Chapter IV

Brownian motion

In this Chapter, we study the very general paradigm provided by Brownian motion. Originally, this motion is that a "heavy" particle, called *Brownian particle*, immersed in a fluid consisting of much lighter particles—in Robert Brown's[(bd)] original observations, this was some pollen grain in water. Due to the successive collisions with the fluid constituents, the Brownian particle is constantly moving, going in always changing, apparently random directions, even if the fluid itself is at rest: the position $\vec{x}(t)$ and the velocity $\vec{v}(t)$ of the Brownian particle are then naturally modeled as stochastic processes, driven by a fluctuating force.

The interest of this rather specific physical problem lies in the fact that the dynamical equation governing the motion of the Brownian particle actually also applies to many stochastic collective properties of a macroscopic system as they approach their equilibrium values. Accordingly, the techniques used for solving the initial question extend to a much wider class of problems in nonequilibrium statistical physics and even beyond. This notwithstanding, we shall throughout this chapter retain the terminology of the original physical problem.

In Sec. IV.1, we introduce the approach pioneered by Paul Langevin,[(be)] which describes the dynamics of the Brownian particle velocity on time scales larger than the typical autocorrelation time of the fluctuating force acting on the particle by explicitly solving the evolution equation for given initial conditions. We then adopt in Sec. IV.2 an alternative description, in which we rather focus on the time evolution of the probability distribution of the velocity. That approach is quite generic and can be used for any Markov process, so that we discuss a straightforward extension in an appendix (IV.A). Next, we investigate a generalization of the Langevin model, in which the friction force exerted by the fluid on the Brownian motion is non-local in time, i.e. we take into consideration memory effects in the autocorrelation function of the fluctuating Langevin force (Sec. IV.3). Eventually, we introduce in Sec IV.4 quantum-mechanical models analogous to classical Brownian motion, in that the spectral properties of some of their operators are similar to those of the Langevin model of Sec. IV.1 or the generalization of Sec. IV.3.

For the sake of simplicity, we consider throughout this Chapter one-dimensional Brownian motion only, with the exception of Sec. IV.4. The generalization to motion in two or more dimensions is straightforward.

IV.1 Langevin dynamics

Following P. Langevin's modeling, the dynamics of a Brownian particle much heavier than the constituents of the medium in which it evolves can be viewed as resulting from the influence of two complementary forces, namely an instantaneous friction force and a fluctuating force. After writing down the corresponding dynamical equation for the velocity of the Brownian particle (§ IV.1.1), we study its solution for given initial conditions (§ IV.1.2), as well as the resulting time evolution of the displacement from the initial position (§ IV.1.3). We then turn in § IV.1.4 to the dynamics of the fluctuations of the velocity for a Brownian particle at equilibrium with its environment. Eventually,

[(bd)] R. Brown, 1773–1858 [(be)] P. Langevin, 1872–1946 anticipating on applications of the Brownian-motion paradigm to other problems, we introduce the spectral function associated to the Langevin model at equilibrium (§ IV.1.5), as well as the linear response of the model to an external force (§ IV.1.6).

IV.1.1 Langevin model

Let M denote the mass of the Brownian particle and $v(t)$ its velocity.

IV.1.1 a Langevin equation

The classical model introduced by P. Langevin [48] consists in splitting the total force exerted by the fluid constituents on the Brownian particle into two contributions:

- First, a Brownian particle in motion with a velocity v with respect to the fluid sees more fluid constituents coming from the direction in which it is moving as from the other direction. The larger v is, the more pronounced the imbalance is.

  To account for this effect, one introduces a friction force opposed to the instantaneous direction of motion—i.e. to the velocity *at the same instant*—and increasing with velocity. The simplest possibility is that of a force proportional to $v(t)$, which will be denoted as $-M\gamma v(t)$ with $\gamma > 0$.

  This actually corresponds to the viscous force exerted by a Newtonian fluid on an immersed body, in which case the "friction coefficient" $M\gamma$ is proportional to the shear viscosity η of the fluid.

- The fluid constituents also exert a *fluctuating force* $F_{\rm L}(t)$, due to their random individual collisions with the Brownian particle. This *Langevin force*, also referred to as *noise term*, will be assumed to be independent of the kinematic variables (position and velocity) of the Brownian particle.

