C.2.5 Markov processes

We now introduce a class of stochastic processes that are often encountered in physics—or, one should rather say, which are often used to model physical phenomena due to their simplicity, since they are entirely determined by the two densities $p_{Y,1}$ and $p_{Y,1|1}$.

C.2.5 a Markov property

A *Markov*[(83)] *process* is a stochastic process $Y(t)$ for which for all $n \in \mathbb{N}^*$ and arbitrary ordered times $t_1 < t_2 < \cdots < t_{n-1} < t_n < t_{n+1}$, the conditional probability densities obey the *Markov property*

$$p_{Y,1|n}(t_{n+1}, y_{n+1} \,|\, t_1, y_1; t_2, y_2; \ldots; t_{n-1}, y_{n-1}; t_n, y_n) = p_{Y,1|1}(t_{n+1}, y_{n+1} \,|\, t_n, y_n). \tag{C.22}$$

Viewing t_n as being "now", this property means that the (conditional) probability that the process takes a given value y_{n+1} in the future (at t_{n+1}) only depends on its present value y_n, not on the values it took in the past.

> An even more drastically "memoryless" class of processes is that of *fully random processes*, for which the value taken at a given time is totally independent of the past values. For such a process, the conditional probability densities equal the joint probability densities—i.e. $p_{Y,n|m} = p_{Y,n}$ for all m, n—, and repeated applications of Bayes' theorem (C.13) show that the n-point density factorizes into the product of n single-time densities,
>
> $$p_{Y,n}(t_1, y_1; \ldots; t_n, y_n) = p_{Y,1}(t_1, y_1) \cdots p_{Y,1}(t_n, y_n).$$

One can check that a Markov process is entirely determined by the single-time probability density $p_{Y,1}(t_1, y_1)$ and by the *transition probability* $p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)$, or equivalently by $p_{Y,1}(t_1, y_1)$ and the two-time density $p_{Y,2}(t_1, y_1; t_2, y_2)$.
For instance, the 3-time probability density can be rewritten as

$$\begin{aligned} p_{Y,3}(t_1, y_1; t_2, y_2; t_3, y_3) &= p_{Y,1|2}(t_3, y_3 \,|\, t_1, y_1; t_2, y_2)\, p_{Y,2}(t_1, y_1; t_2, y_2) \\ &= p_{Y,1|1}(t_3, y_3 \,|\, t_2, y_2)\, p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)\, p_{Y,1}(t_1, y_1), \end{aligned} \tag{C.23}$$

where we have used twice Bayes' theorem (C.13) and once the Markov property (C.22).

Remarks:

∗ The Markov property (C.22) characterizes the n-point densities for ordered times. The value for arbitrary t_1, t_2, …, t_n follows from the necessary invariance [property (C.11b)] of $p_{Y,n}$ when two pairs (t_j, y_j) and (t_k, y_k) are exchanged.

∗ The single-time probability density $p_{Y,1}$ and the transition probability $p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)$ are not fully independent of each other, since they have to obey the obvious identity

$$p_{Y,1}(t_2, y_2) = \int p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)\, p_{Y,1}(t_1, y_1)\, \mathrm{d}y_1. \tag{C.24}$$

C.2.5 b Chapman–Kolmogorov equation

Integrating Eq. (C.23) over the intermediate value y_2 of the stochastic process, while taking into account the consistency condition (C.11c), gives

$$p_{Y,2}(t_1, y_1; t_3, y_3) = p_{Y,1}(t_1, y_1) \int p_{Y,1|1}(t_3, y_3 \,|\, t_2, y_2)\, p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)\, \mathrm{d}y_2,$$

where $t_1 < t_2 < t_3$.

(83) … or *Markoff* in the older literature.

Dividing by $p_{Y,1}(t_1, y_1)$, one obtains the *Chapman*[bw]*–Kolmogorov*[bx] *equation*

$$p_{Y,1|1}(t_3, y_3 \,|\, t_1, y_1) = \int p_{Y,1|1}(t_3, y_3 \,|\, t_2, y_2)\, p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)\, \mathrm{d}y_2 \quad \text{for } t_1 < t_2 < t_3, \tag{C.25}$$

which gives a relation—a nonlinear integral-functional equation—fulfilled by the transition probability of a Markov process.

