Chapter VII

Wigner distribution and Wigner–Weyl formalism

Classical mechanics, in its Hamiltonian formulation, describes the dynamics of a system as an evolution in its phase space (§ II.2.1). In particular, it is, at least conceptually, possible to localize exactly the phase-space position of a classical system. In contrast, it is well known that in quantum mechanics the Heisenberg uncertainty principle prohibits this precise localization in phase space. Accordingly, the usual description of a quantum-mechanical system does not rely on phase space. This is somewhat problematic if one wishes to investigate how the classical description is to emerge from the quantum one.

In this Chapter, we present one prominent phase-space-based description of quantum mechanics, namely the Wigner formalism, starting with the distribution characterizing the system (Sec. VII.1), then going onto the observables (Sec. VII.2). The evolution of the system is then discussed in Sec. VII.3. An alternative formulation of quantum mechanics in phase space is shortly introduced in an appendix (Sec. VII.B).

VII.1 Wigner distribution

For simplicity, we consider in this Section and the following two the case of a single particle without spin propagating in 3 dimensions, in absence of a vector potential. Generalizations will be considered in Sec. VII.4

VII.1.1 Definition

Consider a particle without spin, propagating in a region in which there is no vector potential to which it couples, described by the statistical operator (in Schrödinger representation) $\hat{\rho}(t)$. The associated *Wigner*[(cl)] *distribution* (or *quasi-distribution*) is defined as

$$\rho_W(t,\vec{r},\vec{p}) \equiv \frac{1}{(2\pi\hbar)^3}\int \mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}/\hbar}\left\langle \vec{r}+\frac{\vec{x}}{2}\middle|\hat{\rho}(t)\middle|\vec{r}-\frac{\vec{x}}{2}\right\rangle \mathrm{d}^3\vec{x}. \tag{VII.1}$$

More specifically, the right-hand side of Eq. (VII.1) is the position representation of the Wigner distribution.

Equivalently, the momentum representation of the Wigner distribution is

$$\rho_W(t,\vec{r},\vec{p}) \equiv \frac{1}{(2\pi\hbar)^3}\int \mathrm{e}^{\mathrm{i}\vec{q}\cdot\vec{r}/\hbar}\left\langle \vec{p}+\frac{\vec{q}}{2}\middle|\hat{\rho}(t)\middle|\vec{p}-\frac{\vec{q}}{2}\right\rangle \mathrm{d}^3\vec{q}. \tag{VII.2}$$

The equivalence between the two definitions can be proven using completeness relations with

[(cl)] E. P. Wigner, 1902–1995

normalized plane waves with fixed momentum:

$$\langle \vec{y}\,|\vec{q}\,'\rangle = \frac{1}{(2\pi\hbar)^{3/2}}\,\mathrm{e}^{\mathrm{i}\vec{q}\,'\cdot\vec{y}/\hbar}.$$

Remark: A more formal definition of the Wigner distribution is given in Appendix VII.A.

Given the Wigner distribution ρ_W, the corresponding statistical operator can be reconstructed. For instance, taking the inverse Fourier transform of Eq. (VII.1) one quickly finds the matrix element in position representation

$$\langle\vec{r}_1|\hat{\rho}(t)|\vec{r}_2\rangle = \int \mathrm{e}^{\mathrm{i}\vec{p}\cdot(\vec{r}_1-\vec{r}_2)/\hbar}\rho_W\bigg(t,\frac{\vec{r}_1+\vec{r}_2}{2},\vec{p}\bigg)\,\mathrm{d}^3\vec{p} \tag{VII.3}$$

for all $\vec{r}_1$, $\vec{r}_2$. Similarly, inverting Eq. (VII.2) leads to the matrix elements in momentum representation

$$\langle\vec{p}_1|\hat{\rho}(t)|\vec{p}_2\rangle = \int \mathrm{e}^{\mathrm{i}(\vec{p}_2-\vec{p}_1)\cdot\vec{r}/\hbar}\rho_W\bigg(t,\vec{r},\frac{\vec{p}_1+\vec{p}_2}{2}\bigg)\,\mathrm{d}^3\vec{r}. \tag{VII.4}$$

VII.1.2 Properties

The properties (II.23) of the statistical operator translate into properties of the Wigner distribution, which we now list.

1. $\rho_W(t,\vec{r},\vec{p})$ is real for any $(t,\vec{r},\vec{p})$. (VII.5a)

   Proof: A change of variable $\vec{x}\to-\vec{x}$ in Eq. (VII.1) together with the hermiticity of $\hat{\rho}(t)$ yield at once $[\rho_W(t,\vec{r},\vec{p})]^* = \rho_W(t,\vec{r},\vec{p})$. □

2. $\displaystyle\int \rho_W(t,\vec{r},\vec{p})\,\mathrm{d}^3\vec{p} = \langle\vec{r}|\hat{\rho}(t)|\vec{r}\rangle\,.$ (VII.5b)

   That is, the integral of the Wigner distribution over momentum yields the probability to find the particle at position $\vec{r}$ at time t.

   The integral over $\vec{p}$ of the complex exponential in the integrand of definition (VII.1) yields $(2\pi\hbar)^3\delta^{(3)}(\vec{x})$. □

3. $\displaystyle\int \rho_W(t,\vec{r},\vec{p})\,\mathrm{d}^3\vec{r} = \langle\vec{p}|\hat{\rho}(t)|\vec{p}\rangle\,.$ (VII.5c)

   In turn, the integral of the Wigner distribution over position yields the probability that the particle has momentum $\vec{p}$ at time t.

4. Normalization: $\displaystyle\int \rho_W(t,\vec{r},\vec{p})\,\mathrm{d}^3\vec{r}\,\mathrm{d}^3\vec{p} = 1.$ (VII.5d)

   This follows at once from Eq. (VII.5b) or (VII.5c) and the normalization (II.23c) of the statistical operator. □

5. $\displaystyle\int \rho_W(t,\vec{r},\vec{p})^2\,\mathrm{d}^3\vec{r}\,\mathrm{d}^3\vec{p} \le \frac{1}{(2\pi\hbar)^3}.$ (VII.5e)

   Together with the normalization condition, this inequality shows that the Wigner distribution cannot be too sharply peaked, and accordingly it support on phase space cannot be too small.

   Proof: later. □

VII.1.3 Examples

VII.1.3 a Pure state

In case the particle can be described by a pure state $|\Psi(t)\rangle$, so that the corresponding statistical operator is simply $|\Psi(t)\rangle\langle\Psi(t)|$, the associated Wigner distribution can be expressed in terms of wave functions.

