C.3 Spectral decomposition of stationary processes

In this Section, we focus on stationary stochastic processes and introduce an alternative description of their statistical properties, based on Fourier transformation (§ C.3.1). This approach in particularly leads to the Wiener–Khinchin theorem relating the spectral density to the autocorrelation function (§ C.3.2).

C.3.1 Fourier transformations of a stationary process

Consider a stationary process $Y(t)$. In general, a given realization $y(t)$ will not be an integrable function, e.g. because it does not tend to 0 as t goes to infinity. In order to talk of Fourier transformations of the realization, one thus has to first introduce a finite time interval $[0,\mathcal{T}]$ with nonnegative $\mathcal{T}$, and to work with continuations of the restriction of $y(t)$ to this interval, before letting $\mathcal{T}$ go to infinity.

C.3.1 a Fourier transform

Let first $y_{\mathcal{T}}(t)$ denote the function that coincides with $y(t)$ for $0<t<\mathcal{T}$, and vanishes outside the interval. $y_{\mathcal{T}}(t)$ may be seen as the realization of a stochastic process $Y_{\mathcal{T}}(t)$.

One can meaningfully define the Fourier transform of $y_{\mathcal{T}}(t)$ with the usual formula

$$\tilde{y}_{\mathcal{T}}(\omega) \equiv \int y_{\mathcal{T}}(t)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}t = \int_0^{\mathcal{T}} y(t)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}t. \tag{C.36a}$$

$\tilde{y}_{\mathcal{T}}(\omega)$ is now the realization of a stochastic function $\tilde{Y}_{\mathcal{T}}(\omega)$. The inverse transform reads

$$y_{\mathcal{T}}(t) = \int \tilde{y}_{\mathcal{T}}(\omega)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \frac{\mathrm{d}\omega}{2\pi}. \tag{C.36b}$$

Taking the limit $\mathcal{T}\to\infty$ defines for each realization $y(t)$ a corresponding $\tilde{y}(\omega)$. The latter is itself the realization of a process $\tilde{Y}(\omega)$, and one symbolically writes for the stochastic processes themselves

$$\tilde{Y}(\omega) = \int Y(t)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}t, \qquad Y(t) = \int \tilde{Y}(\omega)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \frac{\mathrm{d}\omega}{2\pi}. \tag{C.37}$$

Remark: The reader can check that thanks to the assumed stationarity of the process, we could have started with restrictions of the realizations to any interval of width $\mathcal{T}$, for instance $[-\frac{\mathcal{T}}{2},\frac{\mathcal{T}}{2}]$, without changing the result after taking the limit $\mathcal{T}\to\infty$.

C.3.1 b Fourier series

Alternatively, one can consider the $\mathcal{T}$-periodic function which coincides with $y(t)$ on the interval $[0,\mathcal{T}]$. This $\mathcal{T}$-periodic function can be written as a Fourier series, which of course equals $y(t)$ for $0<t<\mathcal{T}$:

$$y(t) = \sum_{n=-\infty}^{\infty} c_n\, \mathrm{e}^{-\mathrm{i}\omega_n t} \quad \text{for } 0<t<\mathcal{T}, \tag{C.38a}$$

where the angular frequencies and Fourier coefficients are as usual given by

$$\omega_n = \frac{2\pi n}{\mathcal{T}}, \qquad c_n = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} y(t)\,\mathrm{e}^{\mathrm{i}\omega_n t}\,\mathrm{d}t \quad \text{for } n \in \mathbb{Z}. \tag{C.38b}$$

Again, one considers the limit $\mathcal{T} \to \infty$ at the end of the calculations.

For the stochastic process, one similarly writes

$$Y(t) = \sum_{n=-\infty}^{\infty} C_n\,\mathrm{e}^{-\mathrm{i}\omega_n t} \quad \text{for } 0 < t < \mathcal{T}, \tag{C.39a}$$

where C_n is a random variable, of which the Fourier coefficient c_n is a realization

$$C_n = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} Y(t)\,\mathrm{e}^{\mathrm{i}\omega_n t}\,\mathrm{d}t. \tag{C.39b}$$

At fixed $\mathcal{T}$, one has the obvious relationship

$$c_n = \frac{1}{\mathcal{T}}\tilde{y}_{\mathcal{T}}(\omega_n), \tag{C.40a}$$

which for the corresponding stochastic variables reads

$$C_n = \frac{1}{\mathcal{T}}\tilde{Y}_{\mathcal{T}}(\omega_n). \tag{C.40b}$$

An equivalent relation will also holds in the limit $\mathcal{T} \to \infty$.

