Tutorial sheet 10

The exercise part marked with a star is homework.

Discussion topic: Boltzmann equation: assumptions, formulation. Which change(s) should be made to include inelastic scatterings or decays?

26. Free streaming

Show that in the absence of collision term and of external potential—which defines *free streaming*—, the solutions to the (collisionless!) Boltzmann equation satisfy the equation

$$\bar{\mathsf{f}}(t,\vec{r},\vec{p}) = \bar{\mathsf{f}}(t_0,\vec{r}-\vec{v}(t-t_0),\vec{p}) \tag{1}$$

with $\vec{v}$ the velocity corresponding to momentum $\vec{p}$. That is, the solution at time $t > t_0$ has a straightforward expression in terms of the initial condition at t_0.

27. Lorentz gas

In various physical situations—for instance the motion of neutrons in a nuclear reactor or of electrons in a non-degenerate semi-conductor—one can describe the diffusion of (light) particles inside a medium made of more massive constituents as resulting from collisions on *fixed* scattering centers. The diffusing particles are then referred to as a *Lorentz gas*. The evolution of the corresponding (coarse-grained) single-particle phase-space density is governed by a simplified version of the Boltzmann kinetic equation, which will be derived hereafter.

Throughout this problem, it is assumed that no external vector potential is present, so that the linear and canonical momenta $\vec{p}$ of particles are identical. Furthermore, one neglects the collisions between the particles of the Lorentz gas.

i. Conservation laws

One assumes that the collisions between the particles and the scattering centers are instantaneous and local, as well as elastic and invariant under space parity and time reversal.

Write down the relevant conservation laws. Explain why the collision of a particle on a scattering center amounts to a change $\vec{p} \to \vec{p}\,'$ of its momentum, with $|\vec{p}| = |\vec{p}\,'|$. The corresponding differential cross-section will be denoted as $\sigma(\vec{p} \to \vec{p}\,')$. What can you say about reference frames?

ii. Kinetic equation

Let $\bar{\mathsf{f}}(t,\vec{r},\vec{p})$ be the (dimensionless) single-particle phase-space density of the particles of the Lorentz gas. As in the case of the Boltzmann equation, the dynamics of $\bar{\mathsf{f}}$ obeys an equation of the type

$$\frac{\partial\bar{\mathsf{f}}}{\partial t} + \vec{v}\cdot\vec{\nabla}_{\vec{r}}\bar{\mathsf{f}} + \vec{F}\cdot\vec{\nabla}_{\vec{p}}\bar{\mathsf{f}} = \left(\frac{\partial\bar{\mathsf{f}}}{\partial t}\right)_{\text{coll.}}, \qquad \left(\frac{\partial\bar{\mathsf{f}}}{\partial t}\right)_{\text{coll.}} \equiv \left(\frac{\partial\bar{\mathsf{f}}}{\partial t}\right)_{\text{gain}} - \left(\frac{\partial\bar{\mathsf{f}}}{\partial t}\right)_{\text{loss}}, \tag{2}$$

where $\vec{v}$ is the particle velocity, while the collision integral is conveniently expressed as the difference between a gain and a loss term, which we now want to compute.

a) Loss term. Consider a scattering center at position $\vec{r}$. What is the flux density of particles with a momentum between $\vec{p}$ and $\vec{p}+\mathrm{d}^3\vec{p}$ falling on this scattering center? Show that the number of collisions of the type $\vec{p} \to \vec{p}\,'$, where $\vec{p}$, $\vec{p}\,'$ are known up to infinitesimal uncertainties $\mathrm{d}^3\vec{p}$, $\mathrm{d}^3\vec{p}\,'$, in a time interval $\mathrm{d}t$ is given by

$$\bar{\mathsf{f}}(t,\vec{r},\vec{p})\,\frac{\mathrm{d}^3\vec{p}}{(2\pi\hbar)^3}|\vec{v}|\,\sigma(\vec{p}\to\vec{p}\,')\,\mathrm{d}^2\Omega'\,\mathrm{d}t,$$

where Ω' is the solid angle associated with the direction of $\vec{p}\,'$.

Deduce that the loss term is given by

$$\left(\frac{\partial \bar{\mathsf{f}}}{\partial t}\right)_{\rm loss} = n_{\rm d}(\vec{r})\,|\vec{v}|\,\bar{\mathsf{f}}(t,\vec{r},\vec{p}) \int \sigma(\vec{p}\to\vec{p}\,')\,{\rm d}^2\Omega',$$

with $n_{\rm d}$ the density of scattering centers. What is the integral equal to?

***b) Gain term.** The gain term corresponds to collisions of the type $\vec{p}\,' \to \vec{p}$, where the initial and final momenta are known up to infinitesimal uncertainties. Show that between t and $t + {\rm d}t$, there are

$$\bar{\mathsf{f}}(t,\vec{r},\vec{p}\,')\,\frac{{\rm d}^3\vec{p}\,'}{(2\pi\hbar)^3}|\vec{v}\,'|\,\sigma(\vec{p}\,'\to\vec{p})\,{\rm d}^2\Omega\,{\rm d}t,$$

such collisions on a single scattering center at position $\vec{r}$, where Ω is the solid angle associated with the direction of $\vec{p}$. Let Ω' be the solid angle associated with $\vec{p}\,'$. Justify the identity

$$|\vec{v}\,'|\,{\rm d}^3\vec{p}\,'\,{\rm d}^2\Omega = |\vec{v}|\,{\rm d}^3\vec{p}\;{\rm d}^2\Omega'.$$

Show that the gain term can be written as

$$\left(\frac{\partial \bar{\mathsf{f}}}{\partial t}\right)_{\rm gain} = n_{\rm d}(\vec{r})\,|\vec{v}| \int \bar{\mathsf{f}}(t,\vec{r},\vec{p}\,')\,\sigma(\vec{p}\to\vec{p}\,')\,{\rm d}^2\Omega',$$

and thus the collision integral as

$$\left(\frac{\partial \bar{\mathsf{f}}}{\partial t}\right)_{\rm coll.} = n_{\rm d}(\vec{r})\,|\vec{v}| \int \left[\bar{\mathsf{f}}(t,\vec{r},\vec{p}\,') - \bar{\mathsf{f}}(t,\vec{r},\vec{p})\right]\sigma(\vec{p}\to\vec{p}\,')\,{\rm d}^2\Omega'. \qquad (3)$$