I.2.4 **Linear transport phenomena in simple fluids**

As last example of application of the formalism of linear Markovian thermodynamic processes, let us discuss transport phenomena in isotropic "simple" non-relativistic fluids, i.e. fluids made of a single electrically neutral constituent, whose only species of (spherically symmetric) particles have a mass m.

In such a system, the subsystems that coincide with the cells at the level of which local thermodynamic equilibrium and local extensive quantities are defined—the so-called *fluid particles*—can move with respect to each other. The local momenta $\vec{P}(t,\vec{r})$ of the various cells thus differ from each other, so that there is no global rest frame in which every local momentum would vanish. This constitutes a new feature compared to the previous examples, and will require our determining the proper variables, as defined in some fixed reference frame, for describing the system, before we can apply the generic ideas of linear Markovian processes.

Hereafter, the dependence of fields on time t and position $\vec{r}$ will generally not be written. We shall use Cartesian coordinates labeled by indices i, $j\dots$ running from 1 to 3, whose position will have no meaning. The components of $\vec{r}$ will be denoted as x_i.

I.2.4 a **Extensive parameters, intensive quantities and fluxes**

Each fluid cell is characterized by a set of local extensive variables, namely its total energy, particle number, and momentum. Both energy and momentum clearly depend on the reference frame. On the one hand, we shall consider a fixed inertial frame $\mathcal{R}_0$—corresponding to the frame in which the observer who describes the fluid is at rest. In that frame, energy and momentum will be denoted by E and $\vec{P}$.

In addition, we shall also use an inertial frame $\mathcal{R}_{\vec{v}}$ which at time t moves with respect to $\mathcal{R}_0$ with the same velocity $\vec{v} = \vec{v}(t,\vec{r})$ as the fluid particle located at position $\vec{r}$. In that *comoving* frame, the fluid particle is momentarily at rest, so that $\mathcal{R}_{\vec{v}}$ will be referred to as *comoving (local) rest frame*, where "local" conveniently emphasizes that at a given instant, the velocity takes different values at different points, resulting in the existence of different comoving rest frames. In $\mathcal{R}_{\vec{v}}$, the energy of the fluid cell reduces to its internal energy U while its momentum vanishes. Our first task is to find what are the conjugate intensive variables and fluxes in the fixed frame $\mathcal{R}_0$.

Let $M(t,\vec{r}) \equiv N(t,\vec{r})m$ denotes the mass of fluid contained in the cell at position $\vec{r}$ at time t. The energy $E(t,\vec{r})$ is then simply equal to the sum of the internal energy $U(t,\vec{r})$ and the kinetic energy $\vec{P}(t,\vec{r})^2/2M(t,\vec{r})$ of the fluid particle. Thus, the characteristic extensive parameters of a cell in the fixed frame $\mathcal{R}_0$ read

$$E = U + \frac{\vec{P}^2}{2M}, \quad N, \quad P_i,\ i \in \{1,2,3\}, \tag{I.63a}$$

with respective densities (amount per unit volume)

$$e + \frac{1}{2}\rho\,\vec{v}^2, \quad n, \quad \rho\,v_i, \tag{I.63b}$$

where $\rho(t,\vec{r}) = mn(t,\vec{r})$ is the mass density of the fluid.

Writing that the entropy does not depend on the choice of the inertial frame in which it is measured and equating its values in $\mathcal{R}_0$ and $\mathcal{R}_{\vec{v}}$ (see the derivation of $\mathcal{Y}_{P_i}$ in § I.1.3 a), one finds the intensive variables conjugate to the extensive parameters (I.63) are

$$\mathcal{Y}_E = \frac{1}{T}, \quad \mathcal{Y}_N = -\frac{\mu_{\vec{v}}}{T} = -\frac{\mu + \frac{1}{2}m\,\vec{v}^2}{T}, \quad \mathcal{Y}_{P_i} = -\frac{v_i}{T} \tag{I.64}$$

respectively, where μ denotes the chemical potential in the comoving rest frame. Note that the temperature does not depend on the reference frame.

In non-relativistic fluid dynamics, the traditional variables are the mass density ρ (instead of n), the *flow velocity* $\vec{v}$ (instead of the momentum density) and the thermodynamic pressure $\mathcal{P}$ (instead

of the internal energy density). The latter is given by the Gibbs–Duhem[(o)] relation

$$\mathcal{P} = Ts - e + \mu n. \tag{I.65}$$

Since the right-hand side also equals $Ts - \left(e + \frac{1}{2}\rho\,\vec{v}^2\right) + \left(\mu + \frac{1}{2}m\,\vec{v}^2\right)n$, the pressure $\mathcal{P}$ keeps the same value in every inertial frame.