Remark: Instead of the complex Fourier transform, one can also use real transforms, for instance the sine transform as in Ref. [53].

C.3.1 c Consequences of stationarity

The assumed stationarity of the stochastic process, which allowed us to define the Fourier transformations on an arbitrary interval of width $\mathcal{T}$, has further consequences, some of which we now investigate.

First, the single-time average $\langle Y(t)\rangle$ is independent of time, $\langle Y(t)\rangle = \langle Y\rangle$. Averaging the Fourier coefficient (C.39b) over an ensemble of realizations, the sample average and integration over time can be exchanged, which at once leads to

$$\langle C_0\rangle = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} \langle Y\rangle\,\mathrm{d}t = \langle Y\rangle, \qquad \langle C_n\rangle = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} \langle Y\rangle\,\mathrm{e}^{\mathrm{i}\omega_n t}\,\mathrm{d}t = 0 \text{ for } n \neq 0. \tag{C.41}$$

Consider now a two-time average, which, since the process is stationary, only depends on the time difference. For the sake of simplicity, we assume that the stochastic process is real-valued and centered, $\langle Y\rangle = 0$, so as to shorten the expression of the autocorrelation function. The latter reads

$$\kappa(\tau) = \big\langle Y(t)\,Y(t+\tau)\big\rangle = \sum_{n,n'=-\infty}^{\infty} \langle C_n C_{n'}\rangle\,\mathrm{e}^{-\mathrm{i}(\omega_n+\omega_{n'})t}\mathrm{e}^{-\mathrm{i}\omega_n\tau},$$

which can only be independent of t for all values of τ if $\langle C_n C_{n'}\rangle = 0$ for all values of n and n' such that $\omega_n + \omega_{n'} \neq 0$, i.e. [cf. the frequency (C.38b)] when $n' \neq -n$. Using the classical property $C_{-n} = C_n^*$ of Fourier coefficients, this condition can be written as

$$\langle C_n C_{n'}^*\rangle = \big\langle |C_n|^2\big\rangle\,\delta_{n,n'}, \tag{C.42}$$

with $\delta_{n,n'}$ the Kronecker symbol. Fourier coefficients of different frequencies are thus uncorrelated, and the autocorrelation function reads

$$\kappa(\tau) = \sum_{n=-\infty}^{\infty} \big\langle |C_n|^2\big\rangle\,\mathrm{e}^{-\mathrm{i}\omega_n\tau}. \tag{C.43}$$

C.3.2 Wiener–Khinchin theorem

C.3.2 a Spectral density of a stationary process

Consider a centered stationary stochastic process $Y(t)$. Working first on a finite interval $[0, \mathcal{T}]$, one can introduce the Fourier coefficients C_n or alternatively the Fourier transform $\tilde{Y}_{\mathcal{T}}(\omega)$. Using relation (C.40b), Eq. (C.42) reads

$$\frac{1}{\mathcal{T}^2}\left\langle \tilde{Y}_{\mathcal{T}}(\omega_n)\tilde{Y}_{\mathcal{T}}(\omega_{n'})^*\right\rangle = \frac{1}{\mathcal{T}^2}\left\langle \left|\tilde{Y}_{\mathcal{T}}(\omega_n)\right|^2\right\rangle \delta_{n,n'}.$$

The Kronecker symbol can be rewritten under consideration of the expression (C.38b) of the Fourier frequencies as

$$\delta_{n,n'} = \delta(n-n') = \frac{2\pi}{\mathcal{T}}\delta(\omega_n - \omega_{n'}),$$

leading to

$$\left\langle \tilde{Y}_{\mathcal{T}}(\omega_n)\tilde{Y}_{\mathcal{T}}(\omega_{n'})^*\right\rangle = \frac{2\pi}{\mathcal{T}}\left\langle \left|\tilde{Y}_{\mathcal{T}}(\omega_n)\right|^2\right\rangle \delta(\omega_n - \omega_{n'}).$$