Reciprocally, two arbitrary nonnegative functions $p_{Y,1}(t_1, y_1)$, $p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)$ obeying the two identities (C.24) and (C.25) entirely define a unique Markov process.

Remark: The Chapman–Kolmogorov equation follows quite obviously when invoking the Markov property in the more generic relation (C.15), which holds for every stochastic process. In contrast to the latter, Eq. (C.25) is closed, i.e. does not depend on another function.

C.2.5 c Examples of Markov processes

Wiener process

The stochastic process defined by the "initial condition" $p_{Y,1}(t=0, y) = \delta(y)$ for $y \in \mathbb{R}$ and the Gaussian transition probability (for $0 < t_1 < t_2$)

$$p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1) = \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \exp\left[-\frac{(y_2 - y_1)^2}{2(t_2 - t_1)}\right] \tag{C.26a}$$

is called *Wiener process*.

One easily checks that the transition probability (C.26a) satisfies the Chapman–Kolmogorov equation (C.25), and that the probability density at time $t > 0$ is given by

$$p_{Y,1}(t, y) = \frac{1}{\sqrt{2\pi t}}\, \mathrm{e}^{-y^2/2t}. \tag{C.26b}$$

The Wiener process is obviously not a stationary process, since for instance the second moment $\langle [Y(t)]^2 \rangle = t$ depends on time.

Remark: The single-time probability density (C.26b) is solution of the diffusion equation

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2 f}{\partial y^2} \tag{C.27}$$

with diffusion coefficient $D = \frac{1}{2}$.

Poisson process

Consider now the integer-valued stochastic process $Y(t)$ defined by the Poisson-distributed [cf. Eq. (B.11)] transition probability ($0 \le t_1 \le t_2$)

$$p_{Y,1|1}(t_2, n_2 \,|\, t_1, n_1) = \frac{(t_2 - t_1)^{n_2 - n_1}}{(n_2 - n_1)!}\, \mathrm{e}^{-(t_2 - t_1)} \quad \text{for } n_2 \ge n_1 \tag{C.28}$$

and 0 otherwise, and by the single-time probability density $p_{Y,1}(t=0, n) = \delta_{n,0}$. That is, a realization $y(t)$ is a succession of unit steps taking place at arbitrary instants, whose number between two given times t_1, t_2 obeys a Poisson distribution with parameter $t_2 - t_1$.

$Y(t)$ is a non-stationary Markov process, called *Poisson process*.

Remark: In both Wiener and Poisson processes, the probability density of the *increment* ($y_2 - y_1$ resp. $n_2 - n_1$) between two successive instants t_1, t_2 only depends on the time difference $t_2 - t_1$, not on t_1 (or t_2) alone. Such increments are called *stationary*. Since in addition successive increments are independent, both processes are instances of *Lévy*[by] *processes*.

[bw] S. Chapman, 1888–1970 [bx] A. N. Kolmogorov, 1903–1987 [by] P. Lévy, 1886–1971

C.2.5 d Stationary Markov processes

An interesting case in physics is that of stationary Markov processes. For such processes, the transition probability $p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)$ only depends on the time difference $\tau \equiv t_2 - t_1$, which is hereafter reflected in the use of the special notation

$$\mathcal{T}_{Y;\tau}(y_2 \,|\, y_1) \equiv p_{Y,1|1}(t_1 + \tau, y_2 \,|\, t_1, y_1). \tag{C.29}$$

Using this notation, the Chapman–Kolmogorov equation (C.25) takes the form (both τ and τ' are nonnegative)

$$\boxed{\mathcal{T}_{Y;\tau+\tau'}(y_3 \,|\, y_1) = \int \mathcal{T}_{Y;\tau'}(y_3 \,|\, y_2)\, \mathcal{T}_{Y;\tau}(y_2 \,|\, y_1)\, \mathrm{d}y_2.} \tag{C.30}$$

