time like $\mathrm{e}^{\mathrm{i}\omega t}$.

The non-trivial behavior, referred to as "monopole oscillation" or "breathing mode", was observed for the first time a few years ago, in a gas of cold atoms.[1] Note that the collision integral does not appear in the equations (2), which means that they hold irrespective of whether there are inter-particle collisions or not. This is no longer true if the confining potential is not spherically symmetric—in which case the monopole oscillation couples to higher multipolar modes, which are damped by the collisions.[2]

*32. Electrical conductivity in a magnetic field[3]

We consider the problem of electric conduction in a metal subject to a constant and uniform electromagnetic field $(\vec{\mathscr{E}},\vec{\mathscr{B}})$. The conduction electrons (mass $m_{\rm e}$, charge $-e$) form a non-relativistic, highly degenerate ideal Fermi gas obeying the kinetic Lorentz equation (cf. Tutorial sheet 10) in the presence of an external force $\vec{F}$, which is here simply the Lorentz force. We assume that the various densities are uniform and stationary: $\bar{\mathsf{f}}(t,\vec{r},\vec{p}) = \bar{\mathsf{f}}(\vec{p})$, where $\vec{p}$ denotes the linear momentum, and $n(t,\vec{r}) = n$, which

[1] D. S. Lobser *et al.*, *Observation of a persistent non-equilibrium state in cold atoms*, Nature Phys. **11** (2015) 1009.

[2] The interested reader may have a look at D. Guéry-Odelin *et al.*, *Collective oscillations of a classical gas confined in harmonic traps*, Phys. Rev. A **60** (1999) 4851—note that there is a typo in Eq. (9). Further, very recent developments on the topic (requiring some knowledge of hydrodynamics) are reported in M. I. García de Soria *et al.*, *Fate of Boltzmann's Breathers: Stokes Hypothesis and Anomalous Thermalization*, Phys. Rev. Lett. **132** (2024) 027101.

[3] This exercise was shamelessly stolen from the textbook *Equilibrium and non-equilibrium statistical thermodynamics* by M. Le Bellac *et al.*

leads to simplifications on the left-hand side of the Lorentz equation. In addition, we assume that the local equilibrium distribution is a function of energy only: $\bar{\mathsf{f}}^{(0)}(\vec{p}) = \bar{\mathsf{f}}^{(0)}(\varepsilon)$ with $\varepsilon \equiv \vec{p}^{\,2}/2m_e$.

i. We first take $\vec{\mathscr{B}} = \vec{0}$. Calculate $\delta\bar{\mathsf{f}} \equiv \bar{\mathsf{f}} - \bar{\mathsf{f}}^{(0)}$ and show that the electric current density $\vec{J}_{\text{el.}}$ is given in the relaxation time approximation by

$$\vec{J}_{\text{el.}} = -e^2 \int \tau_{\text{r}}(\varepsilon)(\vec{v}\cdot\vec{\mathscr{E}})\vec{v}\, \frac{\mathrm{d}\bar{\mathsf{f}}^{(0)}}{\mathrm{d}\varepsilon}\, \frac{\mathrm{d}^3\vec{p}}{(2\pi\hbar)^3},$$

where τ_{r} is assumed to depend only on energy. Show that, if the local equilibrium distribution is the Fermi distribution (with 2 spin degrees of freedom) at $T = 0$, $\bar{\mathsf{f}}^{(0)}(\varepsilon) = 2\,\Theta(\varepsilon_{\text{F}} - \varepsilon)$ with ε_{F} the Fermi energy, then $\vec{J}_{\text{el.}}$ obeys Ohm's law with the electrical conductivity

$$\sigma_{\text{el.}} = \frac{n\,e^2}{m_e}\tau_{\text{F}}$$

where $\tau_{\text{F}} \equiv \tau_{\text{r}}(\varepsilon_{\text{F}})$.

ii. Let now $\vec{\mathscr{B}} \neq \vec{0}$. How is the electrical conductivity modified if $\vec{\mathscr{B}}$ is parallel to $\vec{\mathscr{E}}$?

iii. Consider the case where the electric field is in the xy-plane and the magnetic field along the z-axis, $\vec{\mathscr{B}} = \mathscr{B}\,\vec{\mathrm{e}}_z$ with $\mathscr{B} > 0$.