Besides the densities (I.63b) and intensive variables (I.64), whose respective gradients are the various affinities, we still have to introduce the flux densities of the extensive parameters. Following the generic expansion (I.29a), these fluxes generally consist of a contribution depending on the affinities, which describes the response of the system to a departure from equilibrium, and of affinity-independent equilibrium fluxes, which we shall now determine.

For that purpose, it is convenient to first establish the form of the equilibrium fluxes in the comoving local rest frame $\mathcal{R}_{\vec{v}}$ before performing a Galilean transformation with velocity $-\vec{v}$ to obtain the expressions in the fixed reference frame $\mathcal{R}_0$.

In an equilibrated isotropic fluid at rest, invariance under rotations implies that the vectorial fluxes of internal energy and of particle number should vanish. Moreover, the momentum flux density, which is a tensor of order 2, must be proportional to the identity tensor, again to satisfy rotational symmetry, as argued in § I.2.2 a. To interpret the proportionality coefficient, one should realize that the flux of the i-th component of linear momentum through a surface perpendicular to the i-axis represents a normal force on that surface. In mechanical equilibrium, this force is balanced by the i-component of the force exerted by the remainder of the fluid on the surface element, i.e. in a fluid at rest through the *hydrostatic pressure*. All in all, one thus finds that the equilibrium fluxes in the comoving local rest frame are

$$\vec{\mathcal{J}}_E^{\text{eq.}}\big|_{\mathcal{R}_{\vec{v}}} = \vec{0}, \quad \vec{\mathcal{J}}_N^{\text{eq.}}\big|_{\mathcal{R}_{\vec{v}}} = \vec{0}, \quad \mathbf{J}_{\vec{P}}^{\text{eq.}}\big|_{\mathcal{R}_{\vec{v}}} = \mathcal{P}\,\mathbf{1}_3, \tag{I.66}$$

where the latter identity can be expressed in term of components as $(J_{\vec{P}}^{\text{eq.}})_{ij} = \mathcal{P}\,\delta_{ij}$.

When performing the Galilean transformation with velocity $-\vec{v}$ to the fixed reference frame $\mathcal{R}_0$, two effects have to be taken into account. First, the transported quantities may be modified, as is the case of energy density or momentum density. Secondly, a flux is defined as the quantity flowing per unit time through a motionless surface, and the motionless surfaces in both frames differ.

> To account consistently for both these effects, one should first consider an infinitesimal Galilean transformation from a frame $\mathcal{R}_{\vec{v}'}$ with velocity $\vec{v}'$ (with respect to $\mathcal{R}_0$) to a frame $\mathcal{R}_{\vec{v}'+d\vec{v}'}$ with velocity $\vec{v}' + d\vec{v}'$. In $\mathcal{R}_{\vec{v}'}$, the characteristic densities at a point moving with velocity $\vec{v}$ in $\mathcal{R}_0$ take the values [cf. Eq. (I.63b)]
>
> $$e + \frac{1}{2}\rho(\vec{v} - \vec{v}')^2, \quad n, \quad \rho(v_i - v_i').$$
>
> Viewed in $\mathcal{R}_{\vec{v}'+d\vec{v}'}$, the densities at the same point become, to first order in $d\vec{v}'$
>
> $$e + \frac{1}{2}\rho(\vec{v} - \vec{v}' - d\vec{v}')^2 = e + \frac{1}{2}\rho(\vec{v} - \vec{v}')^2 - \rho(\vec{v} - \vec{v}')\cdot d\vec{v}', \quad n, \quad \rho(v_i - v_i' - dv_i') = \rho(v_i - v_i') - n\,m\,dv_i'.$$
>
> From the latter formulae, one deduces the variations of the densities in the infinitesimal Galilean transformation, namely
>
> $$d\left(e + \frac{1}{2}\rho\,\vec{w}^2\right) = \rho\,\vec{w}\cdot d\vec{w}, \quad dn = 0, \quad d(\rho\,\vec{w}) = n\,m\,d\vec{w},$$
>
> where we have set $\vec{w} = \vec{v} - \vec{v}'$ and accordingly $d\vec{w} = -d\vec{v}'$.
>
> Consider now the fluxes of the extensive quantities at the same position and time. Recognizing in the infinitesimal variation of the energy density above the product of the momentum density $\rho\,\vec{w}$ with the velocity increment, one deduces that the change in energy flux due to the variation of the transported energy density will involve the momentum flux, again multiplied by the

---

[(o)]P. Duhem, 1861–1916

velocity increment. Similarly, the variation of momentum density involves the particle number density, so that the change in momentum flux will involve the flux of particle number.