Position representation

Let us denote by $\Psi(t,\vec{r}) \equiv \langle\vec{r}\,|\,\Psi(t)\rangle$ the wave function of the particle in position representation. Then definition (VII.1) leads at once to

$$\rho_W(t,\vec{r},\vec{p}) = \frac{1}{(2\pi\hbar)^3}\int \mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}/\hbar}\,\Psi\left(t,\vec{r}+\frac{\vec{x}}{2}\right)\Psi^*\left(t,\vec{r}-\frac{\vec{x}}{2}\right)\mathrm{d}^3\vec{x}. \tag{VII.6}$$

Momentum representation

Denoting now by $\Phi(t,\vec{p}) \equiv \langle\vec{p}\,|\,\Psi(t)\rangle$ the particle wave function in momentum representation, definition (VII.2) gives the Wigner distribution

$$\rho_W(t,\vec{r},\vec{p}) = \frac{1}{(2\pi\hbar)^3}\int \mathrm{e}^{\mathrm{i}\vec{q}\cdot\vec{r}/\hbar}\,\Phi\left(t,\vec{p}+\frac{\vec{q}}{2}\right)\Phi^*\left(t,\vec{p}-\frac{\vec{q}}{2}\right)\mathrm{d}^3\vec{q}. \tag{VII.7}$$

Remark: If the wave function factorizes, e.g. $\Phi(t,\vec{p}) = \Phi_x(t,p_x)\Phi_y(t,p_y)\Phi_z(t,p_z)$ — as is for instance the case for a free particle or for a particle in a potential that depends separately on x, y, z —, then its Wigner distribution $\rho_W(t,\vec{r},\vec{p})$ factorizes as well.

VII.1.3 b Gaussian wave packet

Let us work for simplicity in one dimension only, which has the advantage that the phase space is two-dimensional.

The wave function in position representation of Gaussian wave packet centered at the point (x_0,p_0) in phase space is

$$\Psi(x) = \frac{1}{(2\pi a^2)^{1/4}}\,\mathrm{e}^{-(x-x_0)^2/4a^2}\,\mathrm{e}^{\mathrm{i}p_0x/\hbar}, \tag{VII.8}$$

where a denotes the width in position space. Fourier transforming this expression yields the wave function in momentum representation, which is Gaussian as well:

$$\Phi(p) = \left(\frac{2a^2}{\pi\hbar^2}\right)^{1/4}\mathrm{e}^{-a^2(p-p_0)^2/\hbar^2}\,\mathrm{e}^{-\mathrm{i}px_0/\hbar}, \tag{VII.9}$$

Inserting either Eq. (VII.8) into Eq. (VII.6), or Eq. (VII.9) into Eq. (VII.7), one finds the Wigner distribution

$$\rho_W(x,p) = \frac{1}{\pi\hbar}\,\mathrm{e}^{-(x-x_0)^2/2a^2}\,\mathrm{e}^{-2a^2(p-p_0)^2/\hbar^2}, \tag{VII.10}$$

i.e. a two-dimensional Gaussian in phase space, as illustrated in Fig. VII.1.1. Interestingly, one sees that $\rho_W(x,p)$ equals the product of the squared norms of the wave functions in position and momentum representations:

$$\rho_W(x,p) = |\Psi(x)|^2|\Phi(p)|^2.$$

Accordingly, in the present case the Wigner distribution is positive everywhere, which is not the typical case. Additionally, there is no correlation between position and momentum.

VII.1.3 c Two separated Gaussian wave packets

We superpose two identical one-dimensional wave packets of the form (VII.8), centered at the points $x = \pm x_0$, taking $p_0 = 0$ for simplicity. Ignoring the precise normalization factor, the wave

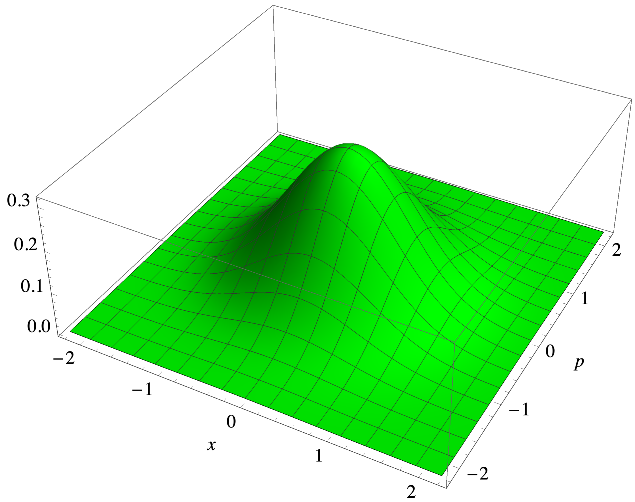

Figure VII.1.1 – Wigner distribution (VII.10) (in units of $\hbar^{-1}$) for a Gaussian wave packet in one dimension. The x resp. p variable is expressed in units of $a\sqrt{2}$ resp. $\hbar/a\sqrt{2}$.

function in position representation is

$$\Psi(x) \simeq \frac{1}{\sqrt{2}}\frac{1}{(2\pi a^2)^{1/4}}\left[\mathrm{e}^{-(x-x_0)^2/4a^2} + \mathrm{e}^{-(x+x_0)^2/4a^2}\right]. \tag{VII.11}$$

Using Eq. (VII.6) with this expression, one finds

$$\rho_W(x,p) \simeq \frac{1}{2\pi\hbar}\mathrm{e}^{-2a^2p^2/\hbar^2}\left[\mathrm{e}^{-(x-x_0)^2/2a^2} + \mathrm{e}^{-(x+x_0)^2/2a^2} + 2\mathrm{e}^{-x^2/2a^2}\cos\frac{2px_0}{\hbar}\right]. \tag{VII.12}$$

The first two terms in the square brackets correspond to contributions of the same form as that of Eq. (VII.10), but in addition there is an oscillatory term that comes from the interference between the two wave packets. Due to this latter term, the Wigner distribution (VII.12) can take negative values, for instance at $x = 0$, choosing an appropriate value of p. This holds even if the two wave packets are largely separated, i.e. $|x_0| \gg a$.

That is, the Wigner distribution may not be interpreted as a probability density on phase space — hence the denomination “quasi-distribution”.

VII.1.3 d Odd wave function

If the wave function of the system, for instance in position representation, is odd, then the associated Wigner distribution is automatically negative at $\vec{r} = \vec{0}$, $\vec{p} = \vec{0}$. Using Eq. (VII.6), one indeed finds

$$\rho_W(\vec{r}=\vec{0},\vec{p}=\vec{0}) = \frac{-1}{(2\pi\hbar)^3}\int\left|\Psi\left(\frac{\vec{x}}{2}\right)\right|^2 \mathrm{d}^3\vec{x} = \frac{-1}{(\pi\hbar)^3},$$

where the second identity follows from a change of integration variable and the normalization of the wave function.