If a Markov process is also stationary, the single-time probability density $p_{Y,1}(y)$ does not depend on time. Invoking a setup in which the probability density would first be time-dependent, i.e. in which the stochastic process Y is not (yet) stationary, $p_{Y,1}$ characterizes the large-time "equilibrium" distribution, reached after a sufficiently large τ, irrespective of the "initial" distribution $y(t)$ at some time $t = t_0$. Taking as initial condition $p_{Y,1}(t{=}t_0, y) = \delta(y - y_0)$, where y_0 is arbitrary, one finds

$$p_{Y,1}(y) = \lim_{\tau \to +\infty} \mathcal{T}_{Y;\tau}(y \,|\, y_0).$$

This follows from the identities

$$\begin{aligned} p_{Y,1}(t_0 + \tau, y) &= \int p_{Y,2}(t_0 + \tau, y; t_0, y')\, \mathrm{d}y' = \int p_{Y,1|1}(t_0 + \tau, y \,|\, t_0, y')\, p_{Y,1}(t_0, y')\, \mathrm{d}y' \\ &= \int \mathcal{T}_{Y;\tau}(y \,|\, y')\, p_{Y,1}(t_0, y')\, \mathrm{d}y', \end{aligned}$$

which with the assumed initial distribution $p_{Y,1}(t_0, y')$ gives the result. □

Ornstein–Uhlenbeck process

An example of stationary Markov process is the *Ornstein*[(bz)]*–Uhlenbeck*[(ca)] *process* [49] defined by the (time-independent) single-time probability density

$$p_{Y,1}(y) = \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-y^2/2} \tag{C.31a}$$

and the transition probability

$$\mathcal{T}_{Y;\tau}(y_2 \,|\, y_1) = \frac{1}{\sqrt{2\pi(1 - \mathrm{e}^{-2\tau})}}\, \exp\left[-\frac{(y_2 - y_1 \mathrm{e}^{-\tau})^2}{2(1 - \mathrm{e}^{-2\tau})}\right]. \tag{C.31b}$$

One can show that the Ornstein–Uhlenbeck process is also Gaussian and that its autocorrelation function is $\kappa(\tau) = \mathrm{e}^{-\tau}$.

Doob's[(cb)] theorem actually states that the Ornstein–Uhlenbeck process is, up to scalings or translations of the time argument, the only nontrivial[(84)] process which is Markovian, Gaussian and stationary.

C.2.5 e Master equation for a Markov process

For an *homogeneous* Markov process $Y(t)$, i.e. a process for which the probability transition $p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)$ only depends on the time difference $\tau \equiv t_2 - t_1$, one can derive under minimal assumptions a linear integrodifferential equation for the transition probability, which constitutes the differential form of the Chapman–Kolmogorov equation for the process.

[(84)] The fully random process mentioned below the introduction of the Markov property (C.22) may also be Gaussian and stationary.

[(bz)] L. Ornstein, 1880–1941 [(ca)] G. Uhlenbeck, 1900–1988 [(cb)] J. L. Doob, 1910–2004

Remark: The assumption on the probability transition does not automatically imply that the process is stationary; yet in analogy with Eq. (C.29) we shall denote it by $\mathcal{T}_{Y;\tau}(y_2 \,|\, y_1)$.

Master equation

Let us assume that for time differences τ much smaller than some time scale τ_c, the transition probability is of the form

$$\mathcal{T}_{Y;\tau}(y_2 \,|\, y_1) = [1 - \gamma(y_1)\tau]\,\delta(y_2 - y_1) + \Gamma(y_2 \,|\, y_1)\tau + o(\tau), \tag{C.32a}$$

where $o(\tau)$ denotes a term that is much smaller than τ in the limit $\tau \to 0$. The nonnegative quantity $\Gamma(y_2 \,|\, y_1)$ is readily interpreted as being the transition rate from y_1 to y_2, and $\gamma(y_1)$ is its integral over y_2

$$\gamma(y_1) = \int \Gamma(y_2 \,|\, y_1)\,\mathrm{d}y_2, \tag{C.32b}$$

thereby ensuring that the integral of the transition probability $\mathcal{T}_{Y;\tau}(y_2 \,|\, y_1)$ over all possible final states y_2 gives unity.