Since both friction and noise terms introduced by this decomposition are actually two aspects of a single underlying phenomenon—the microscopic scatterings of fluid constituents on the Brownian particle—, one can reasonably anticipate the existence of a relationship between them, i.e. between the friction coefficient γ and a characteristic property of $F_{\rm L}(t)$, as we shall indeed find in § IV.1.2 b.

Assuming for the moment that there is no additional force acting on the Brownian particle, i.e. that it is "free",[(60)] the equation of motion reads

$$M\frac{{\rm d}v(t)}{{\rm d}t} = -M\gamma v(t) + F_{\rm L}(t) \quad \text{with} \quad v(t) = \frac{{\rm d}x(t)}{{\rm d}t}. \tag{IV.1}$$

This *Langevin equation* is an instance of a linear *stochastic differential equation*, i.e. an equation including a randomly varying term—here $F_{\rm L}(t)$—with given statistical properties—which we shall specify in the next paragraph in the case of the Langevin force. The solution $v(t)$ to such an equation for a given initial condition is itself a stochastic process.

Accordingly, one should distinguish—although we shall rather sloppily use the same notations—between the stochastic processes $F_{\rm L}(t)$, $v(t)$ and below $x(t)$, and their respective realizations. If $F_{\rm L}(t)$ is a realization of the corresponding stochastic process, then Eq. (IV.1) is an ordinary (linear) differential equation for $v(t)$, including a perfectly deterministic term $F_{\rm L}$. Its solution $v(t)$ for a given initial condition is a well-determined function in the usual sense.

The reader should keep in mind this dual meaning of the notations when going through the following § IV.1.2–IV.1.6.

Remarks:

∗ Strictly speaking, the *classical* collisions of the fluid particles on the Brownian particle are not random, but entirely governed by the deterministic Liouville equation for the total system. The

(60) This assumption will be relaxed in § IV.1.6.

randomness of the macroscopically-perceived Langevin force comes from the fact that it is in practice impossible to fully characterize the microstate of the fluid, which has to be described statistically.

∗ As mentioned at the end of § I.2.1, the relaxation of a thermodynamic extensive variable towards its equilibrium value can be described, provided the system is near equilibrium, by a first order linear differential equation. Such an extensive variable is in fact the expectation value of the sum over many particles of a microscopic quantity, so that it can fluctuate around its average. These fluctuations can be modeled by adding a fluctuating (generalized) force in the evolution equation (I.33), which then becomes of the Langevin type (IV.1):

$$\frac{\mathrm{d}\Delta X_a(t)}{\mathrm{d}t} = -\sum_c \lambda_{ac}\Delta X_c(t) + F_{a,\mathrm{L}}, \tag{IV.2}$$

with $F_{a,\mathrm{L}}$ a fluctuating term.

IV.1.1 b Properties of the noise term

The fluid in which the Brownian particle is immersed is assumed to be in a stationary state, for instance at thermodynamic equilibrium—in which case it is also in mechanical equilibrium, and thus admits a global rest frame, with respect to which $v(t)$ is measured. Accordingly, the Langevin force acting upon the particle is described by a stationary process, that is, the single-time average $\langle F_\mathrm{L}(t)\rangle$ is time-independent, while the two-point average $\langle F_\mathrm{L}(t)F_\mathrm{L}(t')\rangle$ only depends on the difference $t'-t$.

In order for the particle to remain (on average) motionless when it is at rest with respect to the fluid, the single-time average of the Langevin force should actually vanish. Since we assumed $F_\mathrm{L}(t)$ to be independent of the particle velocity, this remains true even when the Brownian particle is moving:

$$\big\langle F_\mathrm{L}(t)\big\rangle = 0. \tag{IV.3a}$$

In consequence, the autocorrelation function (C.5) of the force simplifies to

$$\kappa(\tau) = \big\langle F_\mathrm{L}(t)F_\mathrm{L}(t+\tau)\big\rangle. \tag{IV.3b}$$

As always for stationary processes, $\kappa(\tau)$ only depends on $|\tau|$. $\kappa(\tau)$ is assumed to be integrable, with

$$\int_{-\infty}^{\infty} \kappa(\tau)\,\mathrm{d}\tau \equiv 2D_v M^2, \tag{IV.3c}$$

which defines the parameter D_v.

