

VI.6.1 a Mass conservation

First, instead of particle number, we consider mass, setting $\chi = m$ in formula (VI.59). Since this is a constant, the second, fourth and fifth term drop out, while the average $\langle m \rangle_{\vec{p}}$ in the first term is trivial. The remaining first and third terms then give

$$\frac{\partial}{\partial t} [mn(t, \vec{r})] + \vec{\nabla}_{\vec{r}} \cdot [mn(t, \vec{r}) \langle \vec{v} \rangle_{\vec{p}}] = 0.$$

Introducing the mass density

$$\rho(t, \vec{r}) = mn(t, \vec{r}) \tag{VI.60a}$$

and the average velocity [cf. Eq. (VI.22a)]

$$\vec{v}(t, \vec{r}) \equiv \langle \vec{v} \rangle_{\vec{p}}, \quad (\text{VI.60b})$$

this becomes

$$\frac{\partial \rho(t, \vec{r})}{\partial t} + \vec{\nabla} \cdot [\rho(t, \vec{r}) \vec{v}(t, \vec{r})] = 0, \quad (\text{VI.61})$$

which expresses the local conservation of mass.⁽⁹⁶⁾

VI.6.1 b Momentum balance

Choosing now for χ the i -th component of linear momentum $p^i = mv^i$, the second and fourth terms in the balance equation (VI.59) vanish, leaving

$$\frac{\partial}{\partial t} [mn \langle v^i \rangle_{\vec{p}}] + \vec{\nabla}_{\vec{r}} \cdot [mn \langle v^i \vec{v} \rangle_{\vec{p}}] - nF^i = 0. \quad (\text{VI.62})$$

The term within square brackets in the time derivative is clearly $\rho(t, \vec{r}) \mathbf{v}^i(t, \vec{r})$. The mass density also appears in the argument of the spatial-divergence term. The average $\langle v^i \vec{v} \rangle_{\vec{p}}$ can be transformed by noting the identity

$$\langle (v^i - v^i)(v^j - v^j) \rangle_{\vec{p}} = \langle v^i v^j \rangle_{\vec{p}} - v^i v^j$$

and introducing the second-rank *stress tensor* $\boldsymbol{\pi}$, whose components are given by

$$\pi^{ij}(t, \vec{r}) = \rho(t, \vec{r}) \langle [v^i - v^i(t, \vec{r})][v^j - v^j(t, \vec{r})] \rangle_{\vec{p}}. \quad (\text{VI.63})$$

The balance equation (VI.62) then becomes

$$\frac{\partial}{\partial t} [\rho(t, \vec{r}) v^i(t, \vec{r})] + \sum_{j=1}^3 \frac{\partial}{\partial x^j} [\rho(t, \vec{r}) v^i(t, \vec{r}) v^j(t, \vec{r}) + \pi^{ij}(t, \vec{r})] = \frac{1}{m} \rho(t, \vec{r}) F^i(t, \vec{r}). \quad (\text{VI.64a})$$

Using the mass conservation equation (VI.61), this can be rewritten as⁽⁹⁶⁾

$$\frac{\partial v^i(t, \vec{r})}{\partial t} + \vec{v}(t, \vec{r}) \cdot \vec{\nabla} v^i(t, \vec{r}) = \frac{1}{m} F^i(t, \vec{r}) - \frac{1}{\rho(t, \vec{r})} \sum_{j=1}^3 \frac{\partial \pi^{ij}(t, \vec{r})}{\partial x^j}. \quad (\text{VI.64b})$$

VI.6.1 c Balance equation for internal energy

Consider eventually $\chi(t, \vec{r}, \vec{p}) = [\vec{p} - m\vec{v}(t, \vec{r})]^2/2m = \frac{1}{2}m[\vec{v} - \vec{v}(t, \vec{r})]^2$, which represents the kinetic energy of the particles in a frame locally comoving with their average velocity. Differentiation with respect to time or momentum followed by an average over \vec{p} easily allow one to check that the second and fifth terms in Eq. (VI.59) are zero. There remains

$$\frac{1}{2} \frac{\partial}{\partial t} \langle mn(\vec{v} - \vec{v})^2 \rangle_{\vec{p}} + \frac{1}{2} \vec{\nabla}_{\vec{r}} \cdot \langle mn(\vec{v} - \vec{v})^2 \vec{v} \rangle_{\vec{p}} - \frac{1}{2} mn \langle \vec{v} \cdot \vec{\nabla}_{\vec{r}}(\vec{v} - \vec{v})^2 \rangle_{\vec{p}} = 0. \quad (\text{VI.65})$$