VII.2 Wigner transform of an operator

As in Sec. VII.1, we consider a system consisting of a single particle without spin propagating in three dimensions. The Hilbert space spanned by the usual ket-vectors $\{|\Psi\rangle\}$ for that system is denoted by $\mathscr{H}$.

VII.2.1 Wigner and Weyl transforms

Let $\hat{A}$ denote an operator on the Hilbert space $\mathscr{H}$. Its *Wigner transform* is the function A_W defined on the particle phase space by

$$A_W(\vec{r},\vec{p}) \equiv \int e^{-i\vec{p}\cdot\vec{x}/\hbar} \left\langle \vec{r}+\frac{\vec{x}}{2}\middle|\hat{A}\middle|\vec{r}-\frac{\vec{x}}{2}\right\rangle d^3\vec{x}, \qquad \text{(VII.13a)}$$

or equivalently

$$A_W(\vec{r},\vec{p}) \equiv \int e^{i\vec{q}\cdot\vec{r}/\hbar} \left\langle \vec{p}+\frac{\vec{q}}{2}\middle|\hat{A}\middle|\vec{p}-\frac{\vec{q}}{2}\right\rangle d^3\vec{q}. \qquad \text{(VII.13b)}$$

The former resp. latter expression is the position resp. momentum representation of the Wigner transform.

Remark: Coming back to Eqs. (VII.1)–(VII.2), the Wigner distribution ρ_W is thus $1/(2\pi\hbar)^3$ times the Wigner transform of the statistical operator $\hat{\rho}$, so that the notations are slightly inconsistent.

As in Eqs. (VII.3)–(VII.4), one can invert the Wigner transform and thus define a mapping $A_W \to \hat{A}$ from a phase-space function to an operator on $\mathscr{H}$, given by its matrix elements in position representation:

$$\langle\vec{r}_1|\hat{A}|\vec{r}_2\rangle = \int e^{i\vec{p}\cdot(\vec{r}_1-\vec{r}_2)/\hbar} A_W\!\left(\frac{\vec{r}_1+\vec{r}_2}{2},\vec{p}\right) d^3\vec{p} \qquad \text{(VII.14a)}$$

or in momentum representation

$$\langle\vec{p}_1|\hat{A}|\vec{p}_2\rangle = \int e^{i(\vec{p}_2-\vec{p}_1)\cdot\vec{r}/\hbar} A_W\!\left(\vec{r},\frac{\vec{p}_1+\vec{p}_2}{2}\right) d^3\vec{r}. \qquad \text{(VII.14b)}$$

This “inverse Wigner transform” is often called *Weyl*[(cm)] *transform*.

Remark: One can also give a basis-independent definition of the Weyl transform, by means of the Weyl operator (VII.48).

VII.2.2 Properties of the Wigner transform

Similar to the properties (VII.5) of the Wigner distribution, one can derive properties of the Wigner transforms of general operators.

Using Eqs. (VII.13), one shows that the Wigner transform A_W of a Hermitian operator $\hat{A}$ is real-valued. Conversely, Eqs. (VII.14) show that the Weyl transform of a real-valued phase space function is Hermitian.

The Wigner transform of the identity operator on $\mathscr{H}$ is the phase-space function identically equal to 1.

If $\hat{A}$ and $\hat{B}$ are two operators on $\mathscr{H}$, then the trace of their product can be expressed as an integral over phase space involving their Wigner transforms:

$$\mathrm{Tr}\big(\hat{A}\hat{B}\big) = \int A_W(\vec{r},\vec{p})\,B_W(\vec{r},\vec{p})\,\frac{d^3\vec{r}\,d^3\vec{p}}{(2\pi\hbar)^3}. \qquad \text{(VII.15)}$$

Proof: Using the basis $\{|r_1\rangle\}$ of position eigenstates, the trace first reads

$$\mathrm{Tr}\big(\hat{A}\hat{B}\big) = \int \langle\vec{r}_1|\hat{A}\hat{B}|\vec{r}_1\rangle\, d^3\vec{r}_1.$$

Inserting a completeness relation, this becomes

$$\mathrm{Tr}\big(\hat{A}\hat{B}\big) = \int \langle\vec{r}_1|\hat{A}|\vec{r}_2\rangle\langle\vec{r}_2|\hat{B}|\vec{r}_1\rangle\, d^3\vec{r}_1\, d^3\vec{r}_2.$$

[(cm)] H. Weyl, 1885–1955

The matrix elements in position representation can be expressed using Eq. (VII.14a):

$$\mathrm{Tr}\big(\hat{A}\hat{B}\big) = \int \mathrm{e}^{\mathrm{i}\vec{p}\cdot(\vec{r}_1-\vec{r}_2)/\hbar} A_W\bigg(\frac{\vec{r}_1+\vec{r}_2}{2},\vec{p}\bigg)\, \mathrm{e}^{-\mathrm{i}\vec{p}\,'\cdot(\vec{r}_1-\vec{r}_2)/\hbar} B_W\bigg(\frac{\vec{r}_1+\vec{r}_2}{2},\vec{p}\,'\bigg) \frac{\mathrm{d}^3\vec{r}_1\,\mathrm{d}^3\vec{r}_2\,\mathrm{d}^3\vec{p}\,\mathrm{d}^3\vec{p}\,'}{[(2\pi\hbar)^3]^2}.$$

Substituting $\vec{r} \equiv (\vec{r}_1+\vec{r}_2)/2$, $\vec{r}\,' \equiv \vec{r}_1-\vec{r}_2$, the integral over $\vec{r}\,'$ is straightforward and yields $(2\pi\hbar)^3\delta^{(3)}(\vec{p}-\vec{p}\,')$. In turn, the integral over $\vec{p}\,'$ becomes trivial, and gives Eq. (VII.15). □

Expectation value of an observable

As an important special case of Eq. (VII.15), one can obtain the expectation value of an observable. If the system is in a state — including a mixture of states — described by the statistical operator $\rho(t)$, to which corresponds the Wigner distribution $\rho_W(t,\vec{r},\vec{p})$, then the expectation value at time t of an observable $\hat{A}$ is given by

$$\big\langle \hat{A}(t)\big\rangle = \int \rho_W(t,\vec{r},\vec{p})\, A_W(\vec{r},\vec{p})\,\mathrm{d}^3\vec{r}\,\mathrm{d}^3\vec{p}. \tag{VII.16}$$

This formula is clearly reminiscent of Eq. (II.6a) for the expectation value of an observable of a classical system. Yet one should keep in mind the importance difference that ρ_W is in general not a probability distribution on phase space.