Let τ_c be the *autocorrelation time* over which $\kappa(\tau)$ decreases. τ_c is typically of the order of the time interval between two collisions of the fluid particles on the Brownian particle. If τ_c happens to be much smaller than all other time scales in the system, then the autocorrelation function can meaningfully be approximated by a Dirac distribution

$$\kappa(\tau) = 2D_v M^2\delta(\tau). \tag{IV.3d}$$

More generally, one may write

$$\kappa(\tau) = 2D_v M^2\delta_{\tau_c}(\tau), \tag{IV.3e}$$

where δ_{τ_c} is an even function, peaked around the origin with a typical width of order τ_c, and whose integral over $\mathbb{R}$ equals 1.

Remarks:

∗ Throughout this Chapter, expectation values denoted with angular brackets $\langle\cdots\rangle$—as e.g. in Eq. (IV.3a) or (IV.3b)—represent averages over different microscopic configurations of the “fluid” with the same macroscopic properties.

∗ In the case of multidimensional Brownian motion, one usually assumes that the correlation matrix $\kappa^{ij}(\tau)$ of the Cartesian components of the fluctuating force is diagonal, which can be justified in the case the underlying microscopic dynamics involve interactions depending only on inter-particle distances (see §IV.4.1 a).

∗ Equations (IV.3) constitute "minimal" assumptions, which will allow us hereafter to compute the first and second moments of the stochastic processes $v(t)$ and $x(t)$, but do not fully specify the Langevin force $F_{\rm L}(t)$. A possibility for determining entirely the statistical properties of $F_{\rm L}(t)$ could be to assume that it is *Gaussian*, in addition to stationary.

Since $F_{\rm L}(t)$ actually results from summing a large number of random processes—the microscopic forces due to individual collisions with the fluid constituents—with the same probability distribution, this assumption of Gaussianity simply reflects the central limit theorem (Appendix B.5).

IV.1.2 Relaxation of the velocity

We now wish to solve the Langevin equation (IV.1), assuming that at the initial time $t = t_0$, the velocity of the Brownian particle is fixed, $v(t_0) = v_0$.

Under this initial condition, the solution to the Langevin equation for $t > t_0$ reads[61]

$$v(t) = v_0\,{\rm e}^{-\gamma(t-t_0)} + \frac{1}{M}\int_{t_0}^{t} F_{\rm L}(t')\,{\rm e}^{-\gamma(t-t')}\,{\rm d}t' \quad \text{for } t > t_0. \tag{IV.4}$$

Since $F_{\rm L}(t')$ is a stochastic process, so is $v(t)$. The first moments of its distribution can easily be computed.

Remark: The integral on the right-hand side of the previous equation has to be taken with a grain of salt, as it is not clear whether $F_{\rm L}$ is integrable.

IV.1.2 a Average velocity

Averaging Eq. (IV.4) over an ensemble of realizations, one finds thanks to property (IV.3a)

$$\langle v(t)\rangle = v_0\,{\rm e}^{-\gamma(t-t_0)} \quad \text{for } t > t_0. \tag{IV.5}$$

That is, the average velocity relaxes exponentially to 0 with a characteristic *relaxation time*

$$\tau_{\rm r} \equiv \frac{1}{\gamma}. \tag{IV.6}$$

Since its average value depends on time, $v(t)$ is not a stationary process.

IV.1.2 b Variance of the velocity. Fluctuation–dissipation theorem

Recognizing the average velocity (IV.5) in the first term on the right-hand side of Eq. (IV.4), one also obtains at once the variance

$$\sigma_v(t)^2 \equiv \Big\langle \big[v(t) - \langle v(t)\rangle\big]^2 \Big\rangle = \frac{1}{M^2}\int_{t_0}^{t}\!\int_{t_0}^{t} \big\langle F_{\rm L}(t')F_{\rm L}(t'')\big\rangle\,{\rm e}^{-\gamma(t-t')}\,{\rm e}^{-\gamma(t-t'')}\,{\rm d}t'\,{\rm d}t'' \quad \text{for } t > t_0. \tag{IV.7}$$

If the simplified form (IV.3d) of the autocorrelation function of the Langevin force holds—which for the sake of consistency necessitates at least $\tau_{\rm c} \ll \tau_{\rm r}$—, this variance becomes

$$\sigma_v(t)^2 = 2D_v \int_{t_0}^{t} {\rm e}^{-2\gamma(t-t')}\,{\rm d}t' = \frac{D_v}{\gamma}\big(1 - {\rm e}^{-2\gamma(t-t_0)}\big) \quad \text{for } t > t_0. \tag{IV.8}$$