In the third term, one can first write $\frac{1}{2} \vec{\nabla}_{\vec{r}}(\vec{v} - \vec{v})^2 = [(\vec{v} - \vec{v}) \cdot \vec{\nabla}_{\vec{r}}] \vec{v} - (\vec{v} - \vec{v}) \times (\vec{\nabla}_{\vec{r}} \times \vec{v})$. One then checks that the third term equals the negative of the sum over all indices i and j of π^{ij} times the derivative $\partial v^j / \partial x^i$.

Let then

$$e(t, \vec{r}) = \frac{1}{2} \rho(t, \vec{r}) \langle [\vec{v} - \vec{v}(t, \vec{r})]^2 \rangle_{\vec{p}} \quad (\text{VI.66a})$$

denote the local density of internal energy and

⁽⁹⁶⁾The subscript on the $\vec{\nabla}$ operator has been suppressed, as it is obvious that it denotes the gradient with respect to position.

$$\vec{\mathcal{J}}_U(t, \vec{r}) = \frac{1}{2} \rho(t, \vec{r}) \left\langle [\vec{v} - \vec{v}(t, \vec{r})]^2 [\vec{v} - \vec{v}(t, \vec{r})] \right\rangle_{\vec{p}} \quad (\text{VI.66b})$$

be the internal energy (or heat) flux in the local rest frame. These quantities allow one to rewrite the first and second term in Eq. (VI.65) as the time derivative of internal-energy density and the divergence of $\vec{\mathcal{J}}_U + e\vec{v}$, respectively.

All in all, one finds⁽⁹⁶⁾

$$\frac{\partial e(t, \vec{r})}{\partial t} + \vec{\nabla} \cdot [\vec{\mathcal{J}}_U(t, \vec{r}) + e(t, \vec{r}) \vec{v}(t, \vec{r})] = - \sum_{i,j=1}^3 \pi^{ij}(t, \vec{r}) \frac{\partial v_j(t, \vec{r})}{\partial x^i}. \quad (\text{VI.67})$$

This expresses the local balance of energy.

The balance equations (VI.61), (VI.64b) and (VI.67) are the same, up to the external force, as the laws of hydrodynamics (I.70), (I.71) and (I.72), which were derived at the macroscopic level. Here, we have expressions of the stress tensor and the internal-energy flux in terms of microscopic quantities, which will allow us to compute them provided we have the form of the single-particle density in the Boltzmann gas.

VI.6.2 Zeroth-order approximation: the Boltzmann gas as a perfect fluid

We first start by assuming that the single-particle distribution in the Boltzmann gas is given by a local equilibrium distribution $\bar{f}^{(0)}(t, \vec{r}, \vec{p})$ of the type (VI.38), characterized by a local number density $n(t, \vec{r})$, a local temperature $T(t, \vec{r})$ and a local average velocity $\vec{v}(t, \vec{r})$.

VI.6.2a Stress tensor and internal-energy flux density

Inserting $\bar{f}^{(0)}(t, \vec{r}, \vec{p})$ in the expressions (VI.63) and (VI.66b), we obtain the corresponding stress tensor $\pi_{ij}^{(0)}$ and flux density of internal energy in the local rest frame $\vec{\mathcal{J}}_U^{(0)}$.