On the other hand, the motion with velocity $d\vec{w} = -d\vec{v}'$ in $\mathcal{R}_{\vec{v}'+d\vec{v}'}$ of a surface which is motionless in $\mathcal{R}_{\vec{v}'}$ contributes to any flux an amount given to first order in $d\vec{w}$ by the product of the corresponding density, as measured in $\mathcal{R}_{\vec{v}'}$, with $d\vec{w}$

All in all, the differences between the values of the equilibrium fluxes measured in $\mathcal{R}_{\vec{v}'+d\vec{v}'}$ and $\mathcal{R}_{\vec{v}'}$ read

$$d\mathcal{J}^{\text{eq.}}_{E,i} = \sum_{j=1}^{3}(\mathsf{J}^{\text{eq.}}_{\vec{P}})_{ij}\,dw_j + \left(e + \frac{1}{2}\rho\,\vec{w}^{\,2}\right)dw_i, \quad d\vec{\mathcal{J}}^{\text{eq.}}_N = n\,d\vec{w}, \quad d(\mathsf{J}^{\text{eq.}}_{\vec{P}})_{ij} = \mathcal{J}^{\text{eq.}}_{N,j}\,m\,dw_i + \rho\,w_i\,dw_j.$$

These equations can be viewed as defining partial differential equations, which can be integrated from $\vec{w} = \vec{0}$ (i.e. from the local rest frame comoving with the fluid at velocity $\vec{v}$) to $\vec{w} = \vec{v}$ (the fixed frame), starting with known initial conditions at velocity $\vec{w} = \vec{0}$—namely the equilibrium fluxes (I.66). One first finds $\vec{\mathcal{J}}^{\text{eq.}}_N$, which is then injected in the equations for $(\mathsf{J}^{\text{eq.}}_{\vec{P}})_{ij}$. Solving the latter, the result can be used in the equations for $\mathcal{J}^{\text{eq.}}_{E,i}$.

In the fixed reference frame $\mathcal{R}_0$, the equilibrium fluxes read

$$\vec{\mathcal{J}}^{\text{eq.}}_E = \left(e + \frac{1}{2}\rho\vec{v}^{\,2} + \mathcal{P}\right)\vec{v}, \quad \vec{\mathcal{J}}^{\text{eq.}}_N = n\vec{v}, \quad \left(J^{\text{eq.}}_{\vec{P}}\right)_{ij} = \mathcal{P}\,\delta_{ij} + \rho\,v_i v_j. \tag{I.67}$$

These equilibrium fluxes have to be complemented with affinity-dependent terms, to yield the total fluxes. For the sake of simplicity, we shall restrict ourselves to Markovian, memoryless transport processes.

By definition of the non-relativistic flow velocity as the average velocity of particles in the fluid cell under consideration, the flux of particle number does not receive any such extra term, and will always remain equal to its equilibrium value, $\vec{\mathcal{J}}_N = \vec{\mathcal{J}}^{\text{eq.}}_N$.

As a consequence (cf. the derivation above), the difference between the values of the momentum flux components $(J_{\vec{P}})_{ij}$ taken in the frames $\mathcal{R}_0$ and $\mathcal{R}_{\vec{v}}$ is the same as for the corresponding equilibrium fluxes, namely $\rho\,v_i v_j$. The only dependence of $(\mathsf{J}_{\vec{P}})_{ij}$ on any affinity thus already affects its value in the local rest frame of the fluid, in which the diagonal tensor $\mathcal{P}\,\delta_{ij}$ becomes an affinity-dependent tensor $\boldsymbol{\pi}$ with components π_{ij} called *stress tensor*.