σ_v^2 thus vanishes at $t = t_0$—the initial condition is the same for all realizations—, then grows, at first almost linearly

$$\sigma_v(t)^2 \simeq 2D_v(t - t_0) \quad \text{for } 0 \le t - t_0 \ll \tau_{\rm r}, \tag{IV.9}$$

[61] Remember that this expression, as well as many other ones in this section, holds both for realizations of the stochastic processes at play—in which case the meaning is clear—and by extension for the stochastic processes themselves.

before saturating at large times

$$\sigma_v(t)^2 \simeq \frac{D_v}{\gamma} \quad \text{for } t - t_0 \gg \tau_r. \tag{IV.10}$$

Equation (IV.9) suggests that D_v is a *diffusion coefficient* in velocity space.

Remark: The above results remain valid even if the simplified form (IV.3d) does not hold, provided the discussion is restricted to times t siginificantly larger than the autocorrelation time τ_c of the Langevin force.

Using definition (IV.3b), the right-hand side of Eq. (IV.7) can be recast as

$$\frac{e^{-2\gamma t}}{M^2}\int_{t_0}^{t}\int_{t_0}^{t}\kappa(t'-t'')\,e^{\gamma(t'+t'')}\,dt'\,dt'' = \frac{e^{-2\gamma t}}{2M^2}\int_{t_0-t}^{t-t_0}\kappa(\tau)\,d\tau\int_{2t_0}^{2t}e^{\gamma(t'+t'')}\,d(t'+t'').$$

The integral over $t'+t''$ is straightforward, while for that over τ, one may for $t - t_0 \gg \tau_c$ extend the boundaries to $-\infty$ and $+\infty$ without changing much the result. Invoking then the normalization (IV.3c), one recovers the variance (IV.8). □

According to Eq. (IV.5), the average velocity vanishes at large times, so that the variance (IV.10) equals the mean square velocity. That is, the average kinetic energy of the Brownian particle tends at large times towards a fixed value

$$\langle E(t)\rangle \simeq \frac{MD_v}{2\gamma} \quad \text{for } t - t_0 \gg \tau_r. \tag{IV.11}$$

In that limit, the Brownian particle is in equilibrium with the surrounding fluid. If the latter is at thermodynamic equilibrium at temperature T—one then refers to it as a *thermal bath* or *thermal reservoir*—then the average energy of the Brownian particle is according to the equipartition theorem equal to $\frac{1}{2}k_BT$, which yields

$$\boxed{D_v = \frac{k_BT}{M}\gamma.} \tag{IV.12}$$

This identity relates a quantity associated with fluctuations—the diffusion coefficient D_v, see Eq. (IV.9)—with a coefficient modeling dissipation, namely γ. This is thus an example of the more general *fluctuation–dissipation theorem* discussed in § III.3.4.

Since D_v characterizes the statistical properties of the stochastic Langevin force [Eq. (IV.3c)], Eq. (IV.12) actually relates the latter to the friction force.

IV.1.3 Evolution of the position of the Brownian particle. Diffusion

Until now, we have focused on the velocity of the Brownian particle. Instead, one could study its position $x(t)$, or equivalently its displacement from an initial position $x(t_0) = x_0$ at time t_0.

Integrating the velocity (IV.4) from the initial instant until time t yields the formal solution

$$x(t) = x_0 + \frac{v_0}{\gamma}\left(1 - e^{-\gamma(t-t_0)}\right) + \frac{1}{M}\int_{t_0}^{t} F_L(t')\,\frac{1 - e^{-\gamma(t-t')}}{\gamma}\,dt' \quad \text{for } t > t_0. \tag{IV.13}$$

$x(t)$, and in turn the displacement $x(t) - x_0$, is also a stochastic process, whose first two moments we shall now compute.

IV.1.3 a Average displacement

First, the average position at time t is simply

$$\langle x(t)\rangle = x_0 + \frac{v_0}{\gamma}\left(1 - e^{-\gamma(t-t_0)}\right) \quad \text{for } t > t_0, \tag{IV.14}$$

thanks to the vanishing expectation value (IV.3a) of the Langevin force.

For $t-t_0 \ll \tau_r$, $\langle x(t)\rangle \simeq x_0+v_0(t-t_0)$, i.e. the motion is approximately uniform. In the opposite limit $t-t_0 \gg \tau_r$, the mean displacement $\langle x(t)\rangle - x_0$ tends exponentially towards the asymptotic value v_0/γ.