Let $\vec{w} \equiv \vec{v} - \vec{v}(t, \vec{r})$ denote the velocity of a particle as measured with respect to that reference frame. The stress tensor reads component-wise⁽⁹⁷⁾

$$\pi_{ij}^{(0)}(t, \vec{r}) = \rho(t, \vec{r}) \left[\frac{m}{2\pi k_B T(t, \vec{r})} \right]^{3/2} \int w_i w_j \exp \left\{ -\frac{m\vec{w}^2}{2k_B T(t, \vec{r})} \right\} d^3 \vec{w}.$$

When $i \neq j$, the integrand is odd in w_i , and thus the integral vanishes: the off-diagonal elements of the stress tensor are zero. Using formula (A.1b) with $n = 2$, one finds that all three diagonal elements are equal to $\mathcal{P}(t, \vec{r}) \equiv n(t, \vec{r}) k_B T(t, \vec{r})$. All in all

$$\pi_{ij}^{(0)} = \mathcal{P}(t, \vec{r}) \delta_{ij} = n(t, \vec{r}) k_B T(t, \vec{r}) \delta_{ij}. \quad (\text{VI.68})$$

Interpreting $\mathcal{P}(t, \vec{r})$ as the local pressure—which can be justified by investigating the force on a surface element, which is then normal—, one recognizes the mechanical equation of state of a classical ideal gas.

Calculating the internal energy density (VI.66a), one at once finds $e = \frac{3}{2} n k_B T = \frac{3}{2} \mathcal{P}$, i.e. the thermal equation of state of a classical ideal gas.

For the internal-energy flux, Eqs. (VI.38) and (VI.66b) give

$$\vec{\mathcal{J}}_U^{(0)}(t, \vec{r}) = \frac{1}{2} \rho(t, \vec{r}) \left[\frac{m}{2\pi k_B T(t, \vec{r})} \right]^{3/2} \int \vec{w}^2 \vec{w} \exp \left\{ -\frac{m\vec{w}^2}{2k_B T(t, \vec{r})} \right\} d^3 \vec{w} = \vec{0} \quad (\text{VI.69})$$

where the rightmost identity is due to the oddness of the integrand. That is, there is no diffusive transport of internal energy to this order of approximation.

⁽⁹⁷⁾Throughout this section we use the straightforward change of variable $\int_{\vec{p}} (\cdot) = \int (\cdot) \frac{d^3 \vec{p}}{(2\pi\hbar)^3} = \frac{m^3}{(2\pi\hbar)^3} \int (\cdot) d^3 \vec{w}$.

Since $\boldsymbol{\pi}^{(0)}$ is diagonal, there is no transport of linear momentum by (shear) viscosity either: the Boltzmann gas behaves like a non-dissipative fluid. As a consequence, the balance equations (VI.61), (VI.64b) and (VI.67) will represent the laws governing the dynamics of such a perfect fluid.

VI.6.2b Dynamics of a perfect fluid

Inserting the stress tensor (VI.68) and the (trivial!) heat flux density (VI.69) in Eqs. (VI.64b) and (VI.67), one finds

$$\frac{\partial \vec{v}(t, \vec{r})}{\partial t} + [\vec{v}(t, \vec{r}) \cdot \vec{\nabla}] \vec{v}(t, \vec{r}) = \frac{1}{m} \vec{F}(t, \vec{r}) - \frac{1}{\rho(t, \vec{r})} \vec{\nabla} \mathcal{P}(t, \vec{r}), \quad (\text{VI.70a})$$

which is the *Euler equation* for the dynamics of a perfect fluid, while the local balance of internal energy reads

$$\frac{\partial e(t, \vec{r})}{\partial t} + \vec{\nabla} \cdot [e(t, \vec{r}) \vec{v}(t, \vec{r})] = -\mathcal{P}(t, \vec{r}) \vec{\nabla} \cdot \vec{v}(t, \vec{r}), \quad (\text{VI.70b})$$

where in the right-side one can make the substitution $\mathcal{P}(t, \vec{r}) = \frac{2}{3}e(t, \vec{r})$.

VI.6.3 First-order approximation: the Boltzmann gas as a Newtonian fluid

Adopting now the relaxation-time approximation (VI.49) and assuming that the single-particle distribution can be written as $\bar{f} = \bar{f}^{(0)} + \bar{f}^{(1)}$ [Eq. (VI.50)], we can find $\bar{f}^{(1)}$, and deduce the modifications of the stress tensor and flux of internal energy, which give us new dynamical equations.