VII.2.3 Examples of Wigner transforms

Let us now give a few examples of Wigner transforms of simple operators.

VII.2.3 a Operators function of position or momentum only

If the operator $\hat{A}$ is a function of the position operator $\hat{\vec{r}}$ only — or of one of its components on some Cartesian basis of coordinates —, i.e. $\hat{A} = f\big(\hat{\vec{r}}\big)$, then the corresponding Wigner transform is given by the same function of the variable $\vec{r}$ only:

$$\hat{A} = f\big(\hat{\vec{r}}\big) \quad\mapsto\quad A_W(\vec{r},\vec{p}) = f(\vec{r}). \tag{VII.17}$$

Similarly, for an operator that only depends on the momentum operator $\hat{\vec{p}}$

$$\hat{A} = g\big(\hat{\vec{p}}\big) \quad\mapsto\quad A_W(\vec{r},\vec{p}) = g(\vec{p}). \tag{VII.18}$$

This rule is clearly relevant for the kinetic part of the Hamilton operator — when the canonical momentum coincides with the kinetic momentum —, while the previous one is relevant for position-dependent potentials. Accordingly, Wigner transforming $\hat{H} = \hat{\vec{p}}^2/2m + V\big(\hat{\vec{r}}\big)$ yields

$$\hat{H} = \frac{\hat{\vec{p}}^2}{2m} + V\big(\hat{\vec{r}}\big) \quad\mapsto\quad H_W(\vec{r},\vec{p}) = \frac{\vec{p}^2}{2m} + V(\vec{r}),$$

i.e. the corresponding classical Hamilton function. In combination with Eq. (VII.16), one can then obtain the expectation value of the energy of the corresponding system.

VII.2.3 b Products of operators

Wigner transforms of products of operators depending on both position and momentum are however more intricate. Indeed, using definition (VII.13a) of the Wigner transform for an operator product $\hat{A}\hat{B}$ and inserting a completeness relation on the basis of position eigenstates $\{|\vec{y}\rangle\}$ between $\hat{A}$ and $\hat{B}$, one finds under consideration of the matrix element (VII.14a)

$$\big(AB\big)_W(\vec{r},\vec{p}) = \int \mathrm{e}^{\mathrm{i}\vec{p}_1\cdot(\vec{r}+\vec{x}/2-\vec{y})/\hbar}\,\mathrm{e}^{\mathrm{i}\vec{p}_2\cdot(-\vec{r}+\vec{x}/2+\vec{y})/\hbar}\,\mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}/\hbar} \times A_W\bigg(\frac{\vec{r}+\vec{y}}{2}+\frac{\vec{x}}{4},\vec{p}_1\bigg) B_W\bigg(\frac{\vec{r}+\vec{y}}{2}-\frac{\vec{x}}{4},\vec{p}_2\bigg) \frac{\mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{y}\,\mathrm{d}^3\vec{p}_1\,\mathrm{d}^3\vec{p}_2}{(2\pi\hbar)^6}. \tag{VII.19a}$$

Clearly, this lengthy expression does not coincide with the product $A_W(\vec{r},\vec{p})B_W(\vec{r},\vec{p})$ in general.

Alternatively, one finds starting from the momentum representation of the Wigner and Weyl transforms

$$\big(AB\big)_W(\vec{r},\vec{p}) = \int \mathrm{e}^{\mathrm{i}(\vec{q}\,'-\vec{p}-\vec{q}/2)\cdot\vec{x}/\hbar}\,\mathrm{e}^{\mathrm{i}(\vec{p}-\vec{q}/2-\vec{q}\,')\cdot\vec{y}/\hbar}\,\mathrm{e}^{\mathrm{i}\vec{q}\cdot\vec{r}/\hbar} \times A_W\bigg(\vec{x},\frac{\vec{q}\,'+\vec{p}}{2}+\frac{\vec{q}}{4}\bigg)B_W\bigg(\vec{y},\frac{\vec{q}\,'+\vec{p}}{2}-\frac{\vec{q}}{4}\bigg)\frac{\mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{y}\,\mathrm{d}^3\vec{q}\,\mathrm{d}^3\vec{q}\,'}{(2\pi\hbar)^6}. \tag{VII.19b}$$

These formulas simplify in case $\hat{A}$ (or $\hat{B}$) is a function of either $\hat{\vec{r}}$ or $\hat{\vec{p}}$ only, in which case of the exponentials is readily integrated to yield a δ-distribution, which makes a second integral trivial. For instance, if $\hat{A}=\hat{\vec{r}}$, then the integral over $\vec{p}_1$ in Eq. (VII.19a) gives $(2\pi\hbar)^3\delta^{(3)}(\vec{r}+\vec{x}/2-\vec{y})$, leading to a trivial integral over $\vec{y}$:

$$\big(rB\big)_W(\vec{r},\vec{p}) = \int \mathrm{e}^{\mathrm{i}\vec{p}_2\cdot\vec{x}/\hbar}\mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}/\hbar}\bigg(\vec{r}+\frac{\vec{x}}{2}\bigg)B_W(\vec{r},\vec{p}_2)\,\frac{\mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{p}_2}{(2\pi\hbar)^3}. \tag{VII.20}$$

The integral over $\vec{x}$ is now easy: when multiplying out the factor $(\vec{r}+\vec{x}/2)$, the term in $\vec{r}$ gives $(2\pi\hbar)^3\delta^{(3)}(\vec{p}_2-\vec{p})$, while the term in $\vec{x}/2$ gives $(\hbar/2\mathrm{i})(2\pi\hbar)^3\vec{\nabla}_{\vec{p}_2}\delta^{(3)}(\vec{p}_2-\vec{p})$. Integrating eventually over $\vec{p}_2$ gives

$$\big(rB\big)_W(\vec{r},\vec{p}) = \vec{r}\,B_W(\vec{r},\vec{p}) - \frac{\hbar}{2\mathrm{i}}\vec{\nabla}_{\vec{p}}B_W(\vec{r},\vec{p}). \tag{VII.21}$$

One can similarly compute the Wigner transform $(Br)_W$ of the product $\hat{B}\hat{\vec{r}}$, which yields a similar result with the $-$ sign on the right hand side replaced by a $+$. All in all, one finds for the commutator of $\hat{\vec{r}}$ and $\hat{B}$

$$\big[\hat{\vec{r}},\hat{B}\big]_W(\vec{r},\vec{p}) = \mathrm{i}\hbar\vec{\nabla}_{\vec{p}}B_W(\vec{r},\vec{p}). \tag{VII.22}$$

This is clearly reminiscent of the result one would obtain in classical mechanics when replacing the commutator, divided by $\mathrm{i}\hbar$, by the Poisson brackets (II.2).[100]

Paralleling Eq. (VII.21), one can show — starting from Eq. (VII.19b) and using matrix elements in momentum representation (VII.14b)

$$\big(pB\big)_W(\vec{r},\vec{p}) = \vec{p}\,B_W(\vec{r},\vec{p}) + \frac{\hbar}{2\mathrm{i}}\vec{\nabla}_{\vec{r}}B_W(\vec{r},\vec{p}), \tag{VII.23}$$

which now involves the gradient with respect to the position variables.