IV.1.3 b Variance of the displacement

The first two terms in the right member of Eq. (IV.13) are exactly the average position (IV.14), that is, the last term equals precisely $x(t)-\langle x(t)\rangle$, or equivalently $[x(t)-x_0]-\langle x(t)-x_0\rangle$. The variance of the position is thus equal to the variance of the displacement, and is given by

$$\sigma_x(t)^2 = \frac{1}{M^2\gamma^2}\int_{t_0}^{t}\!\int_{t_0}^{t}\langle F_{\rm L}(t')F_{\rm L}(t'')\rangle\big[1-{\rm e}^{-\gamma(t-t')}\big]\big[1-{\rm e}^{-\gamma(t-t'')}\big]\,{\rm d}t'\,{\rm d}t'' \quad \text{for } t>t_0. \tag{IV.15}$$

When the autocorrelation function of the Langevin force can be approximated by the simplified form (IV.3d), this yields

$$\sigma_x(t)^2 = \frac{2D_v}{\gamma^2}\int_{t_0}^{t}\big[1-{\rm e}^{-\gamma(t-t')}\big]^2\,{\rm d}t' = \frac{2D_v}{\gamma^2}\left[t-t_0-\frac{2-2{\rm e}^{-\gamma(t-t_0)}}{\gamma}+\frac{1-{\rm e}^{-2\gamma(t-t_0)}}{2\gamma}\right] \quad \text{for } t>t_0. \tag{IV.16}$$

This variance vanishes at $t=t_0$—the initial condition is known with certainty—, grows as $(t-t_0)^3$ for times $0\le t-t_0 \ll \tau_r$, then linearly at large times

$$\sigma_x(t)^2 \simeq \frac{2D_v}{\gamma^2}(t-t_0) \quad \text{for } t-t_0 \gg \tau_r. \tag{IV.17}$$

Since Eq. (IV.16) also represents the variance of the displacement, one finds under consideration of Eq. (IV.13)

$$\begin{aligned}\Big\langle\big[x(t)-x_0\big]^2\Big\rangle &= \sigma_x(t)^2+\langle x(t)-x_0\rangle^2 \\ &= \sigma_x(t)^2+\frac{v_0^2}{\gamma^2}\big(1-{\rm e}^{-\gamma(t-t_0)}\big)^2 \simeq \frac{2D_v}{\gamma^2}(t-t_0) \quad \text{for } t-t_0\gg\tau_r.\end{aligned} \tag{IV.18}$$

The last two equations show that the position of the Brownian particle behaves as the solution of a diffusion equation at large times, with a diffusion coefficient (in position space)

$$D=\frac{D_v}{\gamma^2}, \tag{IV.19}$$

(cf. § I.2.3 b).

IV.1.3 c Viscous limit. Einstein relation

In the limit $M\to 0$, $\gamma\to\infty$ at constant product $\eta_v \equiv M\gamma$, which physically amounts to neglecting the influence of inertia ($M\,{\rm d}v/{\rm d}t$) compared to that of friction ($-\eta_v v$)—hence the denomination "viscous limit"—, the Langevin equation (IV.1) becomes

$$\boxed{\eta_v\frac{{\rm d}x(t)}{{\rm d}t} = F_{\rm L}(t).} \tag{IV.20a}$$

In that limit, the autocorrelation function (IV.3d) of the fluctuating force in the limit of negligibly small autocorrelation time is denoted by

$$\kappa(\tau) = 2D\eta_v^2\,\delta(\tau), \tag{IV.20b}$$

which defines the coefficient D.

In this context, the displacement can be directly computed by integrating Eq. (IV.20a) with the initial condition $x(t_0)=x_0$, yielding

$$x(t) = x_0+\frac{1}{\eta_v}\int_{t_0}^{t}F_{\rm L}(t')\,{\rm d}t' \quad \text{for } t>t_0. \tag{IV.21}$$

With the autocorrelation function (IV.20b), this gives at once

$$\Big\langle\big[x(t)-x_0\big]^2\Big\rangle = 2D(t-t_0) \quad \text{for } t\ge t_0, \tag{IV.22}$$

which now holds at any time $t \geq t_0$, not only in the large time limit as in Eq. (IV.18). That is, the motion of the Brownian particle is now a diffusive motion at all times.