VI.6.3a Single-particle distribution to subleading order

In the framework of relaxation-time approximation with a momentum-independent relaxation time, and making the decomposition $\bar{f} = \bar{f}^{(0)} + \bar{f}^{(1)}$ with $|\bar{f}^{(1)}| \ll \bar{f}^{(0)}$, one finds after dropping out the terms of higher order the relation

$$\bar{f}^{(1)}(t, \vec{r}, \vec{p}) = -\tau_r \left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}_{\vec{r}} + \vec{F} \cdot \vec{\nabla}_{\vec{p}} \right) \bar{f}^{(0)}(t, \vec{r}, \vec{p}), \quad (\text{VI.71})$$

which allows the calculation of $\bar{f}^{(1)}(t, \vec{r}, \vec{p})$ by differentiating the expression (VI.38) of the local equilibrium distribution $\bar{f}^{(0)}(t, \vec{r}, \vec{p})$.

Assuming for simplicity that there is not external force \vec{F} and focusing on stationary solutions, one finds after performing all computations⁽⁹⁸⁾

$$\bar{f}^{(1)} = -\tau_r \left[\frac{\vec{w} \cdot \vec{\nabla} T}{T} \left(\frac{1}{k_B T} \frac{m}{2} \vec{w}^2 - \frac{5}{2} \right) + \frac{m}{k_B T} \sum_{i,j} w^i w^j \frac{\partial v_i}{\partial x^j} - \frac{1}{3} \frac{m \vec{w}^2}{k_B T} \vec{\nabla} \cdot \vec{v} \right] \bar{f}^{(0)},$$

where as above $\vec{w} \equiv \vec{v} - \vec{v}(t, \vec{r})$, while all gradients are with respect to spatial coordinates. This can also be rewritten as

$$\bar{f}^{(1)} = -\tau_r \left[\frac{\vec{w} \cdot \vec{\nabla} T}{T} \left(\frac{1}{k_B T} \frac{m}{2} \vec{w}^2 - \frac{5}{2} \right) + \frac{m}{2k_B T} \sum_{i,j=1}^3 \left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i} \right) \left(w^i w^j - \frac{1}{3} \vec{w}^2 \delta^{ij} \right) \right] \bar{f}^{(0)}. \quad (\text{VI.72})$$

VI.6.3b Stress tensor and internal-energy flux density

Substituting $\bar{f} = \bar{f}^{(0)} + \bar{f}^{(1)}$ in Eq. (VI.63) yields the expression of the stress tensor to first order in τ_r . With the same local equilibrium distribution $\bar{f}^{(0)}$ as in § VI.6.2 and $\bar{f}^{(1)}$ given by Eq. (VI.72),

⁽⁹⁸⁾The detailed derivation—which only necessitates some careful bookkeeping in computing the various partial derivatives, but presents no real difficulty, and does not provide much physical insight—can be found e.g. in Huang [54], chapter 5.5.

one finds

$$\pi_{ij}(t, \vec{r}) = m \int_{\vec{p}} w_i w_j [\bar{f}^{(0)}(t, \vec{r}, \vec{p}) + \bar{f}^{(1)}(t, \vec{r}, \vec{p})] = \pi_{ij}^{(0)}(t, \vec{r}) + \pi_{ij}^{(1)}(t, \vec{r}),$$

with obvious notations, where $\pi_{ij}^{(0)}$ is again given by Eq. (VI.68), $\pi_{ij}^{(0)} = n k_B T \delta_{ij}$.

Inspecting the correction to the single-particle distribution (VI.72), the first term within the square brackets is odd in \vec{w} and thus will not contribute to $\pi_{ij}^{(1)}$, which leaves

$$\pi_{ij}^{(1)}(t, \vec{r}) = -\frac{m^2 \tau_r}{2k_B T} \sum_{k,l=1}^3 \left(\frac{\partial v_k}{\partial x^l} + \frac{\partial v_l}{\partial x^k} \right) \int_{\vec{p}} w_i w_j \left(w^k w^l - \frac{1}{3} \vec{w}^2 \delta^{kl} \right) \bar{f}^{(0)}(t, \vec{r}, \vec{p}).$$

