Tutorial sheet 11

The exercise marked with a star is homework.

Discussion topic: Boltzmann equation: balance equations, H-theorem

28. Free-streaming expansion

Consider a two-dimensional system of free-streaming identical particles (throughout the exercise we ignore the z-direction and the corresponding momentum). At time $t = 0$, the geometry of the system is characterized by its typical (squared) size $R^2 \equiv \langle x^2 + y^2 \rangle_0$ and its "ellipticity"

$$\epsilon_2(t\!=\!0) \equiv \frac{\langle y^2 - x^2 \rangle_0}{\langle x^2 + y^2 \rangle_0}, \tag{1}$$

where (x, y) are Cartesian coordinates — with the origin at the center of the system — and $\langle \cdots \rangle_t$ denotes an average with the single-particle phase distribution $\bar{\mathsf{f}}$ at time t:

$$\langle g(x, y, p_x, p_y) \rangle_t \equiv \frac{\displaystyle\int g(x, y, p_x, p_y)\, \bar{\mathsf{f}}(t, x, y, p_x, p_y)\, \mathrm{d}^4\mathcal{V}}{\displaystyle\int \bar{\mathsf{f}}(t, x, y, p_x, p_y)\, \mathrm{d}^4\mathcal{V}}. \tag{2}$$

At $t = 0$ the particle momenta — and velocities — are assumed to be distributed isotropically in the (x, y)-plane, with $\langle v_x^2 \rangle_0 = \langle v_y^2 \rangle_0 \equiv \langle \boldsymbol{v}^2 \rangle_0 / 2$

As time goes by, the system evolves, and in particular its typical size and ellipticity are changing: compute $\epsilon_2(t)$ at time t.

Hint: Remember the generic equation obeyed by a free-streaming solution (exercise **26.**)! Begin with the time-dependence of $\langle x^2 \rangle_t$ and $\langle y^2 \rangle_t$. In the end, $\epsilon_2(t)$ can be expressed in terms of $\epsilon_2(0)$, R^2 and $\langle \boldsymbol{v}^2 \rangle_0$ — and naturally t.

Ask your tutor to explain you the (possible) relevance of this exercise!

29. Entropy conservation in classical dynamics

Starting from the Liouville equation for the evolution of the (dimensionless) phase-space density $\mathsf{f}_N(t, \{\vec{r}_j\}, \{\vec{p}_j\})$ of a classical systems of N particles with two-body interactions[1]

$$\left[\frac{\partial}{\partial t} + \sum_{j=1}^{N} \left(\vec{v}_j \cdot \vec{\nabla}_{\vec{r}_j} + \vec{F}_j \cdot \vec{\nabla}_{\vec{p}_j}\right) + \sum_{1\leq i<j\leq N} \vec{K}_{ij} \cdot \left(\vec{\nabla}_{\vec{p}_i} - \vec{\nabla}_{\vec{p}_j}\right)\right] \mathsf{f}_N(t, \vec{r}_1, \vec{p}_1, \ldots, \vec{r}_N, \vec{p}_N) = 0, \tag{3}$$

show that the classical entropy

$$S_{\mathrm{cl}}(t) \equiv -k_{\mathrm{B}} \int \mathsf{f}_N(t, \{\vec{r}_j\}, \{\vec{p}_j\}) \ln \mathsf{f}_N(t, \{\vec{r}_j\}, \{\vec{p}_j\})\, \mathrm{d}^{6N}\mathcal{V} \tag{4}$$

is conserved under Hamiltonian time evolution.

Hint: What is time derivative of the classical entropy $S_{\mathrm{cl}}(t)$?

*30. Linearized Boltzmann equation

Consider the kinetic Boltzmann equation in absence of an external potential for neutral particles with mass m interacting elastically with each other. One easily checks that the Maxwell–Boltzmann

[1] This is equation (V.14.d) of the lecture notes.

distribution

$$\bar{\mathsf{f}}^{(0)}(\vec{p}) = n\left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2} \mathrm{e}^{-\vec{p}^2/2mk_BT} \tag{5}$$

with constant T is a solution to the equation, whose integral over momentum[2] gives a uniform particle-number density n. In the following, we consider small perturbations

$$\bar{\mathsf{f}}(t,\vec{r},\vec{p}) = \bar{\mathsf{f}}^{(0)}(\vec{p})\big[1 + h(t,\vec{r},\vec{p})\big] \tag{6}$$

away from the "equilibrium" solution (5), where quadratic terms in h will be systematically neglected.

i. Show that h obeys the linearized Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \vec{v}_1 \cdot \vec{\nabla}_{\vec{r}}\right) h(t,\vec{r},\vec{p}_1) = \mathcal{I}_{\mathrm{coll.}}(h) \tag{7a}$$

where $\mathcal{I}_{\mathrm{coll.}}(h)$ is the (linear) collision operator

$$\mathcal{I}_{\mathrm{coll.}}(h) \equiv \int_{\vec{p}_2}\int_{\vec{p}_3}\int_{\vec{p}_4} \bar{\mathsf{f}}^{(0)}(\vec{p}_2)\Big[h(t,\vec{r},\vec{p}_3) + h(t,\vec{r},\vec{p}_4) - h(t,\vec{r},\vec{p}_1) - h(t,\vec{r},\vec{p}_2)\Big]\widetilde{w}(\vec{p}_1,\vec{p}_2 \to \vec{p}_3,\vec{p}_4), \tag{7b}$$

where the same notations as in the lectures were used.

ii. Mathematical results

Consider spatially homogeneous perturbations $h(t,\vec{p})$. To investigate their behavior it is interesting to look at the eigenfunctions ψ_i and eigenvalues λ_i of the collision operator $\mathcal{I}_{\mathrm{coll.}}$, defined by

$$\mathcal{I}_{\mathrm{coll.}}(\psi_i) = \lambda_i \psi_i. \tag{8}$$

It will be assumed that the integral $\displaystyle\int_{\vec{p}_1} \bar{\mathsf{f}}^{(0)}(\vec{p}_1)\big[\psi_i(\vec{p}_1)\big]^2$ exists for every eigenfunction.

a) Show that $\lambda = 0$ is a fivefold degenerate eigenvalue and give (the) corresponding eigenfunctions $\psi_1(\vec{p}_1), \ldots, \psi_5(\vec{p}_1)$ (disregarding any normalization).

Hint: You do not need to know the transition rate $\widetilde{w}$ to answer this question, which means that the eigenfunctions are determined by fundamental properties of the collisions.

b) Show that all other eigenvalues are negative.

Hint: You may express λ_i in terms of the integral $\displaystyle\int_{\vec{p}_1} \bar{\mathsf{f}}^{(0)}(\vec{p}_1)\psi_i(\vec{p}_1)\mathcal{I}_{\mathrm{coll.}}(\psi_i)$, which you can transform as in the proof of the H-theorem.

c) Consider the linearized Boltzmann equation (7a) in the spatially homogeneous case. Assuming that the eigenfunctions form a complete set, write down the solution $h(t,\vec{p}_1)$ as a linear combination of the $\{\psi_i(\vec{p}_1)\}$. How can you interpret the quantities $-1/\lambda_i$ for the non-vanishing eigenvalues?

[2]... with integration measure $\mathrm{d}^3\vec{p}/(2\pi\hbar)^3$