Consider now the Chapman–Kolmogorov equation (C.30). Rewriting $\mathcal{T}_{Y;\tau'}(y_3 \,|\, y_2)$ with the help of Eq. (C.32a), i.e. under the assumption that $\tau' \ll \tau_c$, and leaving aside the negligible term $o(\tau')$, one finds

$$\mathcal{T}_{Y;\tau+\tau'}(y_3 \,|\, y_1) = \big[1 - \gamma(y_3)\,\tau'\big]\,\mathcal{T}_{Y;\tau}(y_3 \,|\, y_1) + \tau' \int \Gamma(y_3 \,|\, y_2)\,\mathcal{T}_{Y;\tau}(y_2 \,|\, y_1)\,\mathrm{d}y_2.$$

Note that we need not make any assumption on τ here. Taylor-expanding $\mathcal{T}_{Y;\tau+\tau'}$ with respect to τ' and dividing both sides by τ', this gives in the limit $\tau' \to 0$ the integrodifferential equation

$$\frac{\partial \mathcal{T}_{Y;\tau}(y_3 \,|\, y_1)}{\partial \tau} = -\gamma(y_3)\mathcal{T}_{Y;\tau}(y_3 \,|\, y_1) + \int \Gamma(y_3 \,|\, y_2)\,\mathcal{T}_{Y;\tau}(y_2 \,|\, y_1)\,\mathrm{d}y_2,$$

where the derivative on the left-hand side has to be taken with a grain of salt in case τ' may (for physical reasons pertaining to the system being considered) actually not become vanishingly small.

Using Eq. (C.32b) for $\gamma(y_3)$ and relabeling the variables ($y_1 \to y_0$, $y_2 \to y'$, $y_3 \to y$), this can be rewritten as

$$\boxed{\frac{\partial \mathcal{T}_{Y;\tau}(y \,|\, y_0)}{\partial \tau} = \int \big[\Gamma(y \,|\, y')\,\mathcal{T}_{Y;\tau}(y' \,|\, y_0) - \Gamma(y' \,|\, y)\,\mathcal{T}_{Y;\tau}(y \,|\, y_0)\big]\,\mathrm{d}y'.} \tag{C.33}$$

This evolution equation—which is fully equivalent to the Chapman–Kolmogorov equation—for the transition probability $\mathcal{T}_{Y;\tau}$ is called *master equation*. It has the structure of a balance equation, with a gain term, involving the rate $\Gamma(y \,|\, y')$, and a loss term depending on the rate $\Gamma(y' \,|\, y)$. It is a linear integrodifferential equation, of first order with respect to τ.

Evolution equation for the single-time probability density

From the master equation (C.33) for the transition probability, one can deduce an equation governing the dynamics of the single-time probability density, which turns out to possess exactly the same structure.

Rewriting Eq. (C.24) in the form

$$p_{Y,1}(\tau, y) = \int \mathcal{T}_{Y;\tau}(y \,|\, y_0)\,p_{Y,1}(t{=}0, y_0)\,\mathrm{d}y_0, \tag{C.34}$$

and differentiating with respect to τ, one obtains with the help of the master equation

$$\frac{\partial p_{Y,1}(\tau, y)}{\partial \tau} = \int \big[\Gamma(y \,|\, y')\,\mathcal{T}_{Y;\tau}(y' \,|\, y_0) - \Gamma(y' \,|\, y)\,\mathcal{T}_{Y;\tau}(y \,|\, y_0)\big]\,p_{Y,1}(t{=}0, y_0)\,\mathrm{d}y_0\,\mathrm{d}y'.$$

Performing the integration over y_0 and using relation (C.34), this yields

$$\frac{\partial p_{Y,1}(\tau, y)}{\partial \tau} = \int \left[\Gamma(y \,|\, y')\, p_{Y,1}(\tau, y') - \Gamma(y' \,|\, y)\, p_{Y,1}(\tau, y)\right] \mathrm{d}y', \tag{C.35}$$

formally identical to the equation for $\mathcal{T}_{Y;\tau}$, and accordingly also referred to as *master equation*.

Remark: When applied to a physical system, Eq. (C.35) allows the computation of the single-time probability density at any time from an initial distribution $p_{Y,1}(t=0, y)$ and the transition rates Γ.