In the integral, only the indices k, l such that every component of \vec{w} appears with an even power contribute: either $k = i$ and $j = l$, or $k = j$ and $l = i$, or $i = j$ (diagonal terms) and $k = l$. Using then formula (A.1b) with $n = 2$ or 3 , one finds

$$\pi_{ij}^{(1)}(t, \vec{r}) = -\eta(t, \vec{r}) \left[\frac{\partial v_i(t, \vec{r})}{\partial x^j} + \frac{\partial v_j(t, \vec{r})}{\partial x^i} - \frac{2}{3} \delta_{ij} \vec{\nabla} \cdot \vec{v}(t, \vec{r}) \right] \quad (\text{VI.73a})$$

with

$$\eta(t, \vec{r}) = n(t, \vec{r}) k_B T(t, \vec{r}) \tau_r, \quad (\text{VI.73b})$$

and all in all

$$\pi_{ij} = \mathcal{P} \delta_{ij} - \eta \left[\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i} - \frac{2}{3} (\vec{\nabla} \cdot \vec{v}) \delta_{ij} \right]. \quad (\text{VI.73c})$$

Comparing with Eqs. (I.73b) and (I.73d), one recognizes the stress tensor of a Newtonian fluid, whose shear viscosity η is given by Eq. (VI.73b), while its bulk viscosity ζ vanishes.

Inserting now $\bar{f} = \bar{f}^{(0)} + \bar{f}^{(1)}$ in the internal energy flux (VI.66b), $\bar{f}^{(0)}$ does not contribute—as in Eq. (VI.69)—, nor does the second term within the square brackets in Eq. (VI.72). There remains

$$\begin{aligned} \vec{J}_U(t, \vec{r}) &= \vec{J}_U^{(0)}(t, \vec{r}) + \vec{J}_U^{(1)}(t, \vec{r}) \\ &= -\frac{m^2 \tau_r}{2T(t, \vec{r})} \sum_{j=1}^3 \frac{\partial T(t, \vec{r})}{\partial x_j} \int_{\vec{p}} \left(\frac{1}{k_B T} \frac{m}{2} \vec{w}^2 - \frac{5}{2} \right) w_j \vec{w}^2 \vec{w} \bar{f}^{(0)}(t, \vec{r}, \vec{p}). \end{aligned}$$

Performing the integration, one eventually finds

$$\vec{J}_U(t, \vec{r}) = \vec{J}_U^{(1)}(t, \vec{r}) = -\kappa(t, \vec{r}) \vec{\nabla} T(t, \vec{r}) \quad (\text{VI.74a})$$

with

$$\kappa(t, \vec{r}) = \frac{5}{2} \frac{n(t, \vec{r}) k_B^2 T(t, \vec{r}) \tau_r}{m}. \quad (\text{VI.74b})$$

One recognizes Fourier's law, with κ the heat conductivity.

Remark: The transport coefficients η and κ , Eqs. (VI.73b) and (VI.74b) are both proportional to the relaxation time τ_r . As mentioned in § VI.4.3, this time is (at least) of the order of the mean free time τ_{mfp} between two successive collisions of a particle, say $\tau_r \sim \tau_{\text{mfp}}$. The latter, divided by some typical particle velocity, gives the mean free path ℓ_{mfp} , i.e. the typical length traveled by a particle between two successive collisions. In turn, ℓ_{mfp} is inversely proportional to the particle density and to the total interaction cross-section, $\ell_{\text{mfp}} \sim 1/n\sigma_{\text{tot}}$. As a consequence, η and κ in a Boltzmann gas are in first approximation independent of density—yet the latter should be small enough that only two-to-two collisions take place—and inversely proportional to the cross-section: the more ideal a gas is (small σ_{tot}), the more dissipative (large transport coefficients) it is. An ideal gas is thus not a perfect fluid!

VI.6.3c Dynamics of a Newtonian fluid

Eventually, one can substitute the stress tensor (VI.73c) and the internal-energy flux (VI.74a) in the balance equation for linear momentum [Eq. (VI.64b)]. Straightforward calculations give

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = \frac{1}{m} \vec{F} - \frac{1}{\rho} \vec{\nabla} \mathcal{P} + \frac{\eta}{\rho} \left[\Delta \vec{v} + \frac{1}{3} \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) \right], \quad (\text{VI.75})$$

which is the *Navier–Stokes equation* [Eq. (I.79)] for a Newtonian fluid with vanishing bulk viscosity. Inserting the stress tensor and heat flux in the balance equation for internal energy, one accordingly recovers Eq. (I.80) with $\zeta = 0$.

Bibliography

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