Defining now the *spectral density* $S(\omega)$ as

$$S(\omega) = \lim_{\mathcal{T}\to\infty}\frac{1}{\mathcal{T}}\left\langle \left|\tilde{Y}_{\mathcal{T}}(\omega)\right|^2\right\rangle, \tag{C.44}$$

the above identity becomes in the limit $\mathcal{T}\to\infty$

$$\left\langle \tilde{Y}(\omega)\,\tilde{Y}(\omega')^*\right\rangle = 2\pi\delta(\omega-\omega')\,S(\omega), \tag{C.45}$$

where the discrete frequencies ω_n, $\omega_{n'}$ have been replaced by values ω, ω' from a continuous interval.

Coming back to Fourier representations defined on a finite-size time interval, let $\mathcal{I}_\omega \equiv [\omega, \omega+\Delta\omega]$ denote an interval in frequency space, over which $\tilde{Y}_{\mathcal{T}}(\omega)$ is assumed to be continuous and to vary only moderately. One introduces a function $\sigma(\omega)$ such that

$$\sigma(\omega)\,\Delta\omega \equiv \sum_{\omega_n\in\mathcal{I}_\omega}\left\langle |C_n|^2\right\rangle = \sum_{\omega_n\in\mathcal{I}_\omega}\frac{1}{\mathcal{T}^2}\left\langle \left|\tilde{Y}_{\mathcal{T}}(\omega_n)\right|^2\right\rangle.$$

From Eq. (C.38b), the number of modes ω_n inside the interval $\mathcal{I}_\omega$ is $\Delta\omega/(2\pi/\mathcal{T}) = \mathcal{T}\,\Delta\omega/2\pi$. Taking the limit $\mathcal{T}\to\infty$, this gives

$$\sigma(\omega) = \lim_{\mathcal{T}\to\infty}\frac{1}{2\pi\mathcal{T}}\left\langle \left|\tilde{Y}(\omega)\right|^2\right\rangle = \frac{1}{2\pi}S(\omega)$$

which shows the relation between the spectral density and the sum of the (squared) amplitudes of the Fourier modes.

C.3.2 b Wiener–Khinchin theorem

Consider the autocorrelation function (C.43). In the limit $\mathcal{T}\to\infty$, the discrete sum is replaced by an integral, yielding

$$\kappa(\tau) = \lim_{\mathcal{T}\to\infty}\frac{1}{2\pi\mathcal{T}}\int\left\langle \left|\tilde{Y}_{\mathcal{T}}(\omega)\right|^2\right\rangle e^{-i\omega\tau}\,d\omega.$$

With the help of the spectral density (C.44), this also reads

$$\kappa(\tau) = \int S(\omega)\,e^{-i\omega\tau}\,\frac{d\omega}{2\pi}. \tag{C.46a}$$

That is, the autocorrelation function is the (inverse) Fourier transform of the spectral density, and reciprocally

$$S(\omega) = \int \kappa(\tau)\, \mathrm{e}^{\mathrm{i}\omega\tau}\, \mathrm{d}\tau. \tag{C.46b}$$

The relations (C.46) are known as *Wiener–Khinchin theorem*, and show that the autocorrelation function $\kappa(\tau)$ and the spectral density $S(\omega)$ contain exactly the same amount of information on the stochastic process.

Remarks:

∗ In deriving the theorem, we did not use the stationarity of the stochastic process, but only its *wide-sense stationarity* (or *covariance stationarity*), which only requires that the first and second moments be independent of time, not all of them.

∗ If the stochastic process $Y(t)$ is not centered, then the Wiener–Khinchin-theorem states that its autocorrelation function $\kappa(\tau)$ and the spectral density $S(\omega)$ of the fluctuations around its average value constitute a Fourier-transform pair.
The spectral density of $X(t)$ itself is given by $S(\omega) + 2\pi|\langle Y\rangle|^2\delta(\omega)$, i.e. it includes a singular contribution at $\omega = 0$.

Bibliography for Appendix C

- Pottier, *Nonequilibrium statistical physics* [6], chapter 1 § 6–10.
- van Kampen, *Stochastic processes in physics and chemistry* [53], chapters III & IV.