a) Show that the Lorentz equation in the relaxation time approximation becomes

$$-e\vec{v}\cdot\vec{\mathscr{E}}\,\frac{\mathrm{d}\bar{\mathsf{f}}^{(0)}}{\mathrm{d}\varepsilon} - e\big(\vec{v}\times\vec{\mathscr{B}}\big)\cdot\vec{\nabla}_{\vec{p}}\,\delta\bar{\mathsf{f}} = -\frac{\delta\bar{\mathsf{f}}}{\tau_r(\varepsilon)}.$$

b) We look for a solution of the form

$$\delta\bar{\mathsf{f}} = -\vec{v}\cdot\vec{C}\,\frac{\mathrm{d}\bar{\mathsf{f}}^{(0)}}{\mathrm{d}\varepsilon}$$

with $\vec{C}$ a vector, function of $\vec{\mathscr{E}}$ and $\vec{\mathscr{B}}$ but independent of $\vec{v}$, to be determined. What should $\vec{C}$ be when $\vec{\mathscr{B}} = \vec{0}$? when $\vec{\mathscr{E}} = \vec{0}$? For the latter case, estimate first the average magnetic force on the electrons.

c) Show that $\vec{C}$ satisfies the equation

$$-e\vec{\mathscr{E}} + \vec{\omega}\times\vec{C} = \frac{\vec{C}}{\tau_r(\varepsilon)},$$

with $\vec{\omega} = \omega\,\vec{\mathrm{e}}_z$, where $\omega \equiv e\mathscr{B}/m_e$ is the Larmor frequency of the electrons. Explain why $\vec{C}$ is necessarily of the form $\vec{C} = \alpha\vec{\mathscr{E}} + \delta\vec{\mathscr{B}} + \gamma\vec{\mathscr{B}}\times\vec{\mathscr{E}}$, with α, δ, γ real numbers. Find the expression of $\vec{C}$ and show that

$$\delta\bar{\mathsf{f}} = \frac{e\tau_{\text{r}}}{1+\omega^2\tau_{\text{r}}^2}\big(\vec{\mathscr{E}} + \tau_{\text{r}}\vec{\omega}\times\vec{\mathscr{E}}\big)\cdot\vec{v}\,\frac{\mathrm{d}\bar{\mathsf{f}}^{(0)}}{\mathrm{d}\varepsilon}.$$

d) Calculate the electric current and the components $(\sigma_{\text{el.}})_{ij}$, $i,j = x,y$ of the electrical conductivity tensor. Verify that $(\sigma_{\text{el.}})_{xy} = -(\sigma_{\text{el.}})_{yx}$ and comment on this relation.

*33. Electrical conductivity in a magnetic field: Hall effect

This exercise is a sequel to the previous one, yet tackles the problem differently. We again consider the problem of electric conduction in a metal. We now assume that the conductor subject to the electric and magnetic fields $\vec{\mathscr{E}}$, $\vec{\mathscr{B}}$ is a rectangular parallelepiped, with its sides along the coordinate axes. Let L, l, and d be the respective lengths of the sides parallel to the x-, y and z-directions.

i. Drude–Lorentz model

To model the effect of collisions in a simple way, one introduces an "average equation of motion" for the conduction electrons—i.e., an evolution equation for their average velocity $\langle\vec{v}\rangle$

$$\frac{\mathrm{d}\langle\vec{v}\rangle}{\mathrm{d}t} = -\frac{\langle\vec{v}\rangle}{\tau_{\text{r}}} - \frac{e}{m_e}\big(\vec{\mathscr{E}} + \langle\vec{v}\rangle\times\vec{\mathscr{B}}\big).$$

Give a physical interpretation for this equation. Check that in the stationary regime one has

$$\langle v_x \rangle = -\frac{e\tau_{\rm r}}{m_{\rm e}}\, \mathscr{E}_x - \omega\tau_{\rm r}\langle v_y \rangle\,, \qquad \langle v_y \rangle = -\frac{e\tau_{\rm r}}{m_{\rm e}}\, \mathscr{E}_y + \omega\tau_{\rm r}\langle v_x \rangle\,,$$

with ω the Larmor frequency defined in exercise **32**. Show that if one takes $\tau_{\rm r} = \tau_{\rm F}$, one recovers the same expression for the conductivity tensor as in exercise **32**.

ii. Calculate in terms of $\mathscr{E}_x$ the value $\mathscr{E}_{\rm H}$ of $\mathscr{E}_y$ which cancels $(J_{\rm el.})_y$. Verify that the transport of electrons in that situation is the same as in the case $\vec{\mathscr{B}} = \vec{0}$, in other words $(J_{\rm el.})_x = \sigma_{\rm el.}\mathscr{E}_x$. The field intensity $\mathscr{E}_{\rm H}$ is called *Hall field*, and the *Hall resistance* is defined by

$$R_{\rm H} \equiv \frac{V_{\rm H}}{I}$$

where $V_{\rm H}$ is the *Hall voltage*, $V_{\rm H}/l = \mathscr{E}_{\rm H}$, and I the total electric current along the x direction. Show that $R_{\rm H}$ is given by

$$R_{\rm H} = \frac{\mathscr{B}}{\mathscr{n}de},$$

with $\mathscr{n}$ the density of conduction electrons and $\mathscr{B} \equiv |\vec{\mathscr{B}}|$. By noting that $R_{\rm H}$ is independent of the relaxation time, find its expression using an elementary argument.