Eventually, this stress tensor plays a role in the change of frame for the energy flux, with $\boldsymbol{\pi}\cdot\vec{v}$ replacing $\mathcal{P}\vec{v}$. Additionally, one also has to allow for an affinity-dependent contribution $\vec{\mathcal{J}}_U$ to the energy flux in the local rest frame. Altogether, one obtains the fluxes

$$\vec{\mathcal{J}}_E = \left(e + \frac{1}{2}\rho\vec{v}^{\,2}\right)\vec{v} + \vec{\mathcal{J}}_U + \boldsymbol{\pi}\cdot\vec{v}, \quad \vec{\mathcal{J}}_N = n\vec{v}, \quad \left(J_{\vec{P}}\right)_{ij} = \pi_{ij} + \rho\,v_i v_j, \tag{I.68}$$

where the functional dependences of $\vec{\mathcal{J}}_U$ and the stress tensor $\boldsymbol{\pi}$ on the various affinities depend on the model under consideration, i.e. concretely on the specific fluid under study.

Remark: One can also show that angular momentum conservation implies that the stress tensor is symmetric.

I.2.4 b Conservation laws

Having obtained the fluxes in the fixed inertial frame $\mathcal{R}_0$, we can now insert their expressions and those of the densities (I.63b) in the general local balance equation (I.18a), where the source/sink term will vanish since energy, particle number and momentum are conserved.

Starting with particle number density, or equivalently—to respect the tradition—with the mass density $\rho = nm$, the simple flux $m\vec{\mathcal{J}}_N = \rho\,\vec{v}$ leads to the continuity equation

$$\boxed{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\,\vec{v}\right) = 0,} \tag{I.69}$$

which expresses local mass conservation.

In a second step, we can consider the balance equation for momentum, for the sake of simplicity component by component. The momentum density $\rho\mathsf{v}_i$ and flux $\mathsf{J}_{\vec{P}}^{ij}$ give for every $i \in \{1, 2, 3\}$

$$\frac{\partial(\rho\mathsf{v}_i)}{\partial t} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\big(\pi_{ij} + \rho\,\mathsf{v}_i\mathsf{v}_j\big) = 0.$$

A simple calculation using the mass balance equation (I.69) to cancel two terms allows one to rewrite this relation as

$$\rho\bigg[\frac{\partial\mathsf{v}_i}{\partial t} + \big(\vec{\mathsf{v}}\cdot\vec{\nabla}\big)\mathsf{v}_i\bigg] + \sum_{j=1}^{3} \frac{\partial\pi_{ij}}{\partial x_j} = 0. \tag{I.70}$$

Since the stress tensor component π_{ij} represents the amount of momentum in direction i flowing per unit time through a unit surface perpendicular to direction j, it equals according to Newton's[p] second law the i-th component of the force per unit area on a surface normal to the direction j. Following Newton's third law, this equals the negative of the force per unit area exerted by the remainder of the fluid on the cell under study, so that this local balance equation actually expresses the fundamental principle of Newtonian dynamics.

Eventually, the energy density (I.63b) and flux (I.68) yield for the local balance equation for energy in the fixed reference frame

$$\frac{\partial}{\partial t}\bigg(e + \frac{1}{2}\rho\,\vec{\mathsf{v}}^2\bigg) + \vec{\nabla}\cdot\vec{\mathcal{J}}_E = \frac{\partial}{\partial t}\bigg(e + \frac{1}{2}\rho\,\vec{\mathsf{v}}^2\bigg) + \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\bigg[\bigg(e + \frac{1}{2}\rho\vec{\mathsf{v}}^2\bigg)\mathsf{v}_i + \vec{\mathcal{J}}_U + \sum_{j=1}^{3}\pi_{ij}\mathsf{v}_j\bigg] = 0.$$

To simplify this equation, one views $\frac{1}{2}\rho\,\vec{\mathsf{v}}^2$ as ρ times $\frac{1}{2}\vec{\mathsf{v}}^2$ and then differentiates with the product rule. The terms proportional to $\frac{1}{2}\vec{\mathsf{v}}^2$ vanish thanks to the continuity equation (I.69). Those proportional to ρ can be rewritten with the help of the identity $\vec{\nabla}\big(\frac{1}{2}\vec{\mathsf{v}}^2\big) = \big(\vec{\mathsf{v}}\cdot\vec{\nabla}\big)\vec{\mathsf{v}} + \vec{\mathsf{v}}\times\big(\vec{\nabla}\times\vec{\mathsf{v}}\big)$, and one recognizes the sum over i of $\rho\mathsf{v}_i$ multiplied with the term within square brackets in Eq. (I.70). A further application to the product rule leads then to

$$\frac{\partial e}{\partial t} + \vec{\nabla}\cdot\big(e\,\vec{\mathsf{v}} + \vec{\mathcal{J}}_U\big) + \sum_{i,j=1}^{3}\pi_{ij}\frac{\partial\mathsf{v}_j}{\partial x_i} = 0. \tag{I.71}$$