Applying the latter result with the product $\hat{\vec{p}}\hat{B}$ in lieu of $\hat{B}$, one finds

$$\big(p^2B\big)_W(\vec{r},\vec{p}) = \vec{p}^{\,2}\,B_W(\vec{r},\vec{p}) - \mathrm{i}\hbar\vec{p}\cdot\vec{\nabla}_{\vec{r}}B_W(\vec{r},\vec{p}) - \frac{\hbar^2}{4}\triangle_{\vec{r}}B_W(\vec{r},\vec{p}), \tag{VII.24}$$

with $\triangle_{\vec{r}}$ the Laplacian with respect to $\vec{r}$. One similarly determines the Wigner transform of the product $\hat{B}\hat{\vec{p}}^2$, from which one eventually deduces the Wigner transform of the commutator:

$$\big[\hat{\vec{p}}^2,\hat{B}\big]_W(\vec{r},\vec{p}) = -2\mathrm{i}\hbar\vec{p}\cdot\vec{\nabla}_{\vec{r}}B_W(\vec{r},\vec{p}). \tag{VII.25}$$

As we shall presently see, this formula is useful when considering the evolution of the Wigner distribution.

VII.3 Evolution equation of the Wigner distribution

Starting from the Liouville–von Neumann equation (II.25) for the time evolution of the statistical operator, the Wigner transform of both sides yields the evolution equation

$$\mathrm{i}\hbar\frac{\partial\rho_W(t,\vec{r},\vec{p})}{\partial t} = \frac{1}{(2\pi\hbar)^3}\big[\hat{H},\hat{\rho}\big]_W(t,\vec{r},\vec{p}). \tag{VII.26}$$

[100] See the formula at the very end of the Appendices to Chapter III, below Eq. (III.115b).

Given some Hamilton operator $\hat{H}$, one can in principle compute the Wigner transform of its commutator with $\hat{\rho}$, using the results of § VII.2.3.

In the following, we illustrate the time evolution for two special yet generic cases, namely that of a particle evolving in a scalar potential $V(\vec{r})$ either in a pure (§ VII.3.1) or in an "arbitrary" state (§ VII.3.2).

VII.3.1 Particle in a scalar potential: pure states

If the particle is in a pure state, its Wigner distribution can be expressed in terms of its wave function in position representation $\Psi(t,\vec{r})$, see Eq. (VII.6). Differentiating both sides of that equation with respect to time yields

$$\frac{\partial \rho_W(t,\vec{r},\vec{p})}{\partial t} = \frac{1}{(2\pi\hbar)^3}\int \mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}/\hbar}\left[\frac{\partial\Psi(t,\vec{r}+\frac{\vec{x}}{2})}{\partial t}\Psi^*\Big(t,\vec{r}-\frac{\vec{x}}{2}\Big) + \Psi\Big(t,\vec{r}+\frac{\vec{x}}{2}\Big)\frac{\partial\Psi^*(t,\vec{r}-\frac{\vec{x}}{2})}{\partial t}\right]\mathrm{d}^3\vec{x}.$$

The time derivatives of Ψ and its complex conjugate can be replaced using the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\Psi(t,\vec{x})}{\partial t} = -\frac{\hbar^2}{2m}\triangle\Psi(t,\vec{x}) + V(\vec{x})\Psi(t,\vec{x})$$

and its complex conjugate. One finds at once

$$\frac{\partial \rho_W(t,\vec{r},\vec{p})}{\partial t} = \frac{\partial \rho_W^{(T)}(t,\vec{r},\vec{p})}{\partial t} + \frac{\partial \rho_W^{(V)}(t,\vec{r},\vec{p})}{\partial t} \tag{VII.27}$$

with a kinetic term

$$\frac{\partial \rho_W^{(T)}(t,\vec{r},\vec{p})}{\partial t} \equiv \frac{\mathrm{i}}{16\pi^3 m\hbar^2}\int \mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}/\hbar}\left[\Psi^*\Big(t,\vec{r}-\frac{\vec{x}}{2}\Big)\triangle\Psi\Big(t,\vec{r}+\frac{\vec{x}}{2}\Big) - \Psi\Big(t,\vec{r}+\frac{\vec{x}}{2}\Big)\triangle\Psi^*\Big(t,\vec{r}-\frac{\vec{x}}{2}\Big)\right]\mathrm{d}^3\vec{x} \tag{VII.28}$$

where $\triangle$ denotes the Laplacian with respect to $\vec{r}$, and a potential term

$$\frac{\partial \rho_W^{(V)}(t,\vec{r},\vec{p})}{\partial t} \equiv \frac{1}{\mathrm{i}\hbar}\frac{1}{(2\pi\hbar)^3}\int \mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}/\hbar}\left[V\Big(\vec{r}+\frac{\vec{x}}{2}\Big) - V\Big(\vec{r}-\frac{\vec{x}}{2}\Big)\right]\Psi\Big(t,\vec{r}+\frac{\vec{x}}{2}\Big)\Psi^*\Big(t,\vec{r}-\frac{\vec{x}}{2}\Big)\,\mathrm{d}^3\vec{x}. \tag{VII.29}$$

On the one hand, one can show that the kinetic term (VII.28) can be rewritten as

$$\frac{\partial \rho_W^{(T)}(t,\vec{r},\vec{p})}{\partial t} = -\frac{\vec{p}}{m}\cdot\vec{\nabla}_{\vec{r}}\,\rho_W(t,\vec{r},\vec{p}) \tag{VII.30}$$

A nontrivial trick in the proof is to write the Laplacian $\triangle$ with respect to $\vec{r}$, when applied to a function of $\vec{r}\pm\frac{\vec{x}}{2}$, as $\pm 2\vec{\nabla}_{\vec{r}}\cdot\vec{\nabla}_{\vec{x}}$, i.e. as the inner product of the gradients with respect to $\vec{r}$ and $\vec{x}$. After that, a partial integration over $\vec{x}$ yields the result.