Combining now Eq, (IV.19) with the fluctuation–dissipation relation (IV.12) and the relation $\eta_v = m\gamma$, one obtains

$$D = \frac{k_B T}{\eta_v}. \tag{IV.23}$$

If the Brownian particles are charged, with electric charge q, then the friction coefficient η_v is related to the electrical mobility $\mu_{\text{el.}}$ by $\mu_{\text{el.}} = q/\eta_v$ (see § IV.1.6 below), so that Eq. (IV.23) becomes the *Einstein relation*

$$D = \frac{k_B T}{q}\,\mu_{\text{el.}} \tag{IV.24}$$

[cf. Eq. (I.50)].

IV.1.4 Autocorrelation function of the velocity at equilibrium

In this paragraph and the following one, we assume that the Brownian particle is in equilibrium with the surrounding environment. This amounts to considering that a large amount of time has passed since the instant t_0 at which the initial condition was fixed, or equivalently that t_0 is far back in the past, $t_0 \to -\infty$.

Taking the latter limit in Eq. (IV.4), the velocity at time t reads

$$v(t) = \frac{1}{M}\int_{-\infty}^{t} F_{\text{L}}(t'')\,\mathrm{e}^{-\gamma(t-t'')}\,\mathrm{d}t'', \tag{IV.25}$$

where we have renamed the integration variable t'' for later convenience. As could be anticipated, v_0 no longer appears in this expression: the initial condition has been “forgotten”.

One easily sees that the average value $\langle v(t)\rangle$ vanishes, and is thus in particular time-independent. More generally, one can check with a straightforward change of variable that $v(t)$ at equilibrium is a stationary stochastic process, thanks to the assumed stationarity of $F_{\text{L}}(t)$. We shall now compute the autocorrelation function of $v(t)$, which characterizes its fluctuations.

Starting from the velocity (IV.25), one first finds the correlation function between the velocity and the fluctuating force

$$\langle F_{\text{L}}(t)v(t+\tau)\rangle = \frac{1}{M}\int_{-\infty}^{t+\tau} \big\langle F_{\text{L}}(t)F_{\text{L}}(t'')\big\rangle \mathrm{e}^{-\gamma(t+\tau-t'')}\,\mathrm{d}t''.$$

In the case where the simplified form (IV.3d) of the autocorrelation function of the Langevin force holds, this becomes

$$\big\langle F_{\text{L}}(t)v(t+\tau)\big\rangle = 2D_v M\int_{-\infty}^{t+\tau} \delta(t-t'')\,\mathrm{e}^{-\gamma(t+\tau-t'')}\,\mathrm{d}t'' = \begin{cases} 2D_v M\,\mathrm{e}^{-\gamma\tau} & \text{for } \tau > 0, \\ 0 & \text{for } \tau < 0. \end{cases} \tag{IV.26}$$

That is, the velocity of the Brownian particle at a given time is only correlated to past values of the Langevin force, and the correlation dies out on a typical time scale of order $\gamma^{-1} = \tau_{\text{r}}$.

The autocorrelation function of the velocity is then easily deduced from

$$\big\langle v(t)v(t')\big\rangle = \frac{1}{M}\int_{-\infty}^{t} \big\langle F_{\text{L}}(t'')v(t')\big\rangle \mathrm{e}^{-\gamma(t-t'')}\,\mathrm{d}t'',$$

which follows from Eq. (IV.25). In the regime where the approximation (IV.26) is valid, that is neglecting the autocorrelation time τ_{c} of the Langevin force, this yields

$$\big\langle v(t)v(t')\big\rangle = \frac{D_v}{\gamma}\,\mathrm{e}^{-\gamma|t-t'|}. \tag{IV.27}$$

This autocorrelation function only depends on the modulus of the time difference, as expected for a stationary stochastic process, and decreases exponentially with an autocorrelation time given by the relaxation time $\tau_{\rm r}$. Note that for $t' = t$, we recover the large-time limit (IV.10) of the variance of the velocity.

If the environment is at thermal equilibrium at temperature T, relation (IV.12) gives

$$\langle v(t)v(t') \rangle = \frac{k_B T}{M}\, \mathrm{e}^{-\gamma|t-t'|}. \tag{IV.28}$$

Remark: Inspecting the average velocity (IV.5) and autocorrelation function (IV.27), one sees that they obey the same first-order linear differential equation, with the same characteristic relaxation time scale $\tau_{\rm r}$.