Interestingly, the kinetic energy density no longer appears in this equation, which can be interpreted as describing the change in the internal energy due to dissipation $(\vec{\mathcal{J}}_U)$ and to the work of forces exerted on neighboring fluid cells (π_{ij}).

I.2.4 c Linear Markovian transport processes in a simple fluid

Let us now explicitly consider a model for the fluid, by assuming that the fluxes (I.68)—and more precisely the energy flux $\vec{\mathcal{J}}_U$ and the stress tensor $\boldsymbol{\pi}$—are linear functions of the affinities, i.e. of the first derivatives of the intensive variables (I.64) with respect to space coordinates. This assumption defines *Newtonian fluids*, which are those governed by the resulting dynamical laws derived from Eqs. (I.70) and (I.71).

In the linear regime, one only needs to introduce first-order kinetic coefficients L_{ab} relating the fluxes to the affinities as in Eq. (I.31). Enumerating the former, the two vectors $\vec{\mathcal{J}}_N$ and $\vec{\mathcal{J}}_U$ and the order 2 tensor $\mathsf{J}_{\vec{P}}$ amount altogether to 15 components. Similarly, the gradients of the intensive variables (I.64) also represent 15 different scalar fields, so that a naive approach would necessitate 15×15 kinetic coefficients L_{ab}. Fortunately, the problem can be considerably simplified thanks to the general principles of § I.2.2 and to system-specific properties, resulting in a small number of coefficients.

---

[p]I. Newton, 1642–1727

A first important remark is that by definition of the flow velocity $\vec{v}$, the particle-number flux $\vec{\mathcal{J}}_N$ always remains equal to its equilibrium contribution $\vec{\mathcal{J}}_N^{\text{eq.}} = n\,\vec{v}$, irrespective of the affinities. This means that all coefficients L_{Na} vanish. Thanks to the Onsager relations, all reciprocal coefficients L_{aN} also vanish. Thus a gradient in chemical potential does not lead to any dissipation in a simple fluid: chemical potential plays no direct role in entropy production.[18]

Using the second consequence listed in § I.2.2 of Curie's symmetry principle applied to isotropic media, the vectorial flux (resp. affinity) for energy cannot couple to the affinity (resp. flux) for momentum, which is a tensor of order 2. That is, the tensors of order 3 coupling E and $\vec{P}$ identically vanish. All in all, this means that there is no indirect transport in a simple fluid.

According to the first of the consequences of isotropy given in § I.2.2, the transport of energy—a scalar quantity—involves a tensor $\mathbf{L}_{EE}$ of order 2 proportional to the identity, i.e. effectively a single kinetic coefficient L_{EE}. To write down this relation explicitly, it is convenient to move to a rest frame in which the fluid is locally at rest, so that the energy flux has a simple expression.

In the Galilean transformation from the fixed frame $\mathcal{R}_0$ to the frame $\mathcal{R}_{\vec{v}_0}$ comoving with the fluid at point M_0, where the fluid velocity at time t_0 is $\vec{v}_0$, the position and velocity respectively transform according to $\vec{r}\,' = \vec{r} - \vec{v}_0(t - t_0)$, which implies the identity of derivatives $\partial/\partial x_i' = \partial/\partial x_i$, and $\vec{v}\,'(t, \vec{r}\,') = \vec{v}(t, \vec{r}) - \vec{v}_0$, where primed resp. unprimed quantities refer to $\mathcal{R}_{\vec{v}_0}$ resp. $\mathcal{R}_0$. Since temperature is the same in all frames, the affinity conjugate to energy is $\vec{\nabla}(1/T)$, so that the relation between this affinity and the energy flux in the comoving rest frame reads

$$\vec{\mathcal{J}}_U = L_{EE}\vec{\nabla}\left(\frac{1}{T}\right), \tag{I.72a}$$

where L_{EE} is as always non-negative.