On the other hand, assuming that the potential V can be expanded in Taylor series, one can transform the potential term (VII.29) into an infinite sum over the odd numbers $2n+1$ with $n \geq 0$ of terms involving products of two types of derivatives of order $2n+1$, namely those of V with respect to $\vec{r}$ and those of ρ_W with respect to $\vec{p}$. In the one-dimensional case, this infinite sum takes the form

$$\frac{\partial \rho_W^{(V)}(t,x,p)}{\partial t} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{\mathrm{i}\hbar}{2}\right)^{2n}\frac{\partial^{2n+1}V(x)}{\partial x^{2n+1}}\frac{\partial^{2n+1}\rho_W(t,x,p)}{\partial p^{2n+1}}. \tag{VII.31}$$

In three dimensions, the leading term with $n=0$ reads

$$\frac{\partial \rho_W^{(V)}(t,x,p)}{\partial t} = \vec{\nabla}_{\vec{r}}V(\vec{r})\cdot\vec{\nabla}_{\vec{p}}\,\rho_W(t,\vec{r},\vec{p}) + \cdots \tag{VII.32}$$

In case the derivatives of the potential of order 2 and higher vanish — that is, for a free particle or a (non necessarily isotropic) harmonic oscillator —, then only this term remains.

Remark: If the potential varies slowly in space, considering only the influence of the leading term $n = 0$ might already constitute a good approximation.

VII.3.2 Particle in a scalar potential: arbitrary states

If the particle under study is described by an arbitrary statistical operator $\hat{\rho}$, one has to use the general evolution equation (VII.26). In the case of a single particle in a scalar potential, its Hamilton operator reads

$$\hat{H} = \frac{\hat{\vec{p}}^{\,2}}{2m} + V(\hat{\vec{r}}). \tag{VII.33}$$

The Wigner transform of the commutator of an arbitrary observable with $\hat{\vec{p}}^{\,2}$ is given by Eq. (VII.25). In analogy with the latter, but keeping in mind that the actual Wigner transform of $\hat{\rho}$ equals $(2\pi\hbar)^3$ times ρ_W [see remark below Eq. (VII.13b)], one obtains

$$\frac{1}{2m}\big[\hat{\vec{p}}^{\,2},\hat{\rho}\big]_W(t,\vec{r},\vec{p}) = \frac{\hbar}{\mathrm{i}m}\vec{p}\cdot\vec{\nabla}_{\vec{r}}\big[(2\pi\hbar)^3\rho_W(t,\vec{r},\vec{p})\big]. \tag{VII.34}$$

In turn, the general formula (VII.19a) simplifies when one of the two observables involved only depends on $\hat{\vec{r}}$, which is the case of the potential term in Eq. (VII.33). In analogy with Eq. (VII.20), one readily finds

$$\big[V(\hat{\vec{r}}),\hat{\rho}\big]_W(t,\vec{r},\vec{p}) = \frac{1}{(2\pi\hbar)^3}\int \mathrm{e}^{\mathrm{i}\vec{q}\cdot\vec{x}/\hbar}\mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}/\hbar}\left[V\left(\vec{r}+\frac{\vec{x}}{2}\right) - V\left(\vec{r}-\frac{\vec{x}}{2}\right)\right](2\pi\hbar)^3\rho_W(t,\vec{r},\vec{q})\,\mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{q}. \tag{VII.35}$$

Gathering Eqs. (VII.26), (VII.34) and (VII.35), one obtains

$$\frac{\partial\rho_W(t,\vec{r},\vec{p})}{\partial t} + \frac{\vec{p}}{m}\cdot\vec{\nabla}_{\vec{r}}\rho_W(t,\vec{r},\vec{p}) = \frac{1}{\mathrm{i}\hbar}\frac{1}{(2\pi\hbar)^3}\int \mathrm{e}^{\mathrm{i}(\vec{q}-\vec{p})\cdot\vec{x}/\hbar}\left[V\left(\vec{r}+\frac{\vec{x}}{2}\right) - V\left(\vec{r}-\frac{\vec{x}}{2}\right)\right]\rho_W(t,\vec{r},\vec{q})\,\mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{q}. \tag{VII.36}$$

One can check that Eq. (VII.28) with the kinetic and potential terms (VII.29) and (VII.30) is actually a special case of this equation.

VII.3.3 Example

We consider a particle without spin propagating in a single dimension, so that the time evolution of its Wigner distribution $\rho_W(t,x,p)$ is governed by [cf. Eqs. (VII.28), (VII.30) and (VII.31)]

$$\frac{\partial\rho_W(t,x,p)}{\partial t} = -\frac{p}{m}\frac{\partial\rho_W(t,x,p)}{\partial x} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{\mathrm{i}\hbar}{2}\right)^{2n}\frac{\partial^{2n+1}V(x)}{\partial x^{2n+1}}\frac{\partial^{2n+1}\rho_W(t,x,p)}{\partial p^{2n+1}}. \tag{VII.37}$$

Let us assume a spatially periodic potential $V(x) = V_0\cos(kx)$, such that its successive derivatives are nontrivial yet readily computed:

$$\frac{\mathrm{d}^l V(x)}{\mathrm{d}x^l} = k^l V_0\cos\left(kx + \frac{l\pi}{2}\right).$$

In particular, the odd derivatives can be expressed in the convenient short form

$$\frac{\mathrm{d}^{2n+1}V(x)}{\mathrm{d}x^{2n+1}} = \left(-k^2\right)^n\frac{\mathrm{d}V(x)}{\mathrm{d}x}$$

which can then be substituted into the second term on the right hand side of Eq. (VII.37):

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{\mathrm{i}\hbar}{2}\right)^{2n}\frac{\mathrm{d}^{2n+1}V(x)}{\mathrm{d}x^{2n+1}}\frac{\partial^{2n+1}\rho_W(t,x,p)}{\partial p^{2n+1}} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{\hbar k}{2}\right)^{2n}\frac{\mathrm{d}V(x)}{\mathrm{d}x}\frac{\partial^{2n+1}\rho_W(t,x,p)}{\partial p^{2n+1}}.$$

Taking the derivative $\mathrm{d}V(x)/\mathrm{d}x$ out of the sum on the right hand side, one recognizes that the latter is (up to a factor $1/\hbar k$) the difference of values of the Wigner distribution at two different momenta:

$$\rho_W\!\left(t,x,p+\frac{\hbar k}{2}\right)-\rho_W\!\left(t,x,p-\frac{\hbar k}{2}\right)=2\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{\hbar k}{2}\right)^{2n+1}\frac{\partial^{2n+1}\rho_W(t,x,p)}{\partial p^{2n+1}}.$$