In turn, the i-th component in $\mathcal{R}_{\vec{v}_0}$ of the affinity $\vec{\mathscr{F}}_{P_j}$ is given by

$$\frac{\partial}{\partial x_i'}\left(-\frac{\mathsf{v}_j'}{T}\right)\bigg|_{t_0,M_0} = \frac{\partial}{\partial x_i}\left(-\frac{\mathsf{v}_j - \mathsf{v}_{0,j}}{T}\right)\bigg|_{t_0,M_0} = -\frac{1}{T}\frac{\partial \mathsf{v}_j}{\partial x_i}\bigg|_{t_0,M_0},$$

where the term proportional to $\partial(1/T)/\partial x_i$ vanishes since it multiplies $\mathsf{v}_j - \mathsf{v}_{0,j}$ taken at time t_0 and at the point M_0. The remaining task is thus to express the components of stress tensor $\boldsymbol{\pi}$, which equals the momentum flux in the comoving local rest frame, as function of the derivatives $\partial \mathsf{v}_k/\partial x_l$. This necessitates a tensor $\mathbf{L}_{\vec{P}\vec{P}}$ of order 4 with components $L_{\vec{P}\vec{P}}^{ijkl}$ such that

$$\pi_{ij} = \sum_{k,l=1}^{3} L_{\vec{P}\vec{P}}^{ijkl}\,\frac{\partial \mathsf{v}_k}{\partial x_l}.$$

Invoking again the local isotropy of the fluid, this tensor can only be a linear combination of the three order-4 tensors invariant under rotations, namely those with (Cartesian) components $\delta_{ij}\delta_{kl}$, $\delta_{ik}\delta_{jl}$ and $\delta_{il}\delta_{jk}$, with three respective kinetic coefficients.

As noted above, $\boldsymbol{\pi}$ must be symmetric ($\pi_{ij} = \pi_{ji}$) to ensure angular momentum conservation, so that the coefficients of $\delta_{ik}\delta_{jl}$ and $\delta_{il}\delta_{jk}$ must be identical. Instead of considering π_{ij} as linear combination of $\delta_{ij}\delta_{kl}$ and $\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$, one traditionally—and equivalently—writes down a linear combination of $\delta_{ij}\delta_{kl}$ and the traceless tensor $\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}$. Introducing the two necessary non-negative coefficients $L_{\vec{P}\vec{P}}^{(1)}$, $L_{\vec{P}\vec{P}}^{(2)}$, the relation between the stress tensor components and the affinities conjugate to momentum reads

$$\pi_{ij} = \mathcal{P}\,\delta_{ij} - \frac{L_{\vec{P}\vec{P}}^{(1)}}{T}\sum_{k,l=1}^{3}\delta_{ij}\delta_{kl}\frac{\partial \mathsf{v}_k}{\partial x_l} - \frac{L_{\vec{P}\vec{P}}^{(2)}}{T}\sum_{k,l=1}^{3}\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}\right]\frac{\partial \mathsf{v}_k}{\partial x_l},$$

---

[18]A gradient in chemical potential, or equivalently in particle number density, at uniform temperature will lead via the equations of state to a gradient in pressure $\mathcal{P}$—cf. the example of an ideal gas—, which results in a macroscopic flow through Eq. (I.70). In turn, this motion will lead to dissipation due to the viscous effects described by the transport coefficients η and ζ.

where we also included the equilibrium part $\mathcal{P}\,\delta_{ij}$. With the help of the traceless symmetric tensor $\boldsymbol{\sigma}$ with components

$$\sigma_{ij} \equiv \frac{1}{2}\left(\frac{\partial \mathsf{v}_i}{\partial x_j} + \frac{\partial \mathsf{v}_j}{\partial x_i}\right) - \frac{1}{3}\left(\vec{\nabla}\cdot\vec{\mathsf{v}}\right)\delta_{ij}, \tag{I.72b}$$

the stress tensor can be rewritten more concisely as

$$\boldsymbol{\pi} = \mathcal{P}\,\mathbb{1}_3 - \frac{L^{(1)}_{\vec{P}\vec{P}}}{T}\left(\vec{\nabla}\cdot\vec{\mathsf{v}}\right)\mathbb{1}_3 - \frac{L^{(2)}_{\vec{P}\vec{P}}}{T}\,\boldsymbol{\sigma} \tag{I.72c}$$

i.e. component-wise

$$\pi_{ij} = \mathcal{P}\,\delta_{ij} - \frac{L^{(1)}_{\vec{P}\vec{P}}}{T}\left(\vec{\nabla}\cdot\vec{\mathsf{v}}\right)\delta_{ij} - \frac{L^{(2)}_{\vec{P}\vec{P}}}{T}\,\sigma_{ij}. \tag{I.72d}$$

This relation and Eq. (I.72a) are the characteristic "constitutive equations" for a Newtonian fluid.

Let us now interpret the three kinetic coefficients L_{EE}, $L^{(1)}_{\vec{P}\vec{P}}$ and $L^{(2)}_{\vec{P}\vec{P}}$ in terms of more traditional transport coefficients. First, Eq. (I.72a) is clearly reminiscent of Fourier's law

$$\vec{\mathcal{J}}_U = -\kappa\vec{\nabla}T \quad \text{with} \quad \kappa = \frac{L_{EE}}{T^2} \tag{I.73}$$

like in the case of an insulator [Eq. (I.37a)]. L_{EE} is thus related to the heat conductivity, which is non-negative, as it should be.

As was already mentioned, π_{ij} is the i-component of the force per unit area acting on a surface normal to the j-direction. Empirically, this force per unit area is related to the gradient along direction j of the i-th component of velocity through *Newton's law of viscosity*[19]

$$\pi_{ij} = -\eta\,\frac{\partial \mathsf{v}_i}{\partial x_j}, \tag{I.74}$$

with η the fluid *shear viscosity*. Identifying this empirical law with relation (I.72d) for $i \neq j$ yields

$$\eta = \frac{L^{(2)}_{\vec{P}\vec{P}}}{2T}. \tag{I.75}$$

Eventually, the parameter $L^{(1)}_{\vec{P}\vec{P}}$ is related to the transport parameter referred to as *volume viscosity* (or at times *second viscosity* or *bulk viscosity*[20]) ζ, which only plays a role in compressible flows ($\vec{\nabla}\cdot\vec{\mathsf{v}} \neq 0$), in particular in the damping of sound waves. To obtain the proper usual form for the equation of motion of a compressible flow, one must set

$$\zeta = \frac{L^{(1)}_{\vec{P}\vec{P}}}{T}. \tag{I.76}$$

With Eqs. (I.75) and (I.76), the stress tensor component (I.72c) becomes

$$\boldsymbol{\pi} = \mathcal{P}\,\mathbb{1} - \zeta\left(\vec{\nabla}\cdot\vec{\mathsf{v}}\right)\mathbb{1} - 2\eta\,\boldsymbol{\sigma}. \tag{I.77}$$

Rewriting Eqs. (I.70) and (I.71) in the linear regime in which the internal energy and momentum fluxes are respectively given by Eqs. (I.73) and (I.77), one obtains in the case of position-independent viscosity coefficients the *Navier*[q]*–Stokes*[r] *equation*

$$\rho\left[\frac{\partial\vec{\mathsf{v}}}{\partial t} + \left(\vec{\mathsf{v}}\cdot\vec{\nabla}\right)\vec{\mathsf{v}}\right] = -\vec{\nabla}\mathcal{P} + \eta\left[\triangle\vec{\mathsf{v}} + \frac{1}{3}\vec{\nabla}\left(\vec{\nabla}\cdot\vec{\mathsf{v}}\right)\right] + \zeta\vec{\nabla}\left(\vec{\nabla}\cdot\vec{\mathsf{v}}\right) \tag{I.78}$$

---

[19]This law is defined for a fluid flowing uniformly along the i-direction, so that the velocity only depends on x_j, which ensures that both $\partial \mathsf{v}_i/\partial x_i$—and thereby $\vec{\nabla}\cdot\vec{\mathsf{v}}$—and $\partial \mathsf{v}_j/\partial x_i$ vanish.

[20]Some authors, as e.g. in Ref. [21], reserve the name "bulk viscosity" to the combination $\zeta + \frac{2}{3}\eta$. The lack of unity in the terminology shows how little this coefficient has actually been studied!

[q]C.L. NAVIER, 1785–1836   [r]G. G. STOKES, 1819–1903

and the energy balance equation

$$\frac{\partial}{\partial t}\left(e + \frac{1}{2}\rho\vec{v}^2\right) + \vec{\nabla}\cdot\left\{\left(e + \frac{1}{2}\rho\vec{v}^2 + \mathcal{P}\right)\vec{v}\right.$$
$$\left. - \eta\left[(\vec{v}\cdot\vec{\nabla})\vec{v} + \vec{\nabla}\left(\frac{\vec{v}^2}{2}\right) - \frac{2}{3}\vec{v}(\vec{\nabla}\cdot\vec{v})\right] - \zeta\vec{v}(\vec{\nabla}\cdot\vec{v}) - \kappa\vec{\nabla}T\right\} = 0. \tag{I.79}$$

I.2.4 d Entropy production

To conclude this application of the thermodynamics of linear irreversible processes to simple fluids, let us write down the balance equation for entropy (I.20).

For that purpose, we first need the expression of the entropy flux density. Using the general formula (I.22) in the comoving local rest frame $\mathcal{R}_{\vec{v}}$, in which $\vec{\mathcal{J}}_N = \vec{0}$ and $\mathcal{Y}_{\vec{P}} = \vec{0}$ since both are in the general case proportional to the fluid velocity, yields

$$(\vec{\mathcal{J}}_S)_{\mathcal{R}_{\vec{v}}} = \frac{1}{T}\,\vec{\mathcal{J}}_U. \tag{I.80a}$$

Transforming back to the fixed frame $\mathcal{R}_0$, the entropy density s is invariant, so that the entropy flux only changes because surfaces at rest in $\mathcal{R}_{\vec{v}}$ are now moving:

$$\vec{\mathcal{J}}_S = \frac{1}{T}\,\vec{\mathcal{J}}_U + s\,\vec{v} \tag{I.80b}$$

Inserting in the general formula (I.30) for entropy production the fluxes (I.68), from which one subtracts their equilibrium values (I.67), and the corresponding affinities, one finds[21]

$$\sigma_S = \vec{\nabla}\left(\frac{1}{T}\right)\cdot\vec{\mathcal{J}}_U + \sum_{i,j=1}^{3}\left(-\frac{1}{T}\frac{\partial\mathsf{v}_j}{\partial x_i}\right)(\pi_{ij} - \mathcal{P}\delta_{ij}). \tag{I.81}$$

In the linear approximation where the energy and momentum fluxes are respectively given by Eqs. (I.73) and (I.77), this becomes

$$\sigma_S = \kappa T^2\left[\vec{\nabla}\left(\frac{1}{T}\right)\right]^2 + \sum_{i,j=1}^{3}\left(-\frac{1}{T}\frac{\partial\mathsf{v}_j}{\partial x_i}\right)\left[-2\eta\sigma_{ij} - \zeta(\vec{\nabla}\cdot\vec{v})\delta_{ij}\right].$$

The rightmost term between square brackets is symmetric under the exchange of the indices i and j, so that one can replace the factor $\partial\mathsf{v}_j/\partial x_i$ in front by half of the symmetrized version, i.e. according to definition (I.72b) by $\sigma_{ij} + \frac{1}{3}(\vec{\nabla}\cdot\vec{v})\delta_{ij}$. Canceling out the minus signs, one thus obtains

$$\sigma_S = \kappa T^2\left[\vec{\nabla}\left(\frac{1}{T}\right)\right]^2 + \frac{1}{T}\sum_{i,j=1}^{3}\left[\sigma_{ij} + \frac{1}{3}(\vec{\nabla}\cdot\vec{v})\delta_{ij}\right]\left[2\eta\sigma_{ij} + \zeta(\vec{\nabla}\cdot\vec{v})\delta_{ij}\right].$$

The remaining product can then be expanded. Multiplying an order-2 tensor by δ_{ij} and summing over i and j amounts to taking the trace of the tensor. Since σ_{ij} is traceless, the products $\sigma_{ij}\delta_{ij}$ yield 0, while $\delta_{ij}\delta_{ij}$ gives 3. In the end, there remains

$$\sigma_S = \kappa T^2\left[\vec{\nabla}\left(\frac{1}{T}\right)\right]^2 + \frac{\zeta}{T}(\vec{\nabla}\cdot\vec{v})^2 + \frac{2\eta}{T}\sum_{i,j=1}^{3}(\sigma_{ij})^2. \tag{I.82}$$

Since the three transport coefficients κ, η and ζ are non-negative, the entropy production rate is also always non-negative—as needed for the second law of thermodynamics.

---

[21]This is most obvious in the comoving frame, holds however in a general frame.