Defining

$$\frac{\delta\rho_W(t,x,p)}{\delta p}\equiv\frac{1}{\hbar k}\left[\rho_W\!\left(t,x,p+\frac{\hbar k}{2}\right)-\rho_W\!\left(t,x,p-\frac{\hbar k}{2}\right)\right], \tag{VII.38}$$

one can thus write

$$\frac{\partial\rho_W(t,x,p)}{\partial t}=-\frac{p}{m}\frac{\partial\rho_W(t,x,p)}{\partial x}+\frac{\mathrm{d}V(x)}{\mathrm{d}x}\frac{\delta\rho_W(t,x,p)}{\delta p}. \tag{VII.39}$$

Assume now that the particle propagating in the potential has a well-defined velocity, so that it is described by a wave packet with a narrow width Δp in momentum — and accordingly, a wide spread in position. More precisely, let us assume that $\Delta p \ll \hbar k$. Then for the values of p at which the Wigner distribution ρ_W takes an appreciable value, the quantity (VII.38) becomes vanishingly small, so that the “potential” term in Eq. (VII.39) is effectively zero: the evolution equation of ρ_W is the same as in the case of a free particle! This is why electrons (disregarding their spin degrees of freedom) propagating in the conduction bands of a solid behave as free (quasi)particles.

VII.4 Generalizations

VII.4.1 System with N particles without spin

The definitions (VII.1)–(VII.2) of the Wigner distribution for a single particle without spin are readily generalized to a quasi-distribution for a system of N such particles. Introducing the basis of position eigenstates $|\{\vec{r}_i\}_{i=1,\dots,N}\rangle$ — remember that the position operators acting on any two different particles commute, so that it makes sense to talk of simultaneous eigenstates —, one defines a real-valued function of time and the phase-space variables of the N particles by

$$\rho_W\big(t,\{\vec{r}_i,\vec{p}_i\}\big)\equiv\frac{1}{(2\pi\hbar)^{3N}}\int \mathrm{e}^{-\mathrm{i}\vec{p}_1\cdot\vec{x}_1/\hbar}\cdots\mathrm{e}^{-\mathrm{i}\vec{p}_N\cdot\vec{x}_N/\hbar}\left\langle\left\{\vec{r}_i+\frac{\vec{x}_i}{2}\right\}\middle|\hat{\rho}(t)\middle|\left\{\vec{r}_i-\frac{\vec{x}_i}{2}\right\}\right\rangle \mathrm{d}^3\vec{x}_1\cdots\mathrm{d}^3\vec{x}_N. \tag{VII.40}$$

Alternatively, using the momentum eigenstates $|\{\vec{p}_i\}_{i=1,\dots,N}\rangle$, one obtains the momentum representation of the N-particle Wigner distribution:

$$\rho_W\big(t,\{\vec{r}_i,\vec{p}_i\}\big)\equiv\frac{1}{(2\pi\hbar)^{3N}}\int \mathrm{e}^{\mathrm{i}\vec{q}_1\cdot\vec{r}_1/\hbar}\cdots\mathrm{e}^{\mathrm{i}\vec{q}_N\cdot\vec{r}_N/\hbar}\left\langle\left\{\vec{p}_i+\frac{\vec{q}_i}{2}\right\}\middle|\hat{\rho}(t)\middle|\left\{\vec{p}_i-\frac{\vec{q}_i}{2}\right\}\right\rangle \mathrm{d}^3\vec{q}_1\cdots\mathrm{d}^3\vec{q}_N. \tag{VII.41}$$

Similar to what was done in Sec. VII.2, one can introduce the Wigner transform of an N-particle operator — see Eqs. (VII.13) — and compute the expectation value of a multiparticle observable using an integral, as in Eq. (VII.16).

VII.4.2 Particles with spin

In case of a single particle with spin s (where $s\in\frac{1}{2}\mathbb{N}$), the space spanned by the eigenvectors of the position (resp. momentum) operator $\hat{\vec{r}}$ (resp. $\hat{\vec{p}}$) for a given eigenvalue $\vec{r}$ (resp. $\vec{p}$) is no longer one-dimensional. A possible basis of that eigenspace will consist of the vectors $\{|\vec{r},\nu\rangle\}$ (resp. $\{|\vec{p},\nu\rangle\}$)

with a given component of the spin along a fixed given direction, say $\nu \in \{-s, -s+1, \dots s-1, s\}$, corresponding to a spin component $\nu\hbar$. Instead of a single Wigner distribution, one defines $(2s+1)^2$ distributions $\rho_W^{\nu\nu'}$ for every pair of values of the spin, for instance, in position representation [cf. Eq. (VII.1)]:

$$\rho_W^{\nu\nu'}(t,\vec{r},\vec{p}) \equiv \frac{1}{(2\pi\hbar)^3}\int \mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}/\hbar}\left\langle \vec{r}+\frac{\vec{x}}{2},\nu\,\middle|\,\hat{\rho}(t)\,\middle|\,\vec{r}-\frac{\vec{x}}{2},\nu'\right\rangle \mathrm{d}^3\vec{x}. \tag{VII.42}$$

Note that $\rho_W^{\nu\nu'}$ is no longer necessarily real-valued when $\nu \neq \nu'$. However, one easily checks that the $(2s+1)\times(2s+1)$-matrix whose (ν,ν')-entry is $\rho_W^{\nu\nu'}(t,\vec{r},\vec{p})$ is Hermitian:

$$\rho_W^{\nu\nu'}(t,\vec{r},\vec{p}) = \left[\rho_W^{\nu'\nu}(t,\vec{r},\vec{p})\right]^*. \tag{VII.43}$$

Combining the ideas of § VII.4.1 and the present paragraph, for a system of N particles with spin s, one would need to introduce $(2s+1)^{2N}$ Wigner distributions, which is quickly intractable.

VII.4.3 Charged particle in a vector potential

In the presence of a vector potential $\vec{A}(t,\vec{r})$, the canonical momentum $\vec{p}$ of a particle with mass m and electric charge q differs from its kinetic momentum: $\vec{p} = m\vec{v} - q\vec{A}$, where $\vec{v}$ denotes the particle velocity. This leads to a first dependence of the Wigner distribution (VII.1) on the gauge in which $\vec{A}$ is considered.

A second source of gauge dependence in the distribution (VII.1) is more directly seen on expression (VII.6) valid for a pure state. Indeed, in a gauge transformation

$$\vec{A}(t,\vec{r}) \to \vec{A}'(t,\vec{r}) = \vec{A}(t,\vec{r}) + \vec{\nabla}_{\vec{r}}\chi(t,\vec{r}) \tag{VII.44}$$

the wave function in position representation $\Psi(t,\vec{r})$ needs to be simultaneously transformed into $\Psi'(t,\vec{r}) = \mathrm{e}^{\mathrm{i}q\chi(t,\vec{r})/\hbar}\Psi(t,\vec{r})$ to ensure gauge invariance, that is, with a position-dependent phase factor. Since the wave function and its complex conjugate are considered at different points in the expression of the Wigner distribution, their phase factors do not cancel, leading to a extra gauge dependence of the definition (VII.1).

While working with a gauge-dependent Wigner distribution is feasible, one may prefer to deal with a gauge-independent quantity. A possible generalization of Eq. (VII.1) of this type is

$$\rho_W(t,\vec{r},\vec{v}) \equiv \frac{1}{(2\pi\hbar)^3}\int \mathrm{e}^{-\mathrm{i}m\vec{v}\cdot\vec{x}/\hbar}\exp\left[\frac{\mathrm{i}q}{\hbar}\int_{\mathcal{P}}\vec{A}(t,\vec{\xi})\cdot\mathrm{d}\vec{\ell}\right]\left\langle \vec{r}+\frac{\vec{x}}{2}\,\middle|\,\hat{\rho}(t)\,\middle|\,\vec{r}-\frac{\vec{x}}{2}\right\rangle \mathrm{d}^3\vec{x} \tag{VII.45}$$

where the line integral in the exponential function runs along a path $\mathcal{P}$ from $\vec{r}-\vec{x}/2$ to $\vec{r}+\vec{x}/2$ — for instance, a straight line.[(101)] Note that if $\vec{A}$ is uniform, then the phase factor involving its line integral equals $\mathrm{e}^{\mathrm{i}q\vec{A}\cdot\vec{x}/\hbar}$ and Eq. (VII.45) coincides with Eq. (VII.1).

Bibliography

- Cohen-Tannoudji, Diu & Laloë, *Quantum mechanics, vol. 3* [64], Appendix VII.

[(101)] If the system propagates in a non-simply connected space, the line integral may depend on the path, in which case one may want to better specify the prescription.

Appendices to Chapter VII

VII.A Formal definition of the Wigner distribution

For simplicity, we only consider the case of a single particle without spin propagating in 3 dimensions and not coupled to a vector potential.

VII.A.1 Preliminary

Consider a Hermitian operator $\hat{A}$ on a Hilbert space $\mathscr{H}$, with a continuous spectrum of eigenvalues $\{\alpha\}$. The corresponding eigenvectors will be denoted by $\{|\alpha, s\rangle\}$, where the extra index s — assumed to be discrete — accounts for the possible degeneracy of the eigenvalue α.

For $a \in \mathbb{R}$, one defines the operator on $\mathscr{H}$

$$\hat{D}_A(a) \equiv \int \mathrm{e}^{\mathrm{i}x(\hat{A}-a)} \frac{\mathrm{d}x}{2\pi}. \tag{VII.46}$$

In a state described by the statistical operator $\hat{\rho}$, the expectation value of $\hat{D}_A(a)$ is as usual

$$\langle \hat{D}_A(a) \rangle = \mathrm{Tr}\big[\hat{\rho}\hat{D}_A(a)\big].$$

Inserting a completeness relation with the eigenvectors of $\hat{A}$, one quickly finds

$$\langle \hat{D}_A(a) \rangle = \sum_s \langle a, s|\hat{\rho}|a, s\rangle = p_A(a), \tag{VII.47}$$

where $p_A(a)$ is the probability density to find a in a measurement of the physical quantity associated with $\hat{A}$.

VII.A.2 Weyl operator, Wigner distribution

Generalizing Eq. (VII.46), one defines for a particle without spin propagating in 3 dimensions the *Weyl operator*

$$\hat{W}(\vec{r}, \vec{p}) \equiv \int \mathrm{e}^{\mathrm{i}[\vec{k}\cdot(\hat{\vec{r}}-\vec{r})+(\hat{\vec{p}}-\vec{p})\cdot\vec{x}/\hbar]} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{\mathrm{d}^3\vec{x}}{(2\pi\hbar)^3}, \tag{VII.48}$$

where $\hat{\vec{r}}$ and $\hat{\vec{p}}$ are the position and momentum operators for the particle. Clearly, $\hat{W}(\vec{r}, \vec{p})$ is Hermitian for any $\vec{r}$, $\vec{p}$.

For every state of the particle under study, the associated Wigner distribution at time t is defined as the expectation of the Weyl operator:

$$\rho_W(t, \vec{r}, \vec{p}) \equiv \langle \hat{W}(\vec{r}, \vec{p}) \rangle_t. \tag{VII.49}$$

If the particle is in a pure state described by a normalized ket vector $|\Psi(t)\rangle$, then

$$\rho_W(t, \vec{r}, \vec{p}) = \langle \Psi(t) | \hat{W}(\vec{r}, \vec{p}) | \Psi(t) \rangle. \tag{VII.50}$$

If the particle is characterized by a statistical operator $\hat{\rho}(t)$, then

$$\rho_W(t, \vec{r}, \vec{p}) = \mathrm{Tr}\big[\hat{\rho}(t)\hat{W}(\vec{r}, \vec{p})\big]. \tag{VII.51}$$

Denoting by $\{|\vec{r}\rangle\}$ the (non-degenerate, since the particle has no spin) eigenvectors of $\hat{\vec{r}}$, one can show that the matrix elements of the Weyl operator in the position representation are

$$\langle\vec{r}_1\big|\hat{W}(\vec{r},\vec{p})\big|\vec{r}_2\rangle = \frac{1}{(2\pi\hbar)^3}\mathrm{e}^{-\mathrm{i}\vec{p}\cdot(\vec{r}_2-\vec{r}_1)/\hbar]}\,\delta^{(3)}\!\left(\frac{\vec{r}_1+\vec{r}_2}{2}-\vec{r}\right). \tag{VII.52}$$

Using this result one can compute the trace in Eq. (VII.51), which leads to Eq. (VII.1).

Eventually, one can invert the Fourier transforms in Eq. (VII.48), which leads to

$$\hat{\rho}(t) = (2\pi\hbar)^3\int \rho_W(t,\vec{r},\vec{p})\,\hat{W}(\vec{r},\vec{p})\,\mathrm{d}^3\vec{r}\,\mathrm{d}^3\vec{p}. \tag{VII.53}$$

One checks at once that this result is consistent with the matrix elements of $\hat{\rho}$, Eqs. (VII.3)